Review Problems

Introduction

Below you will find a compilation of some extra problems which you may find useful in reviewing for your quizzes and exams.

There are four sections of problems. Within each section, the problems are organized randomly. Some problems are especially challenging so don't despair if you find yourself stumped by a portion. The complete solutions appear after the four sections of problems.

Integrals and related problems

1. Determine whether $\int_0^2 \frac{dx}{4x-5}$ is improper. If improper either evaluate, or prove that the integral is divergent.

2. True or False: $\int_{-4}^{4} \frac{dx}{5x^{1/3}-4}$ converges by comparison to $\int_{-4}^{4} \frac{dx}{x^{\frac{1}{3}}}$.

3. Find the arc length of $y = \cosh x$ on the interval $0 \leq x \leq 1$.

4. Evaluate the integral: $\int \tan^4 x \, dx$.

5. Evaluate the integral: $\int \sqrt{2x - x^2} \, dx$.

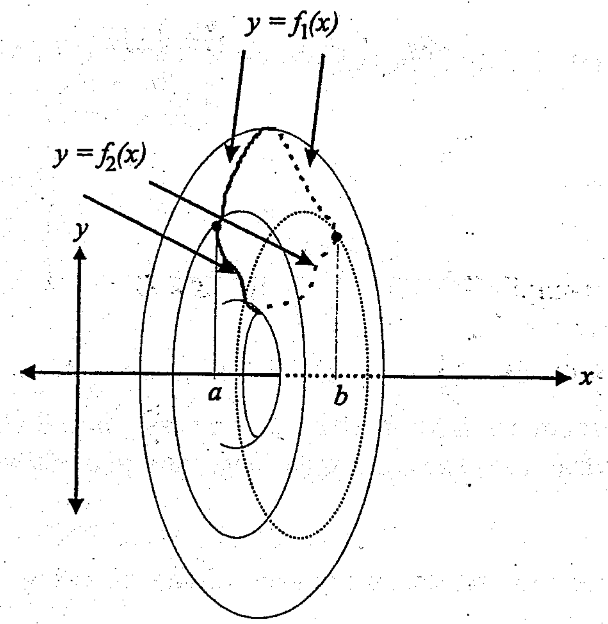
6. (a) Evaluate the integral $\int \frac{dx}{x^2(x+2)}$.

   (b) Evaluate $\int_1^{\infty} \frac{dx}{x^2(x+2)}$ or show that it is divergent.

7. Find the length of the curve defined by:
$$y = \frac{1}{x^2}, \quad 0 < x \leq 1.$$

8. Find the partial fractions decomposition (including the values of A, B, etc.) of $\frac{x^3+2x}{x^3+1}$.

9. Below is pictured a surface of revolution generated by rotating the curves $y = f_1(x)$ and $y = f_2(x)$ around the x-axis.



Find a formula for the surface area of this surface, involving $f_1(x), f_2(x)$, and their derivatives.

10. Integrate: $\displaystyle\int \frac{\sin x}{\cos^{101} x}\, dx$

11. Which of the curves below has *both* of the following properties:
   - its length is *infinite.*
   - the area beneath it and above the x-axis is *finite.*

   (a) $y = \dfrac{1}{\sqrt{x}\cdot|\ln x|}, \quad 0 < x \leq e^{-1}.$

   (b) $y = \dfrac{1}{|\ln x|}, \quad 1 < x < \infty.$

   (c) $y = \dfrac{1}{\sqrt{x}}, \quad e^{-1} < x \leq 1.$

   (d) $y = \dfrac{1}{x^2}, \quad 0 < x \leq e^{-1}.$

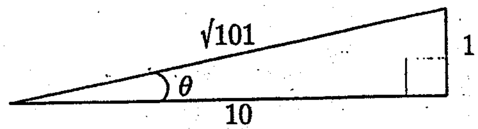
   (e) None of the above.

Sequences and Series

1. Does the following series converge or diverge. (Justify.)

$$\sum_{n=0}^{\infty} \frac{(1,000,000)^n}{n!}$$

2. Below you may use the formula $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. Consider the following figure:



   (a) Using the above, find an expression for θ in terms of an infinite sum.

   (b) Find an approximation for θ with an error less than 0.00001. (You don't need to simplify any fractions that you may have, and needn't express your answer using decimals.)

3. Find a power series representation for each of the following. State the *radius* of convergence in each case.

   (a) $-\ln(1-x)$

   (b) $\ln(1+x)$

   (c) $\ln\left(\frac{1+x}{1-x}\right)$

4. Find the Maclaurin series for $\ln(1-x^3)$. What is the corresponding Taylor polynomial, $T_7(x)$ about $x = 0$?

5. Give an example of each of the following. (No explanation required.)

   (a) An infinite sum whose convergence can be decided by the ratio test.

   (b) An infinite sum whose convergence can be decided by the root test.

   (c) A sequence that is bounded above but diverges.

6. (a) Does $\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^{1.5}}$ converge or diverge?

   (b) Does $\sum_{n=100,000}^{\infty} \frac{1}{n(\ln n)^{1.5}}$ converge or diverge?

7. Determine whether or not the given sum converges. Find its value if it does. (Justify)

   (a) $\sum_{n=1}^{\infty} \left[\sin\left(\frac{n+1}{n}\right) - \sin\left(\frac{n+2}{n+1}\right)\right]$

   (b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

8. For (a) and (b) determine whether the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, absolutely convergent, or divergent.

   (a) $a_n = \dfrac{n^2 - n + 2}{\sqrt[4]{n^{10} + n^5 + 3}}$

   (b) $a_n = (-1)^n \dfrac{1 + e^{-n}}{n}$

9. Find the Taylor series of the function $f(x) = \dfrac{1}{\sqrt{x}}$ about the point $a = 1$.

10. Determine whether each of the following diverges or converges. (Justify.)

    (a) $.9 - .99 + .999 - .9999 + .99999 - .999999 + \cdots$

    (b) $\dfrac{1}{2^2} + \dfrac{2}{3^2} + \dfrac{3}{4^2} + \cdots$

    (c) $\sum_{n=1}^{\infty} (\dfrac{n^2 + 2n}{n^3 + 1} - \dfrac{1}{2})^n$

11. Find the radius of convergence for:

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} 3^{3n} x^n.$$

12. For (a) and (b) determine if the sequence $\{a_n\}$ converges. If it does, find the limit.

    (a) $a_n = \dfrac{1}{n^{-\ln n}}$

    (b) $a_n = \sqrt{\dfrac{(1+n)n}{\sin n + n^2}}$

13. Obtain the Taylor series of $1 - \sin^2 x$ about $x = 0$.
    [Hint: trigonometric identities.]

Differential Equations

1. Find the general solution of the ODE:

$$y' = \cos^2 y \cdot \ln x$$

2. Solve $y' + \cos x \cdot y = \sin x \cdot \cos x$.

3. Find the general solution of $y' = \dfrac{y}{x} + 2$.

4. Sketch a direction field for $\frac{dy}{dt} = \frac{y}{t}$. Then for each initial condition below, graph a solution curve for $t \geq 1$ on the direction field which satisfies the condition.

   (a) $y(1) = 0$.

   (b) $y(1) = 1$.

5. True or false: The families of curves $x = ky^2$ and $\frac{1}{2}x^2 + y^2 = c$ (c and k are constants) are orthogonal trajectories. (Justify.)

6. Solve:

$$y'' - 2y' - 3y = 0; \quad y(0) = 3, \quad y'(0) = 1.$$

7. Solve:

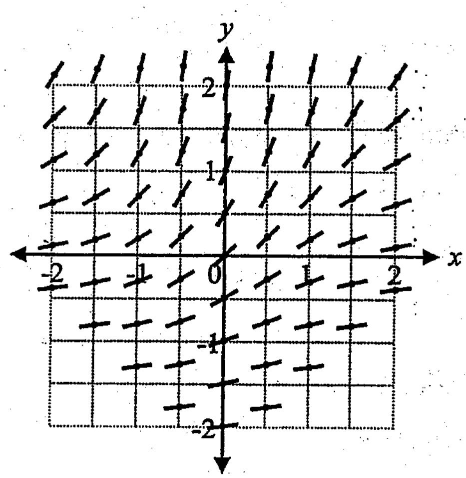
$$y'' - 2y' + 5y = 0.$$

8. Solve:

$$y'' - 6y' + 9y = 0; \quad y(0) = 1, y(1) = e^4 + e^3.$$

9. Below is pictured a direction field for a differential equation of the form

$$y' = f(x, y).$$



Which of the following best describes the function $f(x, y)$?

(a) $\frac{y}{x} + e^{\frac{y}{x}}$

(b) $\frac{e^y}{1+x^2}$

(c) $\csc x$

(d) $\sin y$

10. Consider the linear differential equation

$$y' \cdot \cos x = y \cdot \sin x + e^x \cos x$$

   (a) Which of the following is an *integrating factor* for the differential equation:
      i. $I(x) = e^{\int e^x \cos x \, dx}$
      ii. $I(x) = \sin x$
      iii. $I(x) = e^{-\ln|\cos x|}$
      iv. $I(x) = e^{\cos x}$
      v. $I(x) = \cos x$
      vi. $I(x) = e^{-\cos x}$
      vii. None of the above.

   (b) Find the general solution to the above differential equation.

11. Find the solution of $\dfrac{d^2y}{dx^2} = xy; \quad y(0) = 1, \quad \dfrac{dy}{dx}(0) = 0.$

Complex Numbers

1. Let $z = 1 - i\sqrt{3}$.

   (a) Find $|z|$.

   (b) Find $\arg z$.

   (c) Find z^5 . [Hint DeMoivre, or Euler.]

2. (a) Find all solutions to the equation $x^2 - 2x + 5 = 0$.

   (b) For each solution x, write x^2 and $\dfrac{1}{x}$ in the form $a + ib$.

3. (a) Solve $x^6 = -1$.

   (b) Factor $x^6 + 1$ over $\mathbf{C}$.

   (c) Factor $x^6 + 1$ over $\mathbf{R}$.