Math-54 Fall 2003, Lin. Alg. Diff. Eq.
Midterm #1 Reviewsheet (1.1-3.2), 09-28-03

Below [W] stands for *Worksheet* with Q-Questions, P-Problem, and A-Additional; [H] for *Hill's textbook*, [Q] for *Quiz*, [Kn] for *Knutson's additional question*. $*$ denotes possibly hard questions.

Key Notes

1 Solutions of Linear System

• Concepts (check book to make sure you really know these names): *augmented matrix associated to a linear system, elementary row operation, row echelon (triangular) form.*
• Guassian elimination (trilogy): 1) Write the augmented matrix associated to the given linear system; 2) Reduce to row echelon form (upper triangular form) by elementary row operations; 3) Backsubstitution.
Tricks: free variable selection for many solutions.
Exercise: [W]2: P2,P3; 3: P1; [H]1.2: 7-9,12,26-29
• Number of solutions: 0, 1 and ∞.
Tricks: for 2 variable linear system, use plane geometry; for >= 3 variable system, use row echelon form of the associated augmented matrix.
Exercise: [W]2: P5,A1*; [H]1.1: 13-15,19,29-32*; [Q]1.3,1.4*.

2 The Algebra of Matrices

• Concepts: *matrix, sum and product of matrices, scalar product of a number and a matrix, elementary matrices (important!), diagonal matrix, (unit) upper and lower triangular matrices, the inverse of a matrix, the transpose of a matrix.*
• Basic exercise:
Tricks: matrix size must be compatible in sum (the same size) and products (# col of left = # row of right)
Exercise: [W]3: Q1; [H]1.3: 9-12
• Identity.
Tricks: I is the *only* common solution of $AX = A = XA$ for *all* square matrices A.
Exercise: [W]3:P3.
• Inverse:
Tricks: (1) Any solution of $AX = I = XA$ is an inverse of A (X square matrix); there is at most one inverse (unique if exists).
(2) Equivalent conditions: $Ax = 0$ (x vector) has only trivial solution; $Ax = b$ has a unique solution; A is factorized as product of elementary matrices; A is row-equivalent to I, ([H]Thm 1.50).
(3) A and B invertible $\Rightarrow$ AB invertible and actually $(AB)^{-1} = B^{-1}A^{-1}$.
(4) inverse of a 2×2 matrix and rotation matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}.$$

(5) inverse of elementary matrices and their products.

Exercise: [W]4:Q1*, P4∗, P5*, A1*, A2*. [H]1.5: 40*.
• Commutativity:
Tricks: (1) Any solution for $AX = XA$ is commutative to A (A, X square matrices, X unknown).
(2) scalar matrices aI (a is real) are commutative to arbitrary square matrix A.
(3) disjoint permutation matrices commute each other.
Exercise: [W]3:P6, P7, A2; [H]1.3: 15-16; [Q]2.3.
• Elementary matrices.
Tricks: 1) left multiplication ⇔ row operation; right multiplication ⇔ column operation.
2) inverse of elementary matrices.
3) product of elementary matrices, also see LU decomposition.
4) commutativity of elementary matrices, such as disjoint permutation matrices commute.
Exercise: [W]3: P5; 4: Q2; [H]1.4: 16-20, 1.5: 40*;
• LU decomposition.
Tricks: use products of elementary matrices to represent triangular matrices.
Exercise: [H]1.5: 2-3, 40*; [Kn]9/23: 1, 2; review: additional 1,3,4.
• Matrix equation.
Tricks: (1) $A^2 = 0$ does not imply $A = 0$, such as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

(2) we can have $AB = 0$ with $A \neq 0$ and $B \neq 0$, such as $A = (1, 0)$ and $B = (0, 1)^T$ or

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise: [W]3: A1*, p.11: P3; [Kn]9/23:1.
• Application (Markov chain): [W]3: Q2*, (also see Knutson's lecture notes on 9/25).
• Special matrices: pay attention to 2×2 matrices and its associated geometrical applications, like rotation matrices, diagonal matrices (stretch/compress), shear, (Knutson lecture note 9/9).

3 Vectors

• Concepts: *vector, addition and scalar multiplication of vectors and there geometric meaning, linear combination, span, dot product, norm (length) of vectors, projections, orthogonal, orthonormal, Gram-Schmidt process.*
• Formulas: dot product, angle, norm, projection, Gram-Schmidt process.
• Span: all linear combinations of a set of vectors.
Tricks: (1) geometry in $\mathbb{R}^3$: $\mathbb{R}^3$, plane through 0, line through 0 or 0. [W]6:Q2; [H]3.1:22.
(2) find a vector in the span by solving linear systems. [W]6:P1.
• Dot product: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots u_n v_n$.
Norm: $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{v}}$. [H]3.2:16,17,19.
Angle: $\cos\theta = (\mathbf{u} \cdot \mathbf{v})/(\|\mathbf{u}\|\|\mathbf{v}\|)$. [W]6: A1*.
Bilinearity: $r(\mathbf{u} \cdot \mathbf{v}) = (r\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (r\mathbf{v})$ ($\Rightarrow \|r\mathbf{u}\| = |r|\|\mathbf{u}\|$). [H]3.2:36-37.
Exercise: [W]6:Q4; [Kn]review: 2.
• Orthogonal, Orthnormal, etc.
Orthogonal vectors: $\mathbf{u} \cdot \mathbf{v} = 0$
Orthonormal vectors: $\mathbf{u} \cdot \mathbf{v} = 0$ and $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$
Orthogonal matrix: $A^T A = I = AA^T$ (so $A^{-1} = A^T$ and column/row vectors are orthonormal)

Exercise: [Kn]review: 4, 5.

• Projection and Gram-Schmidt Process.

Projection:

$$\text{proj}_{\mathbf{p}}\mathbf{u} = \text{projection of } \mathbf{u} \text{ on } \mathbf{p} = \frac{\mathbf{u}\cdot\mathbf{p}}{\mathbf{p}\cdot\mathbf{p}}\mathbf{p} = \|\mathbf{u}\|\cos\theta\frac{\mathbf{p}}{\|\mathbf{p}\|}.$$

Gram-Schmidt Process: given a set of (independent) vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, Gram-Schmidt process will find a set of orthogonal (orthonormal) vectors $\{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ ($\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$) such that $\text{span}\{\mathbf{p}_1, \ldots, \mathbf{p}_k\} = \text{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ $(1 \le k \le n)$.

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{u}_1, \\
\mathbf{p}_2 &= \mathbf{u}_2 - \text{proj}_{\mathbf{p}_1}\mathbf{u}_2, \\
\mathbf{p}_3 &= \mathbf{u}_3 - \text{proj}_{\mathbf{p}_2}\mathbf{u}_3 - \text{proj}_{\mathbf{p}_1}\mathbf{u}_3, \\
\vdots \;\; &\;\; \vdots \;\; \vdots \\
\mathbf{p}_n &= \mathbf{u}_n - \text{proj}_{\mathbf{p}_{n-1}}\mathbf{u}_n - \text{proj}_{\mathbf{p}_{n-2}}\mathbf{u}_n - \ldots - \text{proj}_{\mathbf{p}_1}\mathbf{u}_n.
\end{aligned}$$

Normalization $\mathbf{q}_i = \mathbf{p}_i/\|\mathbf{p}_i\|$ $(1 \le i \le n)$ gives an orthonormal set.

Tricks: Pay attention to orthogonal set/orthonormal set required for Gram-Schmidt process. You can rescale $\mathbf{p}_i$ in each step to simplify the computation.

Exercise: [W]6:P5; [H]3.1:41-44, 47-45.

Good Luck!