MATH 1B WORKSHEETS SELECTED SOLUTIONS II

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M Mar 10

2. **Keyword** Convergence/divergence of sequences.

Solution (1)
$$\lim_{n\to\infty} \frac{\sqrt{n}-1}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1-1/\sqrt{n}}{1+1/\sqrt{n}} = 1$$
(2)
$$\lim_{n\to\infty} \sin\frac{n\pi}{2n+1} = \sin(\lim_{n\to\infty} \frac{n\pi}{2n+1}) = \sin(\pi/2) = 1$$
(3) Using l'Hôpital's law,
$$\lim_{n\to\infty} \frac{n^2}{\ln n} = \infty$$
(4)
$$\lim_{n\to\infty} \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{2n\cdot(2n-1)\cdot\dots\cdot(n+1)\cdot n\cdot\dots\cdot 2\cdot 1}{n\cdot\dots\cdot 2\cdot 1} = \lim_{n\to\infty} 2n\cdot\dots\cdot(n+1)$$
Thus the limit is ∞ because the right-handed-side is greater than $2n$ which goes to ∞.

3. **Keyword** $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$

Solution Put $a_{n+1} = (1+a_n)^{1/2}$ with $a_1 = 1$.

First, the sequence is less than 2 by the following induction: Step 1. $a_1 < 2$ is obvious. Step 2. assume $a_n < 2$, then $a_{n+1} = (1+a_n)^{1/2} < (1+2)^{1/2} = \sqrt{3} < 2$. This completes the induction.

Second, the sequence is increasing by the following induction: Step 1. $a_2 = \sqrt{1+a_1} = \sqrt{2} > 1 = a_1$. Step 2. assume $a_n > a_{n-1}$, then because $(1+x)^{1/2}$ is a increasing function, $(1+a_n)^{1/2} > (1+a_{n-1})^{1/2}$, ie. $a_{n+1} > a_n$. This completes the induction.

Therefore, by the monotonicity theorem, a_n converges to some number A. By taking limit on both side of the deduction formula, $A = (1+A)^{1/2}$. This has two solutions, $\frac{-1\pm\sqrt{5}}{2}$, of which the $\frac{-1+\sqrt{5}}{2}$ is the only possible one.
$$\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = \frac{\sqrt{5}-1}{2}$$

M Mar 17 (*I put it wrongly as 'M Jan 17'*)

1. **Keyword** True or false about series convergence

Solution (3) The following series
$$S = \sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sin(1/(n+1))}$$
is not convergent as the sequence of terms goes to 1 (by l'Hôpital's law, or expanding the term as power series.)

2.(2) **Keyword** $\sum_{n=0}^{\infty} \frac{2}{(n+1)(2n+1)}$

(*I changed the problem in the discussion section, but using some power series it is still possible to find the sum even if we don't make the change*)

Solution

$$\sum_{n=0}^{\infty}\frac{2}{(n+1)(2n+1)}=\sum_{n=0}^{\infty}-4(\frac{1}{2n+2}-\frac{1}{2n+1})$$

Thus the right-handed-side can be rewritten as $-4\sum_{n=1}(-1)^n(1/n)$. Note that

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

.

When $x=1$ (believing the convergence) we have the sum of the original series to be $4\ln 2$.

3. **Keyword** limit comparison test.

Solution The convergence of the three series $\sum\frac{4}{n^2-n}$, $\sum\frac{n+1}{n^2-n}$ and $\sum\frac{n^3+1}{n^2-1}$ is easy to determine by doing the limit comparison with dominating parts (ie. $4/n^2$, $1/n$, n respectively).

M Apr 14

1. **Keyword** index shifting.

Solution The three series are transformed as follows:

$$\sum_{n=0}^{\infty}\frac{n^2x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(n-1)^2x^n}{n}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(2n+1)x^{n+2}}{n+1}+\sum_{n=0}^{\infty}\frac{(2n+1)x^{n+1}}{n+1} &= \sum_{n=2}^{\infty}\frac{[2(n-2)+1]x^n}{(n-2)+1}+\sum_{n=1}^{\infty}\frac{[2(n-1)+1]x^n}{(n-1)+1}\\ &= \sum_{n=2}^{\infty}\frac{(2n-3)x^n}{n-1}+\sum_{n=1}^{\infty}\frac{(2n-1)x^n}{n}\\ &= x+\sum_{n=2}^{\infty}(\frac{2n-3}{n-1}+\frac{2n-1}{n})x^n\\ &= x+\sum_{n=2}^{\infty}\frac{1}{n(n-1)}x^n\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2nx^{2n-1}}{5}+\sum_{n=1}^{\infty}\frac{(2n+1)x^{2n}}{5} &= (\frac{2x}{5}+\frac{4x^3}{5}+\cdots)+(\frac{3x^2}{5}+\frac{5x^4}{5}+\cdots)\\ &= \frac{2x}{5}+\frac{3x^2}{5}+\frac{4x^3}{5}+\frac{5x^4}{5}+\cdots\\ &= \sum_{n=1}^{\infty}\frac{(n+1)x^n}{5}\end{aligned}$$

2. **Keyword** expansion using formal transformation

Solution I will only do (4) and (5).

$$\begin{aligned}
(2+x)\arctan x &= (2+x)\int \frac{1}{1+x^2}\mathrm{d}x \\
&= (2+x)\int \sum_{n=0}^{\infty}(-x^2)^n \mathrm{d}x \\
&= (2+x)\sum_{n=0}^{\infty}\int (-1)^n x^{2n}\mathrm{d}x \\
&= C+(2+x)\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{2x^{2n+1}}{2n+1}+(-1)^n\frac{x^{2n+2}}{2n+2}
\end{aligned}$$

Note here $C=0$ because when $x=0$ the function value is 0. Also, the two summation cannot be mixed together using the third trick in problem 1 so we just leave it separated.

$$\begin{aligned}
\ln\sqrt{1-2x-3x^2} &= \frac{1}{2}\{\ln(1+x)+\ln(1-3x)\} \\
&= \frac{1}{2}\{\sum_{n=1}^{\infty}\frac{(-x)^n}{n}+\sum_{n=1}^{\infty}\frac{(3x)^n}{n}\} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n+3^n}{2n}x^n
\end{aligned}$$

Here I used $\ln(1+t)=t-t^2/2+t^3/3-\cdots$ for shortcut, and you can also get it by $\ln(1+t)=\int\frac{\mathrm{d}t}{1+t}$.

4. If $f(x)$ is a function such that $f(c+x)=f(c-x)$ (for any $x\in\mathbb{R}$) with c fixed. What can you say about the Taylor series of $f(x)$ at c? How about if $f(c+x)=-f(c-x)$?

Solution Similar to the case of even/odd functions, the expansion for a function with $f(c+x)=f(c-x)$ at c only has terms with even power, while for a function with $f(c+x)=-f(c-x)$ at c the expansion only has terms with odd power.

M Apr 21

1. **Keyword** $f^{(4)}(0)$ of $f(x)=e^{-x^2}\cos(3x^2)$

Solution

$$\begin{aligned}
f(x) &= (1-x^2+\frac{x^4}{2}+\cdots)(1-\frac{(3x^2)^2}{2}+\cdots) \\
&= 1-x^2+(-1\cdot\frac{3^2}{2}+\frac{1}{2}\cdot 1)x^4+\cdots \\
&= 1-x^2-4x^4+\cdots
\end{aligned}$$

Therefore by the Taylor series formula $f^{(4)}(0)/4!=-4$ so $f^{(4)}(0)=-4!\cdot 4=-96$.

2. **Keyword** $\lim_{x\to 0}\frac{\ln(1+x^2)}{\arctan(2x^2)}$

Solution

$$\frac{\ln(1+x^2)}{\arctan(2x^2)}=\frac{-x^2+x^4/2-x^6/3+\cdots}{2x^2-4x^4/3+8x^6/5+\cdots}=\frac{-1+x^2/2-x^4/3+\cdots}{2-4x^2/3+8x^4/5+\cdots}\to-\frac{1}{2}$$

3. **Keyword** $1/\sqrt{(1+x)}$

Solution

$$\begin{aligned}
(1+x)^{-1/2} &= \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n \\
&= \sum_{n=0}^{\infty} \frac{\frac{-1}{2}\cdot\frac{-3}{2}\cdot\cdots\cdot(-\frac{1}{2}-n+1)}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{\frac{-1}{2}\cdot\frac{-3}{2}\cdot\cdots\cdot(-\frac{2n-1}{2})}{n!} x^n \\
&= \sum_{n=0}^{\infty} (-\frac{1}{2})^n \cdot \frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{n!} x^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!}{2^n n!} x^n
\end{aligned}$$

4. **Keyword** $\int_0^x e^{-t^2}\mathrm{d}t$

Solution

$$\int_0^x e^{-t^2}\mathrm{d}t = \int_0^x (1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\cdots)\mathrm{d}t = x-\frac{x^3}{3}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}+\cdots$$

The general term formula is

$$\frac{(-1)^n x^{2n+1}}{(2n+1)\cdot n!}$$

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To evaluate $\int_0^1 e^{-t^2}\mathrm{d}t$ up to error $< 10^{-8}$, we set $x = 1$ and consider which n makes

$$|\frac{(-1)^n}{(2n+1)\cdot n!}| < 10^{-8}$$

It turns out that $n = 10$.

M Apr 29

3.(1) **Keyword** $y'' + 2y' - 15y = (x+1)e^{2x}$

Solution I will do variation of parameters for example. Since $r^2 + 2r - 15 = 0$, $r = 3, -5$. Considering $y = u_1(x)e^{3x} + u_2(x)e^{-5x}$, we set up a system of linear equations:

$$\begin{aligned}
(1) \qquad e^{3x}u_1' + e^{-5x}u_2' &= 0 \\
(2) \qquad 3e^{3x}u_1' - 5e^{-5x}u_2' &= (x+1)e^{2x}
\end{aligned}$$

To solve the system, use $(1)*3-(2)$ to eliminate u_1':

$$(3e^{-5x} - (-5)e^{-5x})u_2' = -(x+1)e^{2x}$$

Therefore

$$u_2'(x) = -(x+1)e^{7x}/8$$

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Similarly, use $(1)*5+(2)$ to eliminate u_2':

$$(5e^{3x} + 3e^{3x})u_1' = (x+1)e^{2x}$$

.

Therefore

$$u_1'(x) = -(x+1)e^{-x}/8$$

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By integration we have $u_1(x) = \int -(x+1)e^{-x}/8 = (x+2)e^{-x}/8 + C_1,\ u_2(x) = \int -(x+1)e^{7x}/8 = (-x/7 - 6/49)e^{7x}/8 + C_2$. In conclusion,

$$y(x) = (x+2)e^{2x}/8 + C_1e^{3x} + (-x/7 - 6/49)e^{2x}/8 + C_2e^{-5x} = (\frac{3}{28}x + \frac{23}{98})e^{2x} + C_1e^{3x} + C_2e^{-5x}$$

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