Quiz 4 MATH 1B, SPRING 2012

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SOLUTION

Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}.$$

As part of this problem, you should:

1. Find a recurrence relation for the sequence.
2. Prove that the sequence converges (Hint: Use the Monotone Sequence Theorem).
3. Calculate the limit.

1 Recurrence Relation Approach

1. A recurrence relation for the sequence is $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2a_n}$.
2. We use the Monotone Sequence Theorem, so we need to prove the sequence is bounded and monotonic increasing:
   - **Monotone Increasing**: We argue by induction. Base case: $\sqrt{2} < \sqrt{2\sqrt{2}}$.
     Induction Step: Suppose $a_n < a_{n+1}$. Then, this implies $\sqrt{2a_n} < \sqrt{2a_{n+1}}$, since multiplication by 2 and square roots preserve inequality (the functions are increasing on $x > 0$). Substituting equivalent terms from the recurrence relation, we find $a_{n+1} < a_{n+2}$. By induction, the sequence is monotone increasing.
   - **Bounded:** Because the sequence is increasing, it is bounded below by its first term. For the upper bound, we claim that all terms are less than 2. Again, we argue by induction. Base case: $\sqrt{2} < 2$.
     Induction Step: Suppose $a_n < 2 \Rightarrow \sqrt{2a_n} < \sqrt{4} = 2$. Substituting equivalent terms from the recurrence relation, we find $a_{n+1} < 2$. By induction, the sequence is bounded above.
3. Since the sequence converges, we can take the limit on both sides of the recurrence relation, letting L be the limit of the sequence:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} \Rightarrow L = \sqrt{2L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0 \Rightarrow L = 0 \text{ or } 2$$

   Because all terms of the sequence are bigger than $\sqrt{2} > 0$, the limit must be 2.

2 General Term Approach

Many people noticed that you can actually solve this problem by identifying the general term of the sequence.
We can translate the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$$

into

$$\left\{2^{1/2}, 2^{3/4}, 2^{7/8}, 2^{15/16}, \dots\right\}.$$

in which case, the general term is clearly

$$a_n = 2^{1-\frac{1}{2^n}}$$

Taking the limit, we can move the limit of a_n, we can move the limit to the inside of the exponential, because it is a continuous function. So, the limit is

$$\lim_{n\to\infty} 2^{1-\frac{1}{2^n}} = 2^{\lim_{n\to\infty}(1-\frac{1}{2^n})} = 2^1 = 2$$

A major advantage to this approach is that instead of proving convergence before calculating the limit, we prove the sequence converges directly, using the general term.