Integration Review

(1) $\int \sin x - \cos x + 1\, \mathrm{d}x.$

$= -\cos x - \sin x + x + C.$

(2) $\int x + \frac{1}{x}\, \mathrm{d}x.$

$= \frac{x^2}{2} + \ln|x| + C.$

(3) $\int_1^8 x\sqrt[3]{x-1}\, \mathrm{d}x.$ Let $u = x-1$. Then, $du = dx$, $x = u+1$

$= \int_0^7 (u+1)u^{1/3}\, du = \int_0^7 u^{4/3} + u^{1/3}\, du = \frac{3}{7}u^{7/3} + \frac{3}{4}u^{4/3}\Big|_0^7$

$= \frac{3}{7}\cdot 7^{7/3} + \frac{3}{4}\cdot 7^{4/3} = 21\sqrt[3]{7} + \frac{21}{4}\sqrt[3]{7} = \frac{105\sqrt[3]{7}}{4}.$

(4) $\int_0^2 \frac{5x}{x^2+5}\, \mathrm{d}x.$ Let $u = x^2+5$. $du = 2x$.

$= \int_5^9 \frac{\frac{5}{2}\, du}{u} = \frac{5}{2}\ln u \Big|_5^9 = \frac{5}{2}(\ln 9 - \ln 5)$

Integration by Parts

(1) $\int t\sin(t)\,dt$.

$$\begin{bmatrix} u = t & dv = \sin t \; dt \\ du = dt & v = -\cos t \end{bmatrix}$$

$$\Rightarrow \int t \sin t \, dt = -t\cos t + \int \cos t \, dt$$

$$= \boxed{-t\cos t + \sin t + C.}$$

(2) $\int_4^9 \frac{\ln y}{\sqrt{y}}\,dy$.

$$= 2y^{1/2}\ln y - \int \frac{2y^{1/2}}{y}\,dy \qquad \begin{bmatrix} u = \ln y & dv = y^{-\frac{1}{2}}\,dy \\ du = \frac{dy}{y} & v = 2y^{1/2} \end{bmatrix}$$

$$= 2y^{1/2}\ln y - \int 2y^{-1/2}\,dy$$

$$= 2y^{1/2}\ln y - 4y^{1/2}\Big|_4^9 = (6\ln 9 - 12) - (4\ln 4 - 8)$$

$$= \boxed{6\ln 9 - 4\ln 4 - 4.}$$

(3) $\int e^{2x}\sin(x)\,dx$.

$$= \frac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}\cos x\,dx \qquad \begin{bmatrix} u = \sin x & dv = e^{2x}\,dx \\ du = \cos x\,dx & v = \frac{1}{2}e^{2x}. \end{bmatrix}$$

$$\int e^{2x}\cos x\,dx = \qquad \begin{bmatrix} u = \cos x & dv = e^{2x}\,dx \\ du = -\sin x\,dx & v = \frac{1}{2}e^{2x}\,dx \end{bmatrix}$$

$$\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx$$

$$\Rightarrow I = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cos x + \frac{1}{2}I\right) \Rightarrow \frac{5}{4}I = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$

$$\Rightarrow I = \boxed{\int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\sin x - \frac{1}{5}e^{2x}\cos x + C.}$$

(4) $\int \tan^{-1}(x)\,dx$.

$$= x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx \qquad \begin{bmatrix} u = \tan^{-1}(x) & dv = dx \\ du = \frac{dx}{1+x^2} & v = x \end{bmatrix}$$

$$\int \frac{x}{1+x^2}\,dx \qquad \begin{bmatrix} \text{Let } w = 1+x^2. \\ dw = 2x\,dx \end{bmatrix}$$

$$= \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|1+x^2| + C$$

So, $\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \frac{1}{2}\ln|1+x^2| + C.$