Name *Awful Korchiescko* GSI *We ain't got no god.*

Ma 53 Final '08   Prof 

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1. C is the parametric curve in the plane,

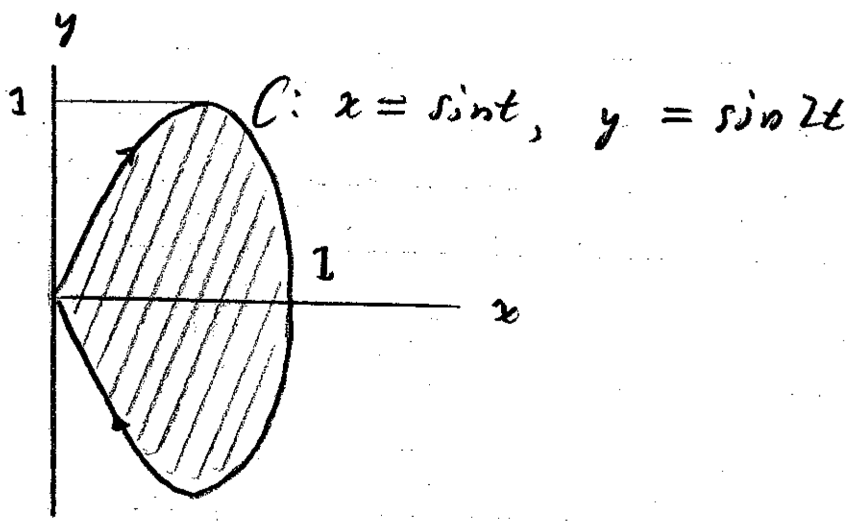
$x(t) = \sin t, \quad y(t) = \sin 2t, \quad 0 \leq t \leq \pi.$

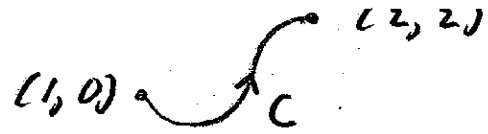
a) Compute the line integral $\int_C y\,dx$.

$$\int_0^\pi \underbrace{\sin 2t}_{y}\ \underbrace{\cos t\,dt}_{dx} = 2\int_0^\pi \cos^2 t \sin t\,dt =$$

$$\frac{2}{3}\left[\cos^3 t\right]_0^\pi = \frac{4}{3}.$$

b) The line integral is the area of a region. Sketch the region.





2. Let C be any curve in $\mathbb{R}^2$ from $(1,0)$ to $(2,2)$. Compute the line integrals

a) $\int_C x\,dx - y\,dy$.

$$= \frac{1}{2}(x^2 - y^2)\Big]_{(1,0)}^{(2,2)} = -\frac{1}{2}$$

b) $\int y\,dx + x\,dy = xy\Big]_{(1,0)}^{(2,2)} = 4 - 0 = 4$

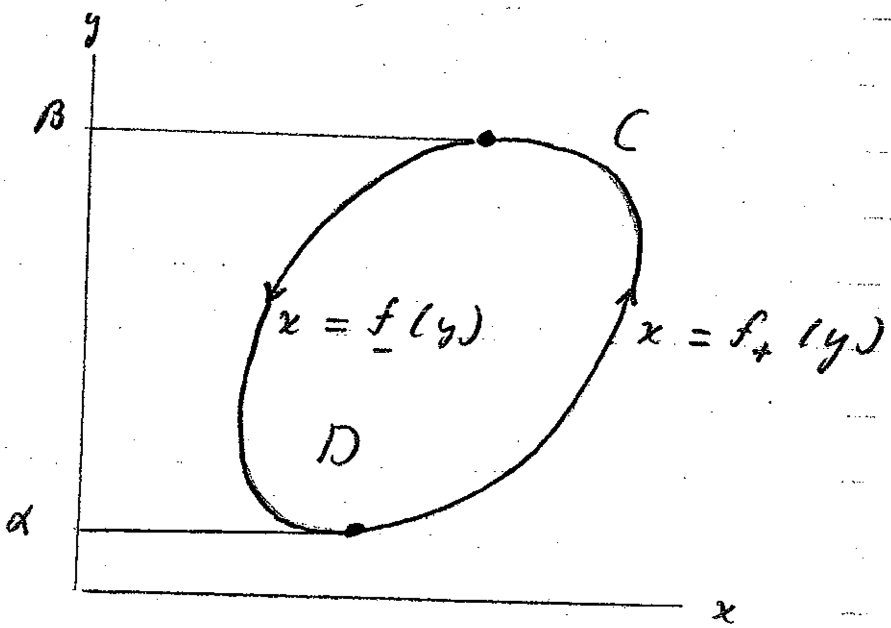
c) Let C be any curve in $\mathbb{R}^3$ from $(1, 0, 0)$ to $(0, 1, 1)$. Compute the line integral

$$\int_C \underbrace{(y+z)}_{f_x}dx + \underbrace{(x+z)}_{f_y}dy + \underbrace{(x+y)}_{f_z}dz.$$

$$f = xy + yz + zx$$

$$\int_C = xy + yz + zx \Big]_{(1,0,0)}^{(0,1,1)} = 1 - 0 = 1$$

3. C is a simple closed curve made of "left" and "right" halves, like this:



We have $f_-(y) < f_+(y)$ in $\alpha < y < \beta$ and $f_- = f_+$ at $y = \alpha$ or b. Shew that for continuously differentiable $Q(x, y)$,

$$\int_C Q\,dy = \iint_D Q_x\,dA,$$

where D is region enclosed by C.

$$\int_C Q\,dy = \int_\alpha^\beta Q(f_+(y), y)\,dy +$$

$$\int_\beta^\alpha Q(f_-(y), y)\,dy =$$

$$\int_\alpha^\beta \{Q(f_+(y), y) - Q(f_-(y), y)\}\,dy =$$

$$\int_\alpha^\beta \int_{f_-(y)}^{f_+(y)} Q_x(x, y)\,dx\,dy =$$

$$\iint_D Q_x(x, y)\,dA.$$

4. a) Compute the upward pointing unit normal of the surface $z = x^2 - y^2$.

$$\underline{r} = x\underline{i} + y\underline{j} + z(x,y)\underline{k}$$

$$\underline{r}_x \times \underline{r}_y = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} =$$

$$-z_x\underline{i} - z_y\underline{j} + \underline{k} =$$

$$-2x\underline{i} + 2y\underline{j} + \underline{k}$$

$$\underline{n} = \frac{\underline{r}_x \times \underline{r}_y}{|\quad|} = \frac{-2x\underline{i} + 2y\underline{j} + \underline{k}}{\sqrt{1 + 4x^2 + 4y^2}}$$

b) Compute the area of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = 1$.

$$S = \iint_{D:\, x^2+y^2<1} \sqrt{1+4(x^2+y^2)}\, dA =$$

$$2\pi \int_0^1 \sqrt{1+4r^2}\, r\, dr =$$

$(u = r^2,$

$du = 2r\,dr)$

$$\pi \int_0^1 \sqrt{1+4u}\, du = \frac{\pi}{4}\frac{2}{3}(1+4u)^{\frac{3}{2}}\Big]_0^1$$

$$= \frac{\pi}{6}(5^{\frac{3}{2}} - 1)$$

5. Compute the surface integral of vector field $\underline{F} = \dfrac{\underline{r}}{|\underline{r}|^3}$ over any plane $z =$ constant $\neq 0$, with $\underline{n} = \underline{k}$.

$$\iint_{S'} \underline{F} \cdot \underline{n} \, d\sigma = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \, dx\,dy =$$

$$2\pi \int_0^{\infty} \frac{z}{(r^2+z^2)^{\frac{3}{2}}} \, r\,dr =$$

$$(u = r^2)$$

$$\pi \int_0^{\infty} \frac{z}{(u+z^2)^{\frac{3}{2}}} \, du = -2\pi \frac{z}{(u+z^2)^{\frac{1}{2}}} \Big]_0^{\infty} = \begin{cases} 2\pi, & z > 0 \\ -2\pi, & z < 0 \end{cases}$$

6. The vector field $\underline{F}(\underline{r})$ on $\mathbb{R}^3$ has radially symmetric divergence, $\underline{\nabla}\cdot\underline{F} = e^{-|\underline{r}|}$. Compute $\iint_S \underline{F}\cdot\underline{n}\,dS$, where S is the sphere of radius R about the origin.

$$\iint_S \underline{F}\cdot\underline{n}\,dS = \iiint_{E:\,\rho<R} \underline{\nabla}\cdot\underline{F}\,dV =$$

$$4\pi\int_0^R \underbrace{r^2}_{u}\,\underbrace{e^{-r}\,dr}_{dv} =$$

$$du = 2r\,dr \qquad v = -e^{-r}$$

$$-4\pi r^2 e^{-r}\Big]_0^R + 8\pi\int_0^R \underbrace{r}_{u}\,\underbrace{e^{-r}\,dr}_{dv} =$$

$$-4\pi R^2 e^{-R} - 8\pi r e^{-r}\Big]_0^R + 8\pi\int_0^R e^{-r}\,dr =$$

$$8\pi\left\{1 - e^{-R} - Re^{-R} - \frac{R^2}{2}e^{-R}\right\}$$

7. a) $\underline{F}$ is a vector field on $\mathbb{R}^3$ with uniform curl, $\underline{\nabla} \times \underline{F} = \underline{i} + \underline{j} + \underline{k}$. C is a circle of radius R in a plane with normal parallel to $\underline{i} - \underline{j} - \underline{k}$. Compute $\int_C \underline{F} \cdot d\underline{r}$.

$$\int_C \underline{F} \cdot d\underline{r} = \iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n} \, dS =$$

$$(\underline{i} + \underline{j} + \underline{k}) \cdot \underbrace{\left(\frac{\underline{i} - \underline{j} - \underline{k}}{\sqrt{3}} \right)}_{\underline{n}} \pi R^2 = -\frac{\pi R^2}{\sqrt{3}}.$$

b) Is there a continuously differentiable vector field $\underline{F}(\underline{r})$ so $\underline{\nabla} \times \underline{F} = \underline{r}$? If not, prove it. If yes, construct one.

$$0 = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{F}) = \underline{\nabla} \cdot \underline{r} = \partial_x(x) + \partial_y(y) + \partial_z(z)$$

$$= 1 + 1 + 1 = 3.$$

contradiction. There <u>ain't</u>.

Name *Awful Kochiesko*

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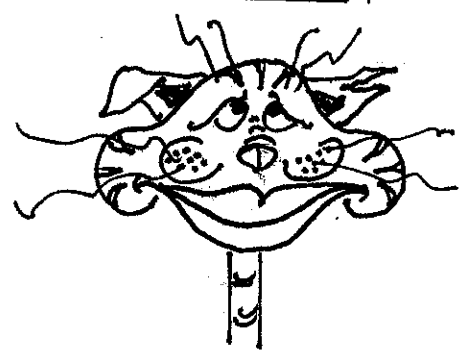


Math 53, Final Exam

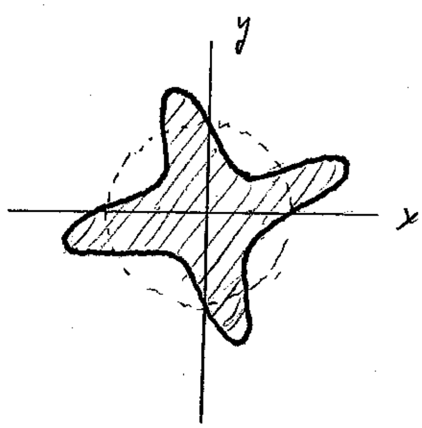
J.Neu, Instructor

1. The region D of the x, y plane is specified in terms of polar coordinates: $0 \leqq r \leqq 2+\sin 4\theta,\ 0 \leqq \theta \leqq 2\pi$.

   a) Sketch D.

   b) Compute $\iint_D \sqrt{x^2+y^2}\, dA$.

a)



b) $$\int_0^{2\pi}\int_0^{2+\sin 4\theta} r\ r\,dr\,d\theta =$$

$$\frac{1}{3}\int_0^{2\pi} (2+\sin 4\theta)^3\, d\theta =$$

$$\frac{1}{3}\int_0^{2\pi} (8 + 3\cdot 4 \underbrace{\sin 4\theta}_{0 \text{ contrib}} + 3\cdot 2\cdot \sin^2 4\theta + \underbrace{\sin^3 4\theta}_{0 \text{ contrib}})\, d\theta =$$

$$\frac{1}{3}\cdot 3\cdot 2\cdot\left(\frac{1}{2}\right)\cdot 2\pi = 2\pi$$

2. E is the region of space above the x, y plane, below the cone $z = \sqrt{x^2+y^2}$ and inside the cylinder $x^2+y^2 \leqq 1$. Compute $\displaystyle\iiint_E x^2 z \, dV$.

$$\int_0^{2\pi} \int_0^1 \int_0^r r^2 \cos^2\theta \, z \, r \, dz \, dr \, d\theta =$$

$$\left(\int_0^{2\pi} \cos^2\theta \, d\theta\right)\left(\int_0^1 \frac{r^4}{2} \, dr\right) = \frac{\pi}{10}$$

3. Compute $\iiint_E \frac{z^2}{(x^2+y^2+z^2)^4}\,dV$ where E is the shell between spheres of radius $\frac{1}{2}$ and 1 centered at the origin.

$$\int_0^{2\pi}\int_0^{\pi}\int_{\frac{1}{2}}^{1} \frac{\rho^2\cos^2\varphi}{\rho^8}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

$$2\pi\left(\int_0^{\pi}\cos^2\varphi\sin\varphi\,d\varphi\right)\left(\int_{\frac{1}{2}}^{1}\frac{d\rho}{\rho^4}\right) =$$

$$2\pi\left[-\frac{1}{3}\cos^3\varphi\right]_0^{\pi}\left[-\frac{1}{3}\frac{1}{\rho^3}\right]_{\frac{1}{2}}^{1} =$$

$$2\pi\left(\frac{2}{3}\right)\frac{1}{3}(8-1) = \frac{28}{9}\pi.$$

4. Compute the line integral $\int_C x dx - 2y dy$ where C is the segment of the curve $x^{10} + y^{10} = 1$ that goes from (1,0) to (0,1). (Hint: the equation of the curve is unpleasant. Perhaps it is irrelevant as well.)

$$x = \partial_x\left(\tfrac{1}{2}x^2 - y^2\right), \quad -2y = \partial_y\left(\tfrac{1}{2}x^2 - y^2\right)$$

$$\int_C x\,dx - y\,dy = \left[\tfrac{1}{2}x^2 - y^2\right]_{(1,0)}^{(0,1)} = -1 - \tfrac{1}{2} = -\tfrac{3}{2}.$$

5. Use Green's theorem to evaluate $\int_C \frac{-ydx + xdy}{x^2+y^2}$ where C is the circle of radius 1 centered about (1,1), with counterclockwise orientation.

$$P \equiv \frac{-y}{x^2+y^2} \Rightarrow P_y = \frac{-1}{x^2+y^2} - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$Q \equiv \frac{x}{x^2+y^2} \Rightarrow Q_x = \frac{x^2-y^2}{x^2+y^2}$$

$$Q_x - P_y \equiv 0 \quad \text{inside } C.$$

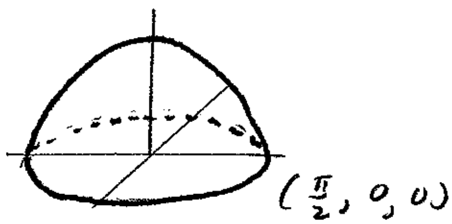
$$\int_C = 0.$$

6. The parametric surface S has vector equation $\mathbf{r} = \mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + \cos u \mathbf{k}$, $0 \leqq u \leqq \frac{\pi}{2}$, $0 \leqq v \leqq 2\pi$.

a) Sketch S.

b) Compute $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} \equiv x\mathbf{i} - y\mathbf{j} + \mathbf{k}$.

a)



b)

$$\underline{r}_u = \cos v\, \underline{i} + \sin v\, \underline{j} - \sin u\, \underline{k}$$

$$\underline{r}_v = -u \sin v\, \underline{i} + u \cos v\, \underline{j}$$

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & -\sin u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = u \sin u \cos v\, \underline{i} + u \sin u \sin v\, \underline{j} + u(\cos^2 v + \sin^2 v)\, \underline{k} =$$

$$u \sin u (\cos v\, \underline{i} + \sin v\, \underline{j}) + u\, \underline{k}$$

$$\underline{F} = u \cos v\, \underline{i} - u \sin v\, \underline{j} + \underline{k}$$

$$\underline{F} \cdot (\underline{r}_u \times \underline{r}_v) = u^2 \sin u (\cos^2 v - \sin^2 v) + u$$

$$\int_S = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \{u^2 \sin u \underbrace{(\cos^2 v - \sin^2 v)}_{\text{zero contrib}} + u\}\, du\, dv = (2\pi) \frac{1}{2} \left(\frac{\pi}{2}\right)^2 = \frac{\pi^3}{4}.$$

7. A cloud of dust is carried by velocity field $\mathbf{U} = (x-y)_{\cdot} + (2y-z)\mathbf{j} + (z-x)\mathbf{k}$. Let $V(t)$ be the volume of cloud as a function of time t. Its rate of change with respect to t is $V'(t) = \int_R \mathbf{U}\cdot\mathbf{n}\,dS$, where R denotes the surface of cloud at time t. What is $V'(t)$ when $V(t) = 2$?

$$V' = \iint_R \underset{\sim}{U}\cdot\underset{\sim}{n}\,dS = \iiint_E \underbrace{\operatorname{div}\underset{\sim}{U}}\,dV = 4\iiint_E dV = 4V.$$

$$\partial_x(x-y) + \partial_y(2y-z) + \partial_z(z-x) = 4$$

when $V = 2$, $V' = 8$.

8. a) Let $f(x, y, z)$ be a real valued function with continuous second partial derivatives. Show that $\text{curl}(\nabla f) = 0$.

b) Let S be a surface with boundary curve C.
Show that $\int_C \nabla f \cdot d\mathbf{x} = 0$ follows from Stokes' theorem.

a) $\underline{\nabla} f = f_x \underline{i} + f_y \underline{j} + f_z \underline{k}$

$$\text{curl } \underline{\nabla} f = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz})\underline{i} - (f_{zx} - f_{xz})\underline{j} + (f_{yx} - f_{xy})\underline{k} = 0.$$

b) $$\int_C \underline{\nabla} f \cdot d\underline{x} = \iint_S \underbrace{\text{curl}(\underline{\nabla} f)}_{\overset{\shortparallel}{0}} \cdot \underline{n} \, dS = 0.$$

Practice

1. D in (x,y) plane = interior of curve

$r = 2\cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad \iint_D \sqrt{x^2+y^2}\, dA$

2. E = region below $z = x^2+y^2$, above x,y plane and inside cylinder $(x-1)^2+y^2 = 1$. $\iiint_E xz\, dV$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \int_0^{r^2} r\cos\theta\, z\, r\, dz\, dr\, d\theta =$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos\theta)^5}{10} \cos\theta\, d\theta = \frac{16}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\theta\, d\theta =$$

$$\left(\frac{1}{2^6}\right)(20)\,\pi = \frac{5\pi}{16}$$

3. $\iiint_E \frac{z^2}{(x^2+y^2+z^2)^3}\, dV$, $E: 1 \le r \le 2$.

$$\int_1^2 \int_0^{\pi} \int_0^{2\pi} \frac{\rho^2 \sin^2\varphi \cos^2\theta}{\rho^6}\, \rho^2 \sin^2\varphi\, d\theta\, d\varphi\, d\rho$$

$$\int_1^2 \frac{d\rho}{\rho^2}\, d\rho \int_0^{\pi} \sin^4\varphi\, d\varphi \int_0^{2\pi} \cos^2\theta\, d\theta$$

4. $\int_C (2x-y)\,dx + (2y-x)\,dy$

$C: \; x^6 + y^{10} = 1 \qquad (-1,1) \to (1,1)$

$f = x^2 - xy + y^2$

$P = 2x - y \qquad Q = 2y - x$

$P_y = -1, \quad Q_x = -1 \quad \Rightarrow f = x^2 - xy + y^2$

$\int_C = f(1,0) - f(-1,0) = 0.$

5. $\int_C \underset{P}{-y\,dx} + \underset{Q}{x\,dy} = \qquad C: \; r = 2 + \sin 4\theta$

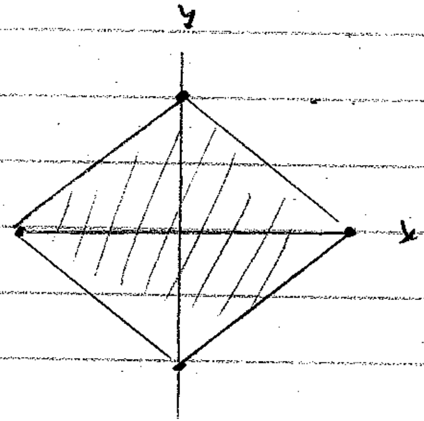
$\int_D (1-(-1))\,dA = 2 \text{ area of } D$

$\int_0^{2\pi} \int_0^{2+\sin 4\theta} 1 \; r\,dr\,d\theta$

6. $\underline{r} = \underbrace{(u+v)}\underline{i} + \underbrace{(u-v)}\underline{j} + \underbrace{uv}\underline{k} \quad -1 \le u, v \le 1.$

$$\left.\begin{aligned} x &= u+v \\ y &= u-v \end{aligned}\right] \quad \begin{aligned} u &= \tfrac{1}{2}(x+y) \\ v &= \tfrac{1}{2}(x-y) \end{aligned}$$

$$z = uv \qquad z = \tfrac{1}{4}(x^2 - y^2) \quad \text{saddle}$$



$$\underline{F} = x\underline{i} - y\underline{j} + z\underline{k} \qquad \iint_S \underline{F} \cdot \underline{n}\, dS$$

$$\underline{r}_u = \underline{i} + \underline{j} + v\underline{k}$$

$$\underline{r}_v = \underline{i} - \underline{j} + u\underline{k}$$

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & v \\ 1 & -1 & u \end{vmatrix} = (u-v)\underline{i} - (u+v)\underline{j} - 2\underline{k}$$

$$\iint_S = \int_{-1}^{1}\int_{-1}^{1} \underbrace{[(u-v)(u+v) + (u+v)(u-v) - 2uv]}_{2(u^2 - v^2 - uv)}\, du\, dv$$

7. A cloud of dust is carried by

velocity field $\underline{v} = (x-y)\underline{i} + (2y-z)\underline{j} + (z-x)\underline{k}$

Let $V(t)$ be volume of cloud as function of

time. Its time rate of change is

$$V'(t) = \int_S \underline{v}\cdot\underline{n}\, dS$$

where S denotes surface of cloud at time t.

What is $V'(t)$ when $V(t) = 2$?

<u>Soln</u> $\iint_S \underline{v}\cdot n\, dS = \iiint_E \operatorname{div} \underline{F}\, dV$

$$\operatorname{div} \underline{F} = \partial_x(x-y) + \partial_y(2y-z) + \partial_z(z-x) = 4$$

Hence, $V'(t) = 4\iiint_E dV = 4V = 8$.

Instructions. Be sure to put your name on each booklet you use. Also, please read the questions carefully. Some ask for more than one thing.

Show all work. Answers without supporting calculation or explanation may not get full credit. The more detail you show, the more credit for what your work deserves will be given.

Each problem is worth 20 points.

1. Consider the line l_1 in the direction of $\mathbf{a}_1 = (0, 1, 3)$ that passes through the point $(1, 2, 2)$ and the line l_2 in the direction of $\mathbf{a}_2 = (1, 0, -1)$ that passes through the point $(-1, -1, 1)$. Do the lines l_1 and l_2 lie in the same plane? If so, find an equation for the plane. If not, show why not.

2. Calculate the (acute) angle between the two planes $x + y = 1$ and $y + z = 1$. Find a parametric equation of the line that is their intersection.

3. Let $\mathbf{g}$ be a function from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that $\mathbf{g}(1, 0, 1) = (1, 1)$ and suppose $D\mathbf{g}(1, 0, 1)$ is the matrix

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & 4 & 4 \end{bmatrix}.$$

Suppose the curve $\gamma(t)$ passes through $(1, 0, 1)$ when $t = 2$ and that the tangent to the curve at the point $(1, 0, 1)$ is $(-1, -3, 1)$. What is the tangent vector to the curve in the plane, $\mathbf{g}(\gamma(t))$, at the point $\mathbf{g}(\gamma(2))$?

4. Suppose f is a differentiable function defined for all real numbers. Define a function of x and y by the formula

$$w = f\left(\frac{xy}{x^2 + y^2}\right).$$

Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0.$$

5. Suppose the components of the plane curve $\mathbf{x}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ are given by the formulae

$$x(s) = \int_0^s \cos g(t)\, dt, \qquad y(s) = \int_0^s \sin g(t)\, dt,$$

where g is a differentiable function and where $s \in [0, 1]$

   (a) Show that $\mathbf{x}$ is parameterized by arclength, i.e., show that $\|\mathbf{x}'(s)\| = 1$ for all s.

   (b) Calculate the curvature $\kappa = \|\frac{d^2\mathbf{x}}{ds^2}\|$ of $\mathbf{x}$.

Bonus Problem Show that the tangent plane to a sphere at any point P is always perpendicular to the vector $\overrightarrow{OP}$, where O is the center of the sphere.

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1. Let f be the function on the plane defined by the formula $f(x, y) = x^3 - 6xy + 3y^2 - 24x + 4$. Find the critical points of f and determine their nature, i.e., for each point, determine if it is a local maximum, a local minimum or a saddle point.

2. Suppose a and b are fixed positive numbers. Find the extreme values of $z = x/a + y/b$ subject to the condition $x^2 + y^2 = 1$. (That is, find the maxima and minima of $z = x/a + y/b$ when $x^2 + y^2 = 1$.)

3. Suppose f is a continuous function defined over a region S in the plane. Suppose that the double integral of f over S can be expressed as the repeated integral

$$\int_{-6}^{2}\left[\int_{(x^2-4)/4}^{2-x} f(x, y)\, dy\right] dx.$$

Make a sketch of the region S and write the double integral as an iterated integral with the order of the variables reversed.

4. Evaluate $\iiint_D xy\, dV$, where D is the solid region in the first octant that is bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and on the sides and bottom by the coordinate planes.

5. Let R be the region in the x, y-plane bounded by the lines, $y = x$ and $y = 2x$, and by the hyperbolas, $xy = 1$ and $xy = 3$. The goal of this problem is to walk you through the process of changing variables to evaluate the double integral: $\iint_R x^3 y\, dA$. The change of variables is given by the transformation T whose formula is $T(u, v) = (\frac{u}{v}, v)$, i.e., $x = \frac{u}{v}$ and $y = v$.

   (a) Sketch the region R and be sure to identify where each of the lines and hyperbolas is located.

   (b) Find the region R^* such that $TR^* = R$. *Be sure to identify the parts of the boundary of R^* that T sends to the various parts of the boundary of R.*

   (c) Calculate the Jacobian of T.

   (d) Convert the integral $\iint_R x^3 y\, dA$ to an integral over R^* and evaluate the integral.

Bonus Problem Let $f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$, where $A > 0$ and $B^2 < AC$.

1. Prove that there is a point (x_1, y_1) at which f has a minimum.

2. Prove that at this point $f(x_1, y_2) = Dx_1 + Ey_1 + F$.

3. Show that at this point

$$f(x_1, y_1) = \frac{1}{AC - B^2}\begin{vmatrix} \mathrm{A} & \mathrm{B} & \mathrm{D} \\ \mathrm{B} & \mathrm{C} & \mathrm{E} \\ \mathrm{D} & \mathrm{E} & \mathrm{F} \end{vmatrix}$$

Afterthoughts:

1. How many people in this room tried to lick their elbows during the test?

2. How many people thought seriously about trying to lick their elbows, but did not want to get caught making the attempt?

3. How many people will attempt to lick their elbows as soon as they get to some place private?

4. And what about you? Did you try to lick your elbow? Are you going to try? Fess up, now.

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Show all work. Answers without supporting calculation or explanation may not get full credit. The more detail you show, the more credit for what your work deserves will be given.

Each problem is worth 20 points.

1. Let $P(x,y) = 5 - xy - y^2$ and $Q(x,y) = x^2 - 2xy$. Evaluate the line integral $\oint_C P\,dx + Q\,dy$, where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

2. When the graph of a function $y = f(x)$, $a \le x \le b$, in the x,y-plane is revolved about the x-axis, the resulting surface in 3-space may be paramaterized by the vector function,
$$\mathbf{X}(s,t) = (s, f(s)\cos t, f(s)\sin t),$$
$a \le s \le b$, $0 \le t < 2\pi$. Calculate $\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}$ and use this calculation to show that the area of the surface is given by the integral
$$2\pi \int_a^b |f(s)|\sqrt{1 + (f'(s))^2}\,ds.$$

3. Let S denote the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$, and let $\mathbf{F}(x,y,z) = x\vec{i} + y\vec{j}$. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

4. Let S be the surface of the unit cube, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and let $\mathbf{F}(x,y,z) = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$. Calcuate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

5. Let ω be the differential form $y\,dx + z\,dy + x\,dz$. Show that $\int_C \omega = \pi a^2\sqrt{3}$, where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$, oriented in a fashion conistent with the outward normal to surface of the sphere.

Bonus Problem The purpose of this problem is to generalize the process of integration by parts to differential forms. Let M be an oriented, parametrized k-manifold in $\mathbb{R}^n$. Assume that the boundary of M, ∂M, is not empty and is given the orientation induced from that on M. Let ω be a differential $(k-1)$-form defined on an open set in $\mathbb{R}^n$ containing M and its boundary, and let f be a continuously differentiable function defined in the same open set.

1. Show that
$$d(f\omega) = df \wedge \omega + f d\omega.$$
(Hint: It may be helpful to assume first that $\omega = F_{i_1 \cdots i_{(k-1)}} dx_{i_1} \wedge dx_{i_2} \wedge \cdots dx_{i_{k-1}}$, where $F_{i_1 \cdots i_{(k-1)}}$ is a smooth function defined in the open set.)

2. Show that
$$\int_M f d\omega = \int_{\partial M} f\omega - \int_M df \wedge \omega.$$

3. Explain why, when $k = n = 1$, the formula in part 2 gives the formula for integration by parts from elementary calculus.