(Based on a worksheet by Rob Bayer)

Second Order Equations

1. For each of the following pairs of functions, determine whether they are linearly independent or dependent:

   (a) $f(x) = e^x,\ g(x) = e^{x+1}$ **DEPENDENT**

   (b) $f(x) = xe^{2x},\ g(x) = e^{2x}$ **INDEPENDENT**

   (c) $f(x) = \ln(x^3),\ g(x) = \ln(x^\pi)$ **DEPENDENT**

   (d) $f(x) = \sin(x),\ g(x) = \cos(x)$ **INDEPENDENT**

2. Which of the following second order differential equations are linear? Homogeneous?

   (a) $e^x y'' + \cos(3x^2)y' + 3y = 0$ **Linear, Homogeneous**

   (b) $y'' + 3y' + 7y = \cos x$ **Linear, Non-homogeneous**

   (c) $y'' + 3xy' + y^2 = 0$ **Not Linear**

   (d) $\tan(y'') + \cos(x)y' = e^x$ **Not Linear**

3. Consider the differential equation $y'' = -y$

   (a) By just thinking about it for a while, come up with two linearly independent solutions to this equation and then use them to find the general solution. (Think about elementary functions whose derivatives cycle.) $\cos x$ **and** $\sin x$ **are two linearly independent solutions.**

   (b) Show that $y = \cos(x + a)$ is a solution for any constant a. **Just compute it!** $y'' = -\cos(x + a) = -y$**.**

   (c) Argue that parts (a) and (b) show that $\cos(x + a) = C_1 \cos x + C_2 \sin x$ for an appropriate choice of C_1 and C_2. **All of the solutions to a second-order linear differential equation are given by sums of constant multiples of two linearly independent solutions. So,** $\cos(x + a)$ **must be expressible in terms of** $\cos(x)$ **and** $\sin(x)$**.**

   (d) Without using trig identities, find C_1 and C_2. (Hint: what should $y(0)$ and $y'(0)$ be? Remember that a is a constant, so C_1 and C_2 can refer to it.) Does this formula look familiar? **Plugging in** $x = 0$**, we get** $\cos a = C_1$**. Taking the derivative and evaluating at** $x = 0$**, we get** $-\sin a = C_2$**. So, we obtain** $\cos(x + a) = \cos a \cos x - \sin a \sin x$**, the angle addition formula.**

Solving Second Order Homogeneous Linear ODEs with Constant Coefficients

1. Find the general solution to $y'' + 3y' - 18y = 0$.

   The auxiliary equation is
   $$r^2 + 3r - 18 = 0 \Rightarrow r = 3 \text{ or } -6.$$
   Therefore the general solution is:
   $$y = c_1 e^{3x} + c_2 e^{-6x}.$$

2. Solve the initial value problem $y'' + 4y' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 3$.

   The auxiliary equation is
   $$r^2 + 4r + 4 = 0 \Leftrightarrow (r + 2)^2 = 0.$$
   Therefore the general solution is:
   $$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$
   Plugging in our initial conditions, we obtain:
   $$1 = c_1, 3 = -2c_1 + c_2 \Rightarrow c_1 = 2, c_2 = 5 \Rightarrow y = e^{-2x} + 5xe^{-2x}.$$

3. Solve the boundary value problem $y'' = y$ with $y(0) = 0$ and $y(2) = 2$.

   The auxiliary equation is
   $$r^2 - 1 = 0 \Leftrightarrow (r + 1)(r - 1) = 0.$$
   Therefore the general solution is:
   $$y = c_1 e^x + c_2 e^{-x}.$$
   Plugging in our boundary conditions, we obtain:
   $$0 = c_1 + c_2, 2 = c_1 e^2 + c_2 e^{-2} \Rightarrow c_1 = \text{csch}(2), c_2 = -\text{csch}(2) \Rightarrow y = \text{csch}(2)e^x - \text{csch}(2)e^{-x}.$$