(Problems selected from worksheets by Rob Bayer.)

(1) Determine whether each of the following sequences are convergent or divergent. For those that are convergent, find the limit.

(a) $a_n = \frac{3n^2+1}{n^2-1}$. **Convergent.** $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \mathbf{3}$.

(b) $a_n = \frac{(n+2)!}{(2n)^2 \cdot n!}$. **Convergent.** $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \frac{\mathbf{1}}{\mathbf{4}}$.

(c) $\{1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \ldots\}$. **Divergent. There are infinitely many terms equal to 1, and another infinite set approaching zero. No sequence can have two limits.**

(d) $a_n = \ln(n^2-3n+1) - \ln(n^2+4)$. **Convergent.** $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \ln(\lim_{\mathbf{n}\to\infty} \frac{\mathbf{n^2-3n+1}}{\mathbf{n^2+4}}) = \ln \mathbf{1} = \mathbf{0}$. **We can move the limit inside because ln is continuous at 1.**

(e) $a_n = n\tan(1/n)$. **Convergent. Apply L'Hopital's rule to the corresponding continuous function.** $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \mathbf{1}$.

(2) True/False. For all problems, a_n and b_n are sequences. If the answer is true, cite a theorem, or explain why. If it is false, give a counterexample, i.e. two sequences for which it is false.

(a) If a_n and b_n converge, then $a_n + b_n$ converges. **True.**

(b) If $a_n + b_n$ converges, then a_n and b_n converge. **False. Counterexample:**$\mathbf{a_n} = \frac{\mathbf{1}}{\mathbf{n}}, \mathbf{b_n} = \frac{\mathbf{-1}}{\mathbf{n}}$

(c) If a_n and b_n converge, then a_n/b_n converges. **False. Counterexample:** $\mathbf{a_n} = \mathbf{1}, \mathbf{b_n} = \frac{\mathbf{1}}{\mathbf{n}}$

(d) If a_n and b_n diverge, then $a_n + b_n$ diverges. **False. Counterexample:** $\mathbf{a_n} = \mathbf{n}, \mathbf{b_n} = -\mathbf{n}$

(e) If $a_n + b_n$ diverges, then a_n and b_n diverge. **False. Counterexample:** $\mathbf{a_n} = \mathbf{n}, \mathbf{b_n} = \frac{\mathbf{1}}{\mathbf{n}}$ **Here, only $\mathbf{a_n}$ diverges.**

(f) If a_n and b_n diverge, then $a_n b_n$ diverges. **False. Counterexample:** $\mathbf{a_n} = \cos \mathbf{n}, \mathbf{b_n} = \sec \mathbf{n}$

(3) For each of the following, give an example of a sequence with the required properties or explain why no such sequence can exist:

(a) Bounded, Monotonic, Convergent.

$$\mathbf{a_n} = \frac{\mathbf{1}}{\mathbf{n}}.\ \ \mathbf{0} \leq \mathbf{a_n} \leq \mathbf{1}.\ \ \mathbf{a_{n+1}} < \mathbf{a_n}.\ \ \lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \mathbf{0}.$$

(b) Bounded, Monotonic, Not Convergent. **By the Monotone Sequence Theorem, any bounded monotonic sequence will converge.**

(c) Bounded, Not Monotonic, Convergent.

$$\mathbf{a_n} = \frac{(\mathbf{-1})^{\mathbf{n}}}{\mathbf{n}}.\ \ -\mathbf{1} \leq \mathbf{a_n} \leq \mathbf{1}.\ \ \lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \mathbf{0}.$$

(d) Bounded, Not Monotonic, Not Convergent.

$$\mathbf{a_n} = (\mathbf{-1})^{\mathbf{n}}.\ \ -\mathbf{1} \leq \mathbf{a_n} \leq \mathbf{1}.$$

(e) Not Bounded, Monotonic, Convergent. **Every convergent sequence is bounded. If $\mathbf{a_n}$ approaches a limit L, then for some N, all of the terms $\mathbf{a_n}, \mathbf{n} > \mathbf{N}$ will be between $(\mathbf{L}-\mathbf{1})$ and $(\mathbf{L}+\mathbf{1})$. Since there area finitely many terms before that, you can take the smallest and biggest, m and M. Then, the bounds will be $\min\{\mathbf{m}, \mathbf{L}-\mathbf{1}\}$ and $\max\{\mathbf{M}, \mathbf{L}+\mathbf{1}\}$.**

(f) Not Bounded, Monotonic, Not Convergent.

$$\mathbf{a_n} = \mathbf{n}.\ \ \mathbf{a_{n+1}} > \mathbf{a_n}.$$

(g) Not Bounded, Not Monotonic, Convergent. **By part (e), the fact that it is not bounded implies that it cannot be convergent.**

(h) Not Bounded, Not Monotonic, Not Convergent.

$$\mathbf{a_n = (-1)^n n.}$$

(4) More Sequences! Determine convergence or divergence, and calculate the limit if convergent:

(a) $a_n = \frac{\cos^2 n + n}{2^n + 3^n}$. **Convergent.** $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \mathbf{0}$.

(b) $a_n = \frac{n^{(-1)^n}}{n + \ln n}$ **Divergent. When $\mathbf{n = 2k}$, the sequence is $\mathbf{\frac{n}{n+\ln n}}$, which approaches 1. When $\mathbf{n = 2k+1}$, the sequence is $\mathbf{\frac{1}{n^2 + n\ln n}}$, which approaches zero.**

(c) $a_n = n^{\frac{\ln 2}{1+\ln n}}$. **Convergent.** $\mathbf{a_n} = \exp(\mathbf{\frac{\ln n \ln 2}{1+\ln n}})$. **Because the exponential is continuous, we can take the limit of the inside.** $\lim_{\mathbf{n}\to\infty} \mathbf{a_n} = \exp(\lim_{\mathbf{n}\to\infty} \mathbf{\frac{\ln n \ln 2}{1+\ln n}}) = \exp(\ln \mathbf{2}) = \mathbf{2}$.