Math 53 Quiz 11

December 4, 2009

1. Evaluate $\iint_S z\, dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.

   Solution. S is the part of the paraboloid that lies under the plane $z = 4$. The plane is not included. Since $z = g(x, y) = x^2 + y^2$, we can use the formula in the book to get

$$\iint_S f dS = \quad \iint_D f(x, y, g(x, y))\sqrt{1 + (\tfrac{\partial g}{\partial x})^2 + (\tfrac{\partial g}{\partial y})^2} dA \tag{1}$$

   in which D is the radius 2 disk on the x, y plane (solve $z = x^2 + y^2$ and $z = 4$ to get the boundary of the disk). Compute the partial derivatives:

$$\frac{\partial g}{\partial x} = 2x \tag{2}$$

$$\frac{\partial g}{\partial x} = 2y \tag{3}$$

$$\tag{4}$$

   Thus we integrate

$$\iint_D (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2} dA = \int_0^{2\pi} \int_0^2 r^2\sqrt{1 + 4r^2}\, r\, dr d\theta \tag{5}$$

$$= 2\pi \int_0^2 r^3\sqrt{1 + 4r^2}\, dr \tag{6}$$

   Optional. To solve this integral, use the substitution $u = \sqrt{1 + 4r^2}$. Thus $u^2 = 1 + 4r^2$, $2udu = 8rdr$, and $r^2 = (u^2 - 1)/4$.

2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$. S is part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.

   Solution. Again the surface is in the form of $g(x, y) = \sqrt{x^2 + y^2}$, so we can use the formula

$$\iint \mathbf{F} \cdot d\mathbf{S} = \iint_D -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\, dA \tag{7}$$

   which in our case, D is the disk of radius 1 centered at the origin on the x, y plane. Since the problem states "downward orientation", and the orientation we get for setting $x = x, y = y, z = g(x, y)$ is the upward orientation, we need to add an negative sign.

$$-\iint_D -x\frac{x}{\sqrt{x^2+y^2}} - y\frac{y}{\sqrt{x^2+y^2}} + (x^2+y^2)^2 dA \tag{8}$$

$$=\iint_D \sqrt{x^2+y^2} - (x^2+y^2)^2 dA \tag{9}$$

$$=\int_0^{2\pi}\int_0^1 (r-r^4)r\,drd\theta \tag{10}$$

$$=2\pi\left[\frac{r^3}{3}-\frac{r^6}{6}\right]_0^1 = 2\pi/6 = \pi/3 \tag{11}$$