Quiz 1

MATH 1B, SPRING 2012

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SECTION: NAME: SOLUTION

Solve the following integral, showing all steps clearly:

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

Let $x = a\sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Then, $dx = a\cos\theta\, d\theta$.

To change the bounds, $x = a\sin\theta = a$ when $\sin\theta = 1$, i.e. when $\theta = \frac{\pi}{2}$. $x = a\sin\theta = 0$ when $\sin\theta = 0$, at $\theta = 0$.

So, our integral becomes:

$$\int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^2(1-\sin^2\theta)}\cdot a\cos\theta\, d\theta.$$

• use $1-\sin^2\theta = \cos^2\theta$

$$= \int_0^{\pi/2} a^4\sin^2\theta\cos\theta\,|\cos\theta|\,d\theta.$$

• On this domain, $\cos\theta > 0$, so $|\cos\theta| = \cos\theta$.

$$= \int_0^{\pi/2} a^4\sin^2\theta\cos^2\theta\, d\theta.$$

• $2\sin\theta\cos\theta = \sin 2\theta$ so, $\sin^2\theta\cos^2\theta = \frac{1}{4}\sin^2 2\theta$

$$= \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta.$$

• $\sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$

$$= \frac{a^4}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{a^4}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{a^4}{8}\cdot\left(\frac{\pi}{2}-0\right) = \frac{\pi a^4}{16}.$$