

Limits and Continuity

1. In section, we proved that $\lim_{x \rightarrow 2} x^2 = 4$. Let's extend this proof to the whole domain of this function: $\forall a \in \mathbb{R}$,

$$\lim_{x \rightarrow a} x^2 = a^2$$

And thus conclude that x^2 is a continuous function on \mathbb{R} .

2. Prove that

$$\lim_{x \rightarrow a} x^4 = a^4$$

Hint: $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2) = (x - a)(x + a)(x^2 + a^2)$

3. Prove that

$$\lim_{x \rightarrow 1} \frac{1}{x} = 1 \tag{1}$$

Hint: to make $1/|x|$ not blow up, it suffices to make sure that $|x|$ stays away from 0.

4. Prove that

$$\lim_{x \rightarrow 1} \sqrt{x} = 1 \tag{2}$$

Hint: First rationalize the expression, then you will end up having to deal with the term $\frac{1}{\sqrt{x}+1}$. Again, to bound such a thing, simply show that $\sqrt{x} + 1$ is not too close to 0.

5. Prove that the function $f(\mathbf{x}) = |\mathbf{x}|^2$ is continuous on \mathbb{R}^n . Hint: Don't forget the triangular inequality.