NOTES ON THE PROOF OF WEIL II

XINWEN ZHU

This is a short note on the proof of Theorem 3.3.1 of [D2]. The main reference is [KW].

1. ℓ-ADIC SHEAVES

We will assume that X is an algebraic scheme (i.e. separated, of finite type) over k some field. Let $\bar{k}$ be the separable closure of k. We will denote $\overline{X} = X \times_k \bar{k}$.

1.1. **Fundamental groups.**

Let FET/X be the category of schemes finite étale over X. Choose $\bar{x}$ a geometric point of X. The fiber functor $F_{\bar{x}} : FET \to SET$ is defined as $F_{\bar{x}}(Y) = \mathrm{Hom}_X(\bar{x}, Y)$. Then the étale fundamental group is defined as

$$\pi_1(X, \bar{x}) = Aut F_{\bar{x}}$$

$\pi_1(X, \bar{x})$ is a profinite group. If $f : X \to Y$ is a morphism, then it induces $f_* : \pi_1(X, \bar{x}) \to \pi_1(Y, f(\bar{x}))$. In the following, we will often omit the "base" point $\bar{x}$ and then f_* is well-defined up to inner automorphisms. If X is the spectrum of a field K, then $\pi_1(K, \bar{K})$ is just the Galois group $Gal(\bar{K}/K)$. If X is geometrically connected, one has the exact sequence

$$1 \to \pi_1(\overline{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to Gal(\bar{k}/k) \to 1$$

The first one is usually called the geometrical fundamental group, and denoted by π_1^{geom}, and the second one is usually called the arithmetic fundamental group and denoted by π_1^{arith}. The above sequence does not necessarily split. However, if $X(k) \neq \emptyset$, then a k-rational point gives a splitting $Gal(\bar{k}/k) \to \pi_1^{arith}$. In general, for $k' \supset k$ a finite extension, a k'-rational point gives $Gal(\bar{k}/k) \supset Gal(\bar{k}/k') \to \pi_1^{arith}$, which partially splits the map $\pi_1^{arith} \to Gal(\bar{k}/k)$ over $Gal(\bar{k}/k')$.

1.2. **ℓ-adic sheaves.**

We will assume k is a finite field and ℓ is a prime number which is invertible in k. For us, a *lisse* $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ is a continuous representation on a finitely dimensional $\overline{\mathbb{Q}}_\ell$-vector space V (sometimes denoted by $\mathcal{F}_{\bar{x}}$ for $\bar{x}$ a geometrical point of X)

$$\rho_{\mathcal{F}} : \pi_1(X) \to GL(V)$$

That is, there is a finite extension $E_\lambda \subset \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$, and a E_λ-vector space W such that: (i) $V = W \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$; (ii) the map $\rho_{\mathcal{F}}$ takes value in $GL(W)$; and (iii) $\rho_{\mathcal{F}}$ is continuous with respect to the Krull topology on both sides.

Recall when k is a field with q elements, then $Fr(x) = x^q$ is a topological generator of $Gal(\bar{k}/k)$. Its inverse, called the geometrical Frobenius, is denoted by σ. Any 1-dimensional lisse sheaf $\mathcal{F}$ on Spec$k$

$$\rho_{\mathcal{F}} : Gal(\bar{k}/k) \to \overline{\mathbb{Q}}_\ell^*$$

is determined by $\rho_{\mathcal{F}}(\sigma)$. It is easy to see that $\rho_{\mathcal{F}}(\sigma)$ takes value in $\mathfrak{o}_\lambda^*$ where $\mathfrak{o}_\lambda$ is the ring of integer of E_λ for some finite extension $E_\lambda \supset \mathbb{Q}_\ell$. In particular, the Tate twist $\overline{Q}_\ell(1)$ is defined by $\rho_{\overline{Q}_\ell(1)}(\sigma) = q^{-1}$. (Maybe before that, the constant sheaf $\overline{\mathbb{Q}}_\ell$ is defined by $\rho_{\overline{Q}_\ell}(\sigma) = 1$.) In general, for w a real number, $\tau : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ a chosen isomorphism of fields, we will define $\overline{\mathbb{Q}}_\ell(w)$ by $\rho_{\overline{\mathbb{Q}}_\ell(w)} = \tau^{-1}q^{-w}$.

Let me indicate what a constructible sheaf should look like in terms of Galois modules. For a G-module V, we will denote V^G the space of invariants and V_G the space of coinvariants. Assume that X is a curve. Then a constructible sheaf on X is equivalent to the following data: a lisse sheaf on some open U of X, i.e. a $\pi_1(U, \bar{\eta})$-module V, η the generic point of X), a $Gal(\overline{k(v)}/k(v))$-module M_v for every $v \in X \setminus U$ a closed point, and $Gal(\overline{k(v)}/k(v))$-equivariant maps $M_v \to V^{I_v}$, where $I_v = Gal(\bar{K}_v/K_v)$ (**???**), K_v the fractional field of the strict Henselization of $\mathcal{O}_{X,v}$, is the ramification group of X at v.

1.3. **Cohomology and duality.** For $\mathcal{F}$ any sheaf on X, there are general notions of cohomology groups $H^i(\overline{X}, \mathcal{F})$ and cohomology groups with compact support $H^i_c(\overline{X}, \mathcal{F})$. Here $\mathcal{F}$ is understood as the pullback to $\overline{X}$. These groups are $Gal(\bar{k}/k)$-modules. Let $\mathcal{F}$ be a lisse sheaf on X, then $H^0(\overline{X}, \mathcal{F}) = V^{\pi_1^{geom}}$, where V is the representation of π_1^{arith} attached to $\mathcal{F}$. One has the Poincaré duality: if X is a smooth, geometrically connected algebraic scheme of dimension r over k, and $\mathcal{F}$ is a lisse sheaf on X with $\mathcal{F}^*$ the dual sheaf (i.e. whose representation is dual to the representation of $\mathcal{F}$), then

$$H^i(\overline{X}, \mathcal{F}) \times H^{2r-i}_c(\overline{X}, \mathcal{F}^*) \to H^{2r}_c(\overline{X}, \overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell(-r)$$

is a nondegenerate pairing of $Gal(\bar{k}/k)$-modules. This implies that

$$H^{2r}_c(\overline{X}, \mathcal{F}) = V_{\pi_1^{geom}}(-r) := V_{\pi_1^{geom}} \otimes \overline{\mathbb{Q}}_\ell(-r)$$

Let us recall the following general fact: if X is affine, then for any constructible sheaf $\mathcal{F}$ on X, $H^i(\overline{X}, \mathcal{F}) = 0$ for $i > r$. Therefore, if X is smooth, then $H^i_c(\overline{X}, \mathcal{F}) = 0$ for $i < r$.

Let us consider the special case X is a geometrically connected, smooth affine curve. We have $H^0(\overline{X}, \mathcal{F}) = \mathcal{F}^{\pi_1^{geom}}$, $H^1(\overline{X}, \mathcal{F}) = H^1(\pi_1^{geom}, V)$, $H^2(\overline{X}, \mathcal{F}) = 0$. Using the Poincaré duality, one could get the descriptions for $H^i_c(\overline{X}, \mathcal{F})$. In particular, $H^0_c(\overline{X}, \mathcal{F}) = 0$.

1.4. **L-series and the Grothendieck trace formula.** Let k be a finite field with q elements. For any X algebraic scheme over k and $\mathcal{F}$ a constructible sheaf, the L-series is defined as

$$L(X, \mathcal{F}, t) = \prod_{x \in |X|} \det(1 - t^{d(x)}\sigma_x, \mathcal{F}_{\bar{x}})^{-1}$$

where $|X|$ is the set of closed points of X, $d(x) = [k(x) : k]$, σ_x is the image of the geometrical Frobenius $\sigma \in Gal(\bar{k}/k(x))$ under the map $Gal(\bar{x}/k(x)) \to \pi_1^{arith}$, and $\bar{x}$ is any geometrical point of X lying over x. It is a good exercise to show that

$$L(X, \mathcal{F}, t) = \exp(\sum_{m \geq 1}(\sum_{x \in X(k_m)} tr(\sigma_x, \mathcal{F}_{\bar{x}}))\frac{t^m}{m})$$

where k_m is the unique field in $\bar{k}$ with $[k_m : k] = m$. The Grothendieck trace formula reads as

$$\sum_{x \in X(k_m)} tr(\sigma_x, \mathcal{F}_{\bar{x}}) = \sum_i (-1)^i tr(\sigma^m, H^i_c(\overline{X}, \mathcal{F}))$$

Therefore,

$$L(X, \mathcal{F}, t) = \prod_i \det(1 - t\sigma, H^i_c(\overline{X}, \mathcal{F}))^{(-1)^{i+1}}$$

If X is a geometrically connected, smooth affine curve, and $\mathcal{F}$ is a lisse sheaf, then

$$\prod_{x \in |X|} \det(1 - t^{d(x)}\sigma_x, \mathcal{F}_{\bar{x}})^{-1} = \frac{\det(1 - t\sigma, H^1_c(\overline{X}, \mathcal{F}))}{\det(1 - t\sigma, H^2_c(\overline{X}, \mathcal{F}))}$$

1.5. **Weil sheaves.** There is a variation of lisse sheaves. Let X be an algebraic scheme over a finite field k. Let $\mathbb{Z} \cong \langle \sigma \rangle \subset Gal(\bar{k}/k)$. Then the Weil group $W(X, \bar{x})$ is defined as the inverse image of $\langle \sigma \rangle$ under the map $\pi_1(X, \bar{x}) \to Gal(\bar{k}/k)$, which is retopologized such that $\mathbb{Z}$ has the discrete topology. One has the following exact sequence

$$1 \to \pi_1(\overline{X}, \bar{x}) \to W(X, \bar{x}) \to \mathbb{Z} \to 1$$

Then a Weil sheaf is a continuous representation of $W(X, \bar{x})$ on some $\overline{\mathbb{Q}}_\ell$-vector space V

$$\rho : W(X, \bar{x}) \to GL(V)$$

The category of lisse sheaves is a full subcategory of the category of Weil sheaves. Consider the Weil sheaves on Spec$k$. Then it is determined by $\rho : \langle \sigma \rangle \to \overline{\mathbb{Q}}_\ell^*$, and therefore by an element $\rho(\sigma) \in \overline{\mathbb{Q}}_\ell^*$. The main difference from lisse sheaves is that $\rho(\sigma)$ is not necessarily an ℓ-adic unit. In general, a Weil sheaf has a filtration whose associated quotients are lisse sheaves twisted by above rank one Weil sheaves on Spec$k$. Therefore, the reason we introduce Weil sheaves here is just a matter of convenience.

2. THEORY OF WEIGHTS

We will assume from now on that k is the finite field with q elements, ℓ some prime which is invertible in k, and $\tau : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ is an isomorphism as field.

2.1. **Weights of sheaves.** Let X be an algebraic scheme over k, and $\mathcal{F}$ be an étale $\overline{\mathbb{Q}}_\ell$-sheaf on X. Let w be a real number.

Then $\mathcal{F}$ is called τ-pure of weight w if for any $x \in |X|$ and any eigenvalue α of σ_x on $\mathcal{F}_{\bar{x}}$, one has

$$|\tau(\alpha)|^2 = q^{d(x)w}$$

The sheaf $\mathcal{F}$ is called to be τ-mixed, if there exists a finite filtration by subsheaves

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(r)} = \mathcal{F}$$

such that all factor sheaves $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ are τ-pure. The sheaf $\mathcal{F}$ is said to be pure of weight w, if for all isomorphisms $\tau : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, the sheaf $\mathcal{F}$ is τ-pure of weight w. The sheaf $\mathcal{F}$ is called mixed, if there exists a finite filtration of $\mathcal{F}$ such that all successive factor sheaves are pure sheaves. For example, $\overline{\mathbb{Q}}_\ell$ is pure of weight zero, $\overline{\mathbb{Q}}_\ell(1)$ is pure of weight -2.

Fix a τ. For any sheaf $\mathcal{F}$ on X, we will define

$$w(\mathcal{F}) = \sup_{x\in|X|}\sup_{\alpha}\frac{\log(|\tau(\alpha)|^2)}{\log(q^{d(x)})}$$

where α ranges over all the eigenvalues of σ_x. Of course, if $\mathcal{F}$ is τ-pure of weight w, $w(\mathcal{F}) = w$.

Proposition 2.1. (Semicontinuity of weights) Let X be a geometrically connected, smooth curve. Let $j : U \hookrightarrow X$ be affine open in X. If $\mathcal{F}$ is lisse over U, then $w(j_*\mathcal{F}) = w(\mathcal{F})$.

Proof. Assume that $w(j_*\mathcal{F}) \geq w(\mathcal{F})$. Certainly we could assume that X is affine (by removing a point in U). Let $Z = X \setminus U$.

$$\tau L(X, j_*\mathcal{F}, t) = \tau L(U, \mathcal{F}, t)\prod_{x\in|Z|}\tau\det(1 - t^{d(x)}\sigma, (j_*\mathcal{F})_{\bar{x}})^{-1}$$

We know that $\tau L(U, \mathcal{F}, t)$ converges in the region $|t| < q^{-\frac{w(\mathcal{F})}{2}-1}$ without any zeros or poles in that region, and some local factor $\det(1 - t^{d(x)}\sigma, (j_*\mathcal{F})_{\bar{x}})^{-1}$ ($x \in |Z|$) will have poles along the circle $|t| = q^{\frac{-w(j_*\mathcal{F})}{2}}$. On the other hand

$$L(X, j_*\mathcal{F}, t) = \frac{\det(1 - t\sigma, H^1_c(\overline{X}, j_*\mathcal{F}))}{\det(1 - t\sigma, H^2_c(\overline{X}, j_*\mathcal{F}))}$$

Since $H^2_c(\overline{X}, j_*\mathcal{F}) = H^2_c(\overline{U}, \mathcal{F})$, and $w(H^2_c(\overline{U}, \mathcal{F})) \leq w(\mathcal{F}) + 2$, the right hand side converges in the region $|t| < q^{\frac{-w(\mathcal{F})}{2}-1}$. Therefore, $w(j_*\mathcal{F}) \leq w(\mathcal{F}) + 2$.

Replace $\mathcal{F}$ by $\mathcal{F}^{\otimes n}$ for n positive integer, we obtain that

$$nw(j_*\mathcal{F}) = w((j_*\mathcal{F})^{\otimes n}) \leq w(j_*(\mathcal{F}^{\otimes n})) \leq w(\mathcal{F}^{\otimes n}) + 2 = nw(\mathcal{F}) + 2$$

The inequality $w((j_*\mathcal{F})^{\otimes n}) \leq w(j_*(\mathcal{F}^{\otimes n}))$ is because $(j_*\mathcal{F})^{\otimes n} \to j_*(\mathcal{F}^{\otimes n})$ is an injective map. The proposition follows by letting $n \to \infty$. □

Remark. The general form of the proposition is as follows. Let X be a scheme of finite type over k, and $j : U \to X$ is affine open. Then for any $\mathcal{F}$ is τ-mixed perverse sheaf on U, $w(j_{!*}\mathcal{F}) \leq w(\mathcal{F})$.

Corollary 2.2 . Let X be as in the proposition. If $\mathcal{F}$ is a τ-mixed irreducible lisse sheaf on X, then $\mathcal{F}$ is τ-pure.

Proof. $\mathcal{F}$ has a filtration such that the successive quotients are τ-pure. Then on some open $j : U \hookrightarrow X$, $\mathcal{F}$ are filtered by lisse sheaves. However, $\mathcal{F}$ remains irreducible on U. Therefore, $j^*\mathcal{F}$ is τ-pure. Since $\mathcal{F} \subset j_*j^*\mathcal{F}$, $w(\mathcal{F}) \leq w(j^*\mathcal{F})$. That is, for $x \in |X| \setminus |U|$, any eigenvalue α of σ_x on $\mathcal{F}_{\bar{x}}$ satisfies $|\tau(\alpha)|^2 \leq q^{d(x)w(j^*\mathcal{F})}$. By considering the dual sheaf $\mathcal{F}^*$, one obtains the inverse inequality. □

2.2. **Sheaves to functions.** There is a general way to get functions from sheaves. Let $\mathcal{F}$ be an étale sheaf over X an algebraic scheme over k. Then

$$f_{\mathcal{F}} : X(k_m) \to \mathbb{C}$$

is defined by

$$f_{\mathcal{F}}(x) = \tau tr(\sigma_x, \mathcal{F}_{\bar{x}})$$

For example, $f_{\overline{\mathbb{Q}}_\ell(n)}$ is the constant function on $X(k_m)$ whose value is q^{-mn}. For $\varphi : X \to Y$ a k_m-morphism.

$$f_{f^*\mathcal{F}} = \varphi^*(\varphi_{\mathcal{F}}), \quad f_{R\varphi_!\mathcal{F}} = \varphi_!(f_{\mathcal{F}})$$

Here $(\varphi_! f)(y) = \sum_{\varphi(x)=y} f(x)$ is the integration along the fibers. The first identity is trivial, and the second is due to the Grothendieck trace formula.

For function $f, g : X(k_m) \to \mathbb{C}$, we can define the inner product

$$(f, g)_m = \sum_{x\in X(k_m)} f(x)\overline{g(x)}$$

and the L^2-norm

$$\|f\|_m^2 = (f, f)$$

The following lemma relates "arithmetic" to "analysis".

Lemma 2.3. There exists a constant $C > 0$, depending only on X and $\mathcal{F}$, such that for every m,

$$\|f_{\mathcal{F}}\|_m^2 \leq Cq^{m(w(\mathcal{F})+\dim X)}$$

Proof. One could assume that X is affine and integral. Then the Noether normalization theorem provides a finite morphism $\varphi : X \to \mathbb{A}^d$, where $d = \dim X$. Therefore, there exists $C_1 > 0$ such $\sharp X(k_m) \leq C_1 q^{md}$. On the other hand, $|\tau tr(\sigma_x, \mathcal{F}_x)|^2 \leq r^2 q^{mw(\mathcal{F})}$, where r is the rank of $\mathcal{F}$. Combining these two facts, one obtains the desired estimate. □

Exercise. Use the same method to prove that $\tau L(X, \mathcal{F}, t)$ is convergent in the region $|t| < q^{\frac{-w(\mathcal{F})}{2}-\dim X}$ and without any zeros nor poles in that region.

Let

$$\|\mathcal{F}\| = \sup\{\rho | \limsup_m \frac{\|f_{\mathcal{F}}\|_m^2}{q^{m(\rho+\dim X)}} > 0\}$$

Or in other words, $\|\mathcal{F}\| = -\frac{\log R}{\log q} - \dim X$, where R is the radius of the convergence of the following series

$$\phi_{\mathcal{F}}(t) = \sum_{m\geq 1} \|f_{\mathcal{F}}\|_m^2 t^m$$

Then above lemma is equivalent to $w(\mathcal{F}) \geq \|\mathcal{F}\|$. The first technical theorem is the following

Theorem 2.4. Let $\mathcal{F}$ be a τ-mixed sheaf on a geometrically connected, smooth affine curve X. If $H_c^0(\overline{X}, \mathcal{F}) = 0$, then $\|\mathcal{F}\| = w(\mathcal{F})$.

Proof. We rewrite $\phi_{\mathcal{F}}(t) = \sum\limits_{x\in|X|} \sum\limits_{n\geq 1} d(x)|tr(\sigma_x^n, \mathcal{F}_{\bar{x}})|^2 t^{nd(x)}$.

Step I. If X is of dimension 0, the theorem holds unconditionally. In this case, one could assume that X has only one closed point, x. The data $\mathcal{F}$ is then equivalent to a complex vector space V with σ an automorphism. Simple calculation shows that

$$\phi_{\mathcal{F}} = t\frac{d}{dt} \log\det(1 - t^{d(x)}\sigma, V \otimes \overline{V})^{-1}$$

where $\overline{V}$ is the complex conjugate of V. Therefore, the radius of convergence is $\min\limits_{\alpha,\beta} |\tau(\alpha)\tau(\beta)|^{-1/d(x)} = q^{w(\mathcal{F})}$ where α, β range over all eigenvalues of σ. The theorem follows.

Step II. We prove the theorem holds for τ-pure lisse sheaf of weight w, i.e. the radius of convergence for $\phi_{\mathcal{F}}(t)$ is q^{-w-1}. Observe that $H^0_c(\overline{X}, \mathcal{F}) = 0$ automatically holds in this case. Denote $\overline{\mathcal{F}}$=$\mathcal{F}^*(-w)$. The eigenvalues of σ_x on $\tau\overline{\mathcal{F}}_{\bar{x}}$ are those $\overline{\tau(\alpha)}$, where $\tau(\alpha)$ are eigenvalues of σ_x on $\tau\mathcal{F}$. $\mathcal{F} \otimes \overline{\mathcal{F}}$ is τ-pure of wight $2w$. One has

$$\phi_{\mathcal{F}} = t\frac{d}{dt} \log \tau L(X, \mathcal{F} \otimes \overline{\mathcal{F}}, t)$$

We know by the exercise that, $\tau L(X, \mathcal{F} \otimes \overline{\mathcal{F}}, t)$ converges for $|t| < q^{-w-1}$ and in that region, $\tau L(X, \mathcal{F} \otimes \overline{\mathcal{F}}, t)$ has neither zeros nor poles. On the other hand, we have

$$\tau L(X, \mathcal{F} \otimes \overline{\mathcal{F}}, t) = \frac{\tau \det(1 - t\sigma, H^1_c(\overline{X}, \mathcal{F} \otimes \overline{\mathcal{F}}))}{\tau \det(1 - t\sigma, H^2_c(\overline{X}, \mathcal{F} \otimes \overline{\mathcal{F}}))}$$

Since $H^2_c(\overline{X}, \mathcal{F} \otimes \overline{\mathcal{F}})$ is τ-pure of weight $2w + 2$, the right hand side either has poles along the circle $|t| = q^{-w-1}$ or is an entire function. If the radius of convergence of the left hand side were great than q^{-w-1}, this would force the right hand side to be an entire function, and so would be the left hand side. However, the left hand side could not be an entire function since each local factor $\tau \det(1 - t^{d(x)}\sigma_x, \mathcal{F}_{\bar{x}} \otimes \overline{\mathcal{F}}_{\bar{x}})^{-1}$ has poles at $t = q^{-w}$. This proves the theorem in this case.

Step III. The theorem holds for τ-mixed lisse sheaves. Given a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

one has $f_{\mathcal{F}} = f_{\mathcal{F}'} + f_{\mathcal{F}''}$, $w(\mathcal{F}) = \max\{w(\mathcal{F}'), w(\mathcal{F}'')\}$, and

$$\phi_{\mathcal{F}} = \phi_{\mathcal{F}'} + \phi_{\mathcal{F}''} + \sum_{m \geq 1} 2\mathfrak{Re}(f_{\mathcal{F}'}, f_{\mathcal{F}''})_m t^m$$

Therefore, we could assume that $\mathcal{F}$ is a semisimple τ-mixed lisse sheaf, since by above observation, semisimplification does not change $w(\mathcal{F})$ and $\phi_{\mathcal{F}}$. Furthermore, since every τ-mixed irreducible lisse sheaf is in fact τ-pure by Corollary 2.2, one could assume that $\mathcal{F}'$ is τ-mixed lisse subsheaf of $\mathcal{F}$ such that $w(\mathcal{F}') = w(\mathcal{F})$, and $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$ is τ-mixed lisse such that $w(\mathcal{F}'') < w(\mathcal{F})$.

Since

$$|\mathfrak{Re}(f_{\mathcal{F}'}, f_{\mathcal{F}''})_m| \leq \|f_{\mathcal{F}'}\|_m \|f_{\mathcal{F}'}\|_m \leq Cq^{m(\frac{w(\mathcal{F}')+w(\mathcal{F}'')}{2}+1)} < Cq^{m(w(\mathcal{F})+1)}$$

by the lemma above, the radius of convergence for $\sum\limits_{m \geq 1} 2\mathfrak{Re}(f_{\mathcal{F}'}, f_{\mathcal{F}''})_m t^m$ is strictly greater than $q^{-w(\mathcal{F})-1}$. By induction, the radius of convergence for the first term is $q^{-w(\mathcal{F})-1}$ and the one for the second term is $q^{-w(\mathcal{F}'')-1} > q^{-w(\mathcal{F})-1}$. The theorem follows.

Step IV. The theorem holds for any τ-mixed sheaf $\mathcal{F}$ such that $H^0_c(\overline{X}, \mathcal{F})$. Choose some affine open $j : U \hookrightarrow X$ such that $\mathcal{F}$ is lisse over U. Let $i : Z = X \setminus U \hookrightarrow$ be the closed complement. One has the following

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

and $\phi_{\mathcal{F}} = \phi_{j_!j^*\mathcal{F}} + \phi_{i_*i^*\mathcal{F}}$. By Step I, $\phi_{i_*i^*\mathcal{F}}$ has radius of convergence$\geq q^{-w(\mathcal{F})} > q^{-w(\mathcal{F})-1}$. Since $j^*\mathcal{F}$ is lisse τ-mixed over U, by Step III, $\phi_{j_!j^*\mathcal{F}}$ has radius of convergence $q^{-w(j^*\mathcal{F})-1}$. It remains to show that $w(j^*\mathcal{F}) = w(\mathcal{F})$, or equivalently, $w(j^*\mathcal{F}) \geq w(\mathcal{F})$. Since $H^0_c(\overline{X}, \mathcal{F}) = 0$, $\mathcal{F} \to j_*j^*\mathcal{F}$ is injective. Therefore, $w(\mathcal{F}) \leq w(j_*j^*\mathcal{F}) = w(j^*\mathcal{F})$ by the semicontinuity of weights (Proposition 2.1). □

3. CRITERION OF PURITY

3.1. determinantal weights.

We begin with

Proposition 3.1. Let $\mathcal{F}$ be a rank one lisse sheaf over an absolutely irreducible smooth curve X. Then there exists some n, such that $\mathcal{F}^{\otimes n}$ is the constant sheaf over $\overline{X}$.

Proof. We prove the proposition for Weil sheaves and therefore for lisse sheaves. Let X' be the complete curve over k with $X \hookrightarrow X'$ an open embedding. Let K be the function field of X. For every $v \in |X'|$, let K_v be the completion of K at v and $\mathfrak{o}_v$ the ring of integer in K_v. Let $\mathbb{A}^*_K$ be the ring of idèles. Then the class field theory for function fields identifies

$$W(X)^{ab} \cong (\prod_{v\in|X|} \mathfrak{o}^*_v)\backslash\mathbb{A}^*_K/K^*$$

and the image of the natural map $\pi_1^{geom} \to W(X) \to W(X)^{ab}$ is $(\prod_{v\in|X|} \mathfrak{o}^*_v)\backslash\mathbb{A}^1_K/K^*$, where A^1_K is the subgroup of the idèles consisting of elements with norm one. It is known that

$$(\prod_{v\in|X'|} \mathfrak{o}^*_v)\backslash\mathbb{A}^1_K/K^* \cong Pic^0(X')(k)$$

is a finite group and each factor $\mathfrak{o}^*_v$ is an extension of a finite group by a pro-p-group ($p = chark$). Therefore, $(\prod_{v\in|X|} \mathfrak{o}^*_v)\backslash\mathbb{A}^1_K/K^*$ is a finite group extended by pro-p-groups. Since $p \neq \ell$. The proposition follows. □

Remark. This proposition in fact holds for any normal absolutely irreducible scheme of finite type over k.

Corollary 3.2. Any lisse sheaf on rank one on a smooth, absolutely irreducible curve is τ-pure.

Let $\mathcal{F}$ be a lisse sheaf on X an absolutely irreducible smooth curve (or in general if you accept the previous remark, normal absolutely irreducible scheme, of finite type) over k. A real number w is called the determinantal weights of $\mathcal{F}$ if there exists some $\mathcal{G}$ an irreducible (Jordan-Hölder) constituent of $\mathcal{F}$, such that $rw = w(\bigwedge^r \mathcal{G})$, where r is the rank of $\mathcal{G}$. Every lisse sheaf has finitely many determinantal weights.

Lemma 3.3. Let $\mathcal{F}$ be a lisse sheaf over X. If α is a eigenvalue of $\sigma : H^2_c(\overline{X}, \mathcal{F}) \to H^2_c(\overline{X}, \mathcal{F})$, then $\frac{\log(|\tau(\alpha)|^2)}{\log(q)} - 2$ is a determinantal weight for $\mathcal{F}$.

Proof. Let V be the representation of π_1^{arith} attached to $\mathcal{F}$. Then $H^2_c(\overline{X}, \mathcal{F}) = V_{\pi_1^{geom}}(-1)$. Therefore, $q^{-1}\alpha$ is an eigenvalue of σ acting on $V_{\pi_1^{geom}}$. □

The main result of determinantal weights for us is the following

Proposition 3.4. Let X be as above. $\mathcal{F}$, $\mathcal{G}$ two lisse sheaves on X.

(i) If $\mathcal{F}$ and $\mathcal{G}$ have single determinantal weights, say w_1 and w_2 respectively, then $\mathcal{F} \otimes \mathcal{G}$ has single determinantal weight $w_1 + w_2$.

(ii) For $w \in \mathbb{R}$, let $r(w)$ denote the sum of ranks of all irreducible constituents of $\mathcal{F}$, which have the same determinant weight w with respect to τ. Then the determinantal weights of $\bigwedge^r \mathcal{F}$ are the numbers

$$\sum_w n(w)w \quad \text{with} \quad \sum_w n(w) = r, 0 \leq n(w) \leq r(w), n(w) \in \mathbb{Z}$$

Proof. (ii) follows from (i). To prove (i), we could assume that $\mathcal{F}$ and $\mathcal{G}$ are irreducible. Let V and W be the representations attached to $\mathcal{F}$ and $\mathcal{G}$ respectively. Let $G \subset GL(V)$ and $G' \subset GL(W)$ be the arithmetic monodromy groups of $\mathcal{F}$ and $\mathcal{G}$ respectively. By Theorem 3.5 in §3.2, there exists some m, and a lifting of $\sigma \in Gal(\bar{k}/k)$ to $\tilde{\sigma} \in G(\overline{\mathbb{Q}}_\ell)$ and $\tilde{\sigma}' \in G'(\overline{\mathbb{Q}}_\ell)$, such that $\tilde{\sigma}^m$ and $(\tilde{\sigma}')^m$ are in the centers of the corresponding algebraic groups. By Schur's lemma, $\tilde{\sigma}^m$ acts on V by scalar α with $|\tau(\alpha)|^2 = q^{mw_1}$ and $(\tilde{\sigma}')^m$ acts on W by scalar α' with $|\tau(\alpha)|^2 = q^{mw_2}$. Then the proposition is clear. □

3.2. **monodromy groups.** Let $\mathcal{F}$ be a Weil sheaf, i.e. a representation $\rho : W(X, \bar{x}) \to GL(V)$, where V is a finitely dimensional vector space over $\overline{\mathbb{Q}}_\ell$. There is some $E \subset \overline{\mathbb{Q}}_\ell$ a finite extension of $\mathbb{Q}_\ell$ and W a vector space over E such that $V = W \otimes_E \overline{\mathbb{Q}}_\ell$. Let G_{geom} be the Zariski closure of $\rho(\pi_1^{geom})$ in $GL(V)$. This is a group scheme naturally defined over E, and called the geometrical monodromy group. By pushing-out the exact sequence $1 \to \pi_1^{geom} \to W(X, \bar{x}) \to \mathbb{Z} \to 1$, one obtains

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{geom} \arrow[r] \arrow[d] & W(X,\bar{x}) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
1 \arrow[r] & G_{geom} \arrow[r] & G \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] & 1 \\
 & & GL(V) & &
\end{tikzcd}$$

G, usually called the arithmetic monodromy group, is also a group scheme (not of finite type) defined over E.

Theorem 3.5. Let $\mathcal{F}$ be a semisimple Weil sheaf over X. Then

(i) G_{geom} is semisimple, not necessarily connected.

(ii) There is some $\tilde{\sigma} \in G(\overline{\mathbb{Q}}_\ell)$ a lifting of $\sigma \in \mathbb{Z} \cong \langle \sigma \rangle$ and some $m > 0$, such that $\tilde{\sigma}^m$ is in the center of $G(\overline{\mathbb{Q}}_\ell)$.

To prove the theorem requires a little bit knowledge of reductive groups. We omit the proof. cf. [D2] 1.3.9, [KW] I.3.3.

3.3. **Real sheaves.** Let $\mathcal{F}$ be a sheaf on an algebraic scheme X over k. We say $\mathcal{F}$ is τ-real if for any $x \in |X|$, $\tau \det(1 - t\sigma_x, \mathcal{F}_{\bar{x}})$ has real coefficients. According to the formula

$$\frac{d}{dt} \log \det(1 - t\sigma_x, \mathcal{F}_{\bar{x}}^{\otimes k}) = \sum_{n \geq 1} (tr(\sigma_x^n))^k t^{n-1}$$

It is clear that for any τ-real sheaf $\mathcal{F}$, $\mathcal{F}^{\otimes k}$ is again real. Furthermore, for any $x \in |X|$, $\tau \det(1 - t\sigma_x, \mathcal{F}_{\bar{x}}^{\otimes 2k})^{-1}$ will have non-negative coefficients.

Exercise. Let $\mathcal{F}$ be a lisse sheaf on X. Show that if $\mathcal{F}$ is τ-real, so is $\bigwedge^r \mathcal{F}$, where $r \leq rk\mathcal{F}$.

The second technical theorem is the following

Theorem 3.6. Any τ-real sheaf is τ-mixed.

Proof. We will prove the theorem for X being a geometrically connected, smooth affine curve. The general case is reduced to this special case by the Bertini Theorem.

Observe that if $j : U \to X$ is open embedding and $i : Z = X \setminus U \to X$ is the complement, then one has

$$0 \to j_! j^*(\mathcal{F}) \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

If both $j_! j^* \mathcal{F}$ and $i_* i^* \mathcal{F}$ are τ-mixed, so is $\mathcal{F}$. therefore, we could assume that $\mathcal{F}$ is lisse.

Let w be the largest determinantal weight of $\mathcal{F}$. We first prove that

$$w(\mathcal{F}) \leq w$$

For this, apply the Grothendieck trace formula to $\mathcal{F}^{\otimes 2k}$,

$$\prod_{x \in |X|} \frac{1}{\tau \det(1 - t^{d(x)} \sigma_x, \mathcal{F}_{\bar{x}}^{\otimes 2k})} = \frac{\tau \det(1 - t\sigma, H_c^1(\overline{X}, \mathcal{F}^{\otimes 2k}))}{\tau \det(1 - t\sigma, H_c^2(\overline{X}, \mathcal{F}^{\otimes 2k}))}$$

Observe that by Proposition 3.4(i) in §3.1, the largest determinantal weights for $\mathcal{F}$ is $2kw$. According to Lemma 3.3 in §3.1, any eigenvalue α of $\sigma : H_c^2(\overline{X}, \mathcal{F}^{\otimes 2k}) \to H_c^2(\overline{X}, \mathcal{F}^{\otimes 2k})$ satisfying $|\tau(\alpha)| \leq q^{kw+1}$ and therefore, the series on the right hand side converges for $|t| < q^{-kw-1}$. Since each local factor on the left hand side has nonnegative coefficients with leading term one, each local factor has to converge in the same domain $|t| < q^{-kw-1}$. That is, every eigenvalue α of σ_x on $\mathcal{F}_{\bar{x}}^{\otimes 2k}$ satisfies $|\tau(\alpha)| \leq q^{kw+1}$. Therefore, for every eigenvalue α of σ_x on $\mathcal{F}_{\bar{x}}$, $|\tau(\sigma)|^2 \leq q^{w+1/k}$. Letting $k \to \infty$, we obtain the desired estimate.

Next, we show that above result will imply the theorem. We will show that each irreducible constituent of $\mathcal{F}$ is τ-pure. Therefore, we could replace $\mathcal{F}$ by its semisimplification. Let

$$w_1 > w_2 > \cdots > w_r$$

be all the determinantal weights of $\mathcal{F}$. For each w_i, let $\mathcal{F}(i)$ be the direct sum of all irreducible constituent of $\mathcal{F}$ with determinantal weight w_i. Then $\mathcal{F} = \oplus_i \mathcal{F}(i)$. Let $r(i)$ denote the rank of $\mathcal{F}(i)$ and for any n with $0 \leq n < r$, put $N = \sum_{i=1}^n r(i)$. Then according to Proposition 3.4(ii) of §3.1, the largest determinantal weight of $\bigwedge^{N+1} \mathcal{F}$ is $w_{n+1} + \sum_{i=1}^n r(i) w_i$. Let $\alpha_1^{(i)}, \ldots, \alpha_{r(i)}^{(i)}$ denote the (generalized) eigenvalue of σ_x on $\mathcal{F}(i)_{\bar{x}}$, counted with multiplicity. Then $\alpha_k^{(n+1)} \prod_{i=1}^n \prod_{j=1}^{r(i)} \alpha_j^{(i)}$ for $k = 1, \ldots, r(n+1)$ are eigenvalues of σ_x on $\bigwedge^{N+1} \mathcal{F}$. Since $\bigwedge^{N+1} \mathcal{F}$ is still τ-real,

$$|\tau(\alpha_k^{(n+1)} \prod_{i=1}^n \prod_{j=1}^{r(i)} \alpha_j^{(i)})|^2 \leq q^{d(x)(w_{n+1} + \sum_{i=1}^n r(i) w_i)}$$

However, by the definition of determinantal weights,

$$|\tau(\prod_{i=1}^n \prod_{j=1}^{r(i)} \alpha_j^{(i)}))|^2 = q^{d(x)(\sum_{i=1}^n r(i) w_i)}$$

Therefore $|\tau(\alpha_k^{(n+1)})|^2 \leq q^{d(x)w_{n+1}}$. Applying the same argument to the dual sheaf $\mathcal{F}^*$, we obtains $|\tau(\alpha_k^{(n+1)})|^2 = q^{d(x)w_{n+1}}$. Therefore, $\mathcal{F}(i)$ is τ-pure of weight w_i. $\square$

Remark. We have the following simple observation. Let X be absolutely irreducible smooth curve over k, and $\mathcal{F}$ a lisse τ-real sheaf. If $H_c^2(\overline{X},\mathcal{F}) = 0$, then $H_c^1(\overline{X},\mathcal{F})$ is τ-real. We will see later that if $\mathcal{F}$ is τ-real and τ-pure, then both $H_c^1(\overline{X},\mathcal{F})$ and $H_c^2(\overline{X},\mathcal{F})$ are τ-real.

4. ℓ-ADIC FOURIER TRANSFORM

4.1. **Artin-Scheier sheaves.** Regard k as a cyclic group with q elements. Fix $\psi : k \to \overline{\mathbb{Q}}_\ell^*$ a nontrivial character. Then the character group of k is identified with k via $x \to (\psi_x : k \to \overline{\mathbb{Q}}_\ell^*)$, where $\psi_x(y) = \psi(xy)$.

Let $\mathbb{A}^1$ be the affine line over k. The Artin-Scheier cover $\pi : \mathbb{A}^1 \to \mathbb{A}^1$ is a Galois cover with Galois group k, defined as $x \mapsto x - x^q$. Then the Artin-Scheier sheaf $\mathcal{L}_\psi$ is the lisse sheaf on $\mathbb{A}^1$ corresponding to the representation $\pi_1(\mathbb{A}^1) \to k \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^*$. Similarly, for each $x \in k$, we have $\mathcal{L}_{\psi_x}$. Observe that $\mathcal{L}_{\psi_0} \cong \overline{\mathbb{Q}}_\ell$. The following basic facts are easy to verify:

1. Let $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ be the multiplication $m(x,y) = xy$. Then $m^*\mathcal{L}_\psi \cong \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi$.
2. $\mathcal{L}_\psi$ is pure of weight 0. More precisely, for $x \in \mathbb{A}^1(L) \cong L$, σ_x acts on $(\mathcal{L}_\psi)_{\bar{x}}$ via $\psi(Tr_{L/k}(x))$, which is always a p'th root of unit.
3. $\pi_*\overline{\mathbb{Q}}_\ell \cong \sum_{x\in k} \mathcal{L}_{\psi_x}$.
4. $H^1(\overline{A^1}, \mathcal{L}_{\psi_x}) = H_c^1(\overline{\mathbb{A}^1}, \mathcal{L}_{\psi_x}) = 0$, and
$$H_c^2(\overline{\mathbb{A}^1}, \mathcal{L}_{\psi_x}) = \begin{cases} 0 & x \neq 0 \\ \overline{\mathbb{Q}}_\ell(-1) & x = 0 \end{cases}$$

4.2. **ℓ-adic Fourier transform.** There are three morphisms $\mathrm{pr}_1, \mathrm{pr}_2, m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, where pr_i is the projection to the i'th factor and m is the multiplication $m(x,y) = xy$. The ℓ-adic Fourier transform $T_\psi : D_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell) \to D_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$ is defined as

$$T_\psi(\mathcal{F}) = R(\mathrm{pr}_1)_!(\mathrm{pr}_2^*\mathcal{F} \otimes m^*\mathcal{L}_\psi)[1]$$

It is clear that for $x \in \mathbb{A}^1(k) \cong k$, and $\bar{x}$ a geometrical point over x,

$$T_\psi(\mathcal{F})_{\bar{x}} = R\Gamma_c(\overline{\mathbb{A}^1}, \mathcal{F} \otimes \mathcal{L}_{\psi_x})[1]$$

Therefore, $T_\psi\overline{\mathbb{Q}}_\ell \cong \delta_0(-1)[-1]$, where $\delta_x = (i_x)_*\overline{\mathbb{Q}}_\ell$. On the other hand, it is trivial to see that $T_\psi\delta_0[-1] = \overline{\mathbb{Q}}_\ell$. It suggests the following

Theorem 4.1. For any complex $K \in D_c^b(\mathbb{A}^1, \overline{\mathbb{Q}}_\ell)$, one has

$$T_{\psi^{-1}}T_\psi K \cong K(-1)$$

Proof is omitted. cf. [KW] I.5.8.

Pass from sheaves to functions. One obtains

$$f_{T_\psi(K)}(x) = -\sum_{y\in k_m} f_K(y)\psi(Tr_{k_m/k}(xy))$$

This is the reason that T_ψ is called the ℓ-adic Fourier transform. For example, take $K = \overline{\mathbb{Q}}_\ell$, then above formula gives the following well-known formula

$$q\delta(x-z) = \sum_{y\in k} \psi((x-z)y)$$

The Plancherel formula are of the form

$$\|f_{T_\psi(K)}\|_m^2 = q^m \|f_K\|_m^2$$

4.3. **The Weil conjecture for the affine line.** Now we are ready to prove

Theorem 4.2. Let $\mathcal{F}$ be a τ-pure lisse sheaf on $\mathbb{A}^1$. Then $w(H_c^1(\overline{X}, \mathcal{F})) \leq w + 1$.

Proof. First, we could assume that $\mathcal{F}$ is irreducible. By Theorem 3.5 in §3.2, we could further assume that $\mathcal{F}$ is geometrically irreducible. We could also assume that $\mathcal{F}$ is not a Tate twist of an Artin-Scheier sheaf, since their cohomology are known to satisfy the theorem. Choose $\psi : k \to \overline{\mathbb{Q}}_\ell$ a nontrivial character. We will prove

(i) $T_\psi(\mathcal{F})$ is a sheaf rather than a complex.
(ii) $H_c^0(\overline{\mathbb{A}^1}, T_\psi(\mathcal{F})) = 0$.
(iii) $T_\psi(\mathcal{F})$ is τ-mixed.

Assuming (i)-(iii), we first conclude the proof of the theorem. We know $T_\psi(\mathcal{F})_0 \cong H_c^1(\overline{\mathbb{A}^1}, \mathcal{F})$. Therefore, it is enough to show that $w(T_\psi(\mathcal{F})) \leq w(\mathcal{F}) + 1$. By (i)-(iii) and Theorem 2.4 in §2, $w(T_\psi(\mathcal{F})) = \|T_\psi(\mathcal{F})\|$ and $w(\mathcal{F}) = \|\mathcal{F}\|$. So, it is enough to show that $\|T_\psi(\mathcal{F})\| \leq \|\mathcal{F}\| + 1$. However, the Plancherel formula gives $\|T_\psi(\mathcal{F})\| = \|\mathcal{F}\| + 1$. The theorem is proved.

It remains to prove (i)-(iii). We begin with (i). To prove that $T_\psi(\mathcal{F})$ is a sheaf, we need to show that

$$H_c^0(\overline{\mathbb{A}^1}, \mathcal{F} \otimes \mathcal{L}_{\psi_x}) = H_c^2(\overline{\mathbb{A}^1}, \mathcal{F} \otimes \mathcal{L}_{\psi_x}) = 0$$

The first cohomology group vanishes since $\mathcal{F} \otimes \mathcal{L}_{\psi_x}$ is lisse. For the second, let (ρ, V) be the representation attached to $\mathcal{F}$. Then $\rho \otimes \psi_x$ is the representation attached to $\mathcal{F} \otimes \mathcal{L}_{\psi_x}$, which is still geometrically irreducible. Since $H_c^2(\overline{\mathbb{A}^1}, \mathcal{F} \otimes \mathcal{L}_{\psi_x}) = V_{\pi_1^{geom}}(-1)$, if $H_c^2(\overline{\mathbb{A}^1}, \mathcal{F} \otimes \mathcal{L}_{\psi_x}) \neq 0$, $\rho \otimes \psi_x$ would have to be geometrically trivial. Whereas, ρ is not a Tate twist of an Artin-Scheier sheaf, this could not be true.

(ii) follows from

$$H_c^0(\overline{\mathbb{A}^1}, T_\psi(\mathcal{F})) \cong \mathcal{H}^{-1}(T_{\psi^{-1}} T_\psi \mathcal{F})_0 = \mathcal{H}^{-1}\mathcal{F}(-1)_0 = 0$$

To prove (iii), we define a real sheaf on $\mathbb{A}^2$ by

$$\mathcal{G} = (\mathrm{pr}_2^* \mathcal{F} \otimes m^* \mathcal{L}_\psi) \oplus (\mathrm{pr}_2^* \mathcal{F}^* \otimes m^* \mathcal{L}_{\psi^{-1}} \otimes \overline{\mathbb{Q}}_\ell(-w))$$

Then $R(\mathrm{pr}_1)_! \mathcal{G}[1] = T_\psi(\mathcal{F}) \oplus T_{\psi^{-1}}(\mathcal{F}^*) \otimes \overline{\mathbb{Q}}_\ell(-w)$. For the same reason as in the proof of (i), $R(\mathrm{pr}_1)_! \mathcal{G}[1]$ is a sheaf rather than a complex. Furthermore, $R(\mathrm{pr}_1)_! \mathcal{G}[1]$ is τ-real, by the remark at the end of §3. According to Theorem 3.6 in §3, $R(\mathrm{pr}_1)_! \mathcal{G}[1]$ is τ-mixed. As a direct summand, $T_\psi(\mathcal{F})$ is τ-mixed. □

5. PROOF OF THE WEIL CONJECTURE

5.1. **Curve case.** The goal is to prove the following theorem

Theorem 5.1. Let X be a geometrically connected, smooth affine curve. Let $\mathcal{F}$ be a τ-pure lisse sheaf on X with weight w. Then $w(H^1_c(\overline{X},\mathcal{F}))\leq w+1$.

Proof. We reduce the theorem to the case $X=\mathbb{A}^1$, which is already proved in §4.

(1) One could extend the base field if necessary.
(2) One could replace X by its open subscheme U because $H^1_c(\overline{U},\mathcal{F}|_U)\to H^1_c(\overline{X},\mathcal{F})$ is always surjective.
(3) One could assume that X is an open subscheme of $\mathbb{A}^1$. Assume X is affine. The Riemann-Roch theorem provides a finite separable morphism $\pi:X\to\mathbb{A}^1$. By shrinking X, one could assume X is finite étale over some open $U\in\mathbb{A}^1$. Replace $X,\mathcal{F}$ by $U,\pi_*\mathcal{F}$.
(4) One could assume that X is $\mathbb{A}^1\setminus\{0\}$. Let $S=\mathbb{A}^1\setminus X$. By extension of base field, one could assume all the points of S are rational. Let Γ be the finite group in $\mathbb{A}^1(k)$ generated by $|S|$. One could replace X by $\mathbb{A}^1-\Gamma$. Then $\mathbb{A}^1/\Gamma\cong\mathbb{A}^1$ is finite étale with Galois group Γ. Under the map, X is finite étale over $\mathbb{A}^1\setminus\{0\}$.
(5) One could assume that X is $\mathbb{A}^1$ since over characteristic p, the map $\mathbb{G}_m\to\mathbb{A}^1$ given by $x\mapsto x^p+\frac{1}{x}$ is finite étale of degree $p+1$.

□

Corollary 5.2. Let X be an absolutely irreducible smooth affine curve over k. If $\mathcal{F}$ is a τ-real and τ-pure lisse sheaf on X, of weight w, then $H^1_c(\overline{X},\mathcal{F})$ and $H^2_c(\overline{X},\mathcal{F})$ are also τ-real.

Proof. This is by the trace formula

$$\prod_{x\in|X|}\frac{1}{\tau\det(1-t^{d(x)}\sigma_x,\mathcal{F}_{\bar{x}})}=\frac{\tau\det(1-t\sigma,H^1_c(\overline{X},\mathcal{F}))}{\tau\det(1-t\sigma,H^2_c(\overline{X},\mathcal{F}))}$$

The zeros for the denominator are on the circle $|t|=q^{-\frac{w}{2}-1}$. The zeros for the numerator are outside the disc $|t|\geq q^{-\frac{w}{2}-\frac{1}{2}}$. Therefore, there are no common factors for the numerator and the denominator, and both should be polynomials with real coefficients. □

5.2. **The Weil conjecture for a morphism.** The goal is to prove the following theorem.

Theorem 5.3. If $f:X\to Y$ is a separated morphism of schemes of finite type over k, and $\mathcal{F}$ is a τ-mixed sheaf of weights $\leq n$, then for every i, $R^if_!\mathcal{F}$ is τ-mixed of weights $\leq n+i$.

Proof. We play the following dévissages.

a) Let $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ be an exact sequence. If the theorem holds for $\mathcal{F}'$ and $\mathcal{F}''$, then it also holds for $\mathcal{F}$. If the theorem holds for $\mathcal{F}$ and $\mathcal{F}''$, then it also holds for $\mathcal{F}'$.
b) Let $j:U\hookrightarrow X$ be an open embedding with $i:S\to X$ the complement. Applying a) to the exact sequence $0\to j_!j^*\mathcal{F}\to\mathcal{F}\to i_*i^*\mathcal{F}\to 0$, we obtain that it is enough to prove theorem for $fj:U\to Y$ and $fi:S\to Y$.

c) Let $j : V \hookrightarrow Y$ be an open embedding with $i : T \to Y$ the complement. It is enough to prove the theorem after the base changes under j and i.

d) If $f = gh$, then it is enough to prove the theorem for g and h. This follows from the Grothendieck-Leray spectral sequence.

e) If $Y' \to Y$ is a universal homeomorphism, then it suffices to prove the theorem after the base change to Y'. If $X' \to X$ is a universal homeomorphism, it is enough to prove the theorem for $X' \to X \to Y$.

Therefore, by b)-e), one could assume that both X and Y are connected smooth and affine, and $f : X \to Y$ is surjective and purely of relative dimension one.

To proceed, one makes use of the following fact: Let C be a curve over a field K. Then there is a finite purely inseparable extension $K' \supset K$, such that there exists an open dense subset $U \in C'_{red}$, smooth over K', where $C' = C \times_K K'$.

Now apply above claim to the case Spec$K$ is the generic point of Y, $C = \mathrm{Spec}K \times_Y X$. Let Y' be the normalization of Y in K', and $X' = X \times_Y Y'$. Since $K' \supset K$ is purely inseparable, $Y' \to Y$ is a universal homeomorphism. Furthermore, there is some dense open subset in X'_{red}, which is smooth over Y' of relative dimension one. Therefore, by b) and e), one could further assume that $f : X \to Y$ is smooth surjective, purely of relative dimension one.

By a) and b), one could assume that $\mathcal{F}$ is lisse, irreducible and τ-pure of weight w. Since $(R^i f_! \mathcal{F})_{\bar{y}} = H^i_c(\overline{X_y}, \mathcal{F})$, by Theorem 5.1, $w(R^i f_! \mathcal{F}) \leq w + i$.

It remains to show that $R^i f_! \mathcal{F}$ is τ-mixed. Since $\mathcal{F}$ is lisse and τ-pure, it is a direct summand of a lisse τ-real sheaf $\mathcal{F}'$. By Corollary 5.2, $R^i f_! \mathcal{F}'$ is also τ-real, and therefore τ-mixed by Theorem 3.6. Therefore, as a direct summand, $R^i f_! \mathcal{F}$ is also τ-mixed. □

We remark here that Theorem 5.3 here will not imply Theorem 3.3.1 in [D2] immediately. The claim there says that if $\mathcal{F}$ is mixed, then $R^i f_! \mathcal{F}$ is also mixed. What we have is the following: if $\mathcal{F}$ is mixed, then it is τ-mixed for all τ, and therefore, $R^i f_! \mathcal{F}$ is τ-mixed for all τ. In the case, there is a finite filtration of $R^i f_! \mathcal{F}$ such that associated quotients are pure sheaves twisted by some rank one sheaves on Spec$k$.

Exercise. Prove that if $\mathcal{F}$ is a sheaf that is τ-mixed for any $\tau : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, then there is a finite filtration of $\mathcal{F}$ such that each associated quotient is τ-pure for any τ.

However, we do obtain [D1] Lemma 1.7.

Theorem. Let X be a smooth proper variety over k. For every i, the eigenvalues of σ on $H^i(\overline{X}, \overline{\mathbb{Q}}_\ell)$ are algebraic numbers, all of whose complex conjugates have absolute value $q^{\frac{i}{2}}$.

Proof. Let X be smooth, geometrically connected. If $\mathcal{F}$ is a τ-pure lisse sheaf of weight w, then $H^i_c(\overline{X}, \mathcal{F})$ is τ-mixed of weight $\leq w+i$ by Theorem 5.3, and $H^i(\overline{X}, \mathcal{F})$ is τ-mixed of weight $\geq w+i$. Therefore, if X is proper, $H^i(\overline{X}, \mathcal{F}$ is τ-pure of weight $w+i$. If $\mathcal{F}$ is pure of weight w, then $H^i(\overline{X}, \mathcal{F})$ is pure of weight $w+i$. Observe that in this case, the eigenvalues of σ on $H^i(\overline{X}, \mathcal{F})$ are automatically algebraic numbers. □

References

[D1] Deligne, P.: *La conjecture de Weil I*, Publications Mathématiques IHÉS 43, 1974.
[D2] Deligne, P.: *La conjecture de Weil II*, Publications Mathématiques IHÉS 52, 1980.

[KW] Kiehl, R., Weissauer, R.: *Weil conjectures, perverse sheaves and ℓ'adic Fourier transform*, Ergebnisse der Mathematik und ihrer Grenzgebiete; 3. Folge Bd 42, Springer Verlag, 2001.