Some remarks on the teaching of fractions in elementary school

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It is widely recognized that there are at least two major bottlenecks in the mathematics education of grades K–8: the teaching of fractions and the introduction of algebra. Both are in need of an overhaul. I hope to make a contribution to the former problem by devising a new approach to elevate teachers' understanding of fractions. The need for a better knowledge of fractions among teachers has no better illustration than the the following story related by Herbert Clemens (1995):

> Last August, I began a week of fractions classes at a workshop for elementary teachers with a graph paper explanation of why $\frac{2}{7} \div \frac{1}{9} = 2\frac{4}{7}$. The reaction of my audience astounded me. Several of the teachers present were simply terrified. None of my protestations about this being a preview, none of my "Don't worry" statements had any effect.

This situation cries out for improvement.

Through the years, there has been no want of attempts from the mathematics education community to improve on the teaching of fractions (Lamon 1999, Bezuk-Cramer 1989, Lappan Bouck 1989, among others), but much

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work remains to be done. In analyzing these attempts and the existing school texts on fractions, one detects certain persistent problematic areas in both the theory and practice, and they can be briefly described as follows:

> (1) The concept of a fraction is never clearly defined in all of K–12.
> (2) The conceptual complexities associated with the common usage of fractions are emphasized from the beginning at the expense of the underlying mathematical simplicity of the concept.
> (3) The rules of the four arithmetic operations seem to be made up on an *ad hoc* basis, unrelated to the usual four operations on positive integers with which students are familiar.
> (4) In general, *mathematical* explanations of essentially all aspects of fractions are lacking.

These four problems are interrelated and are all fundamentally mathematical in nature. For example, if one never gives a clearcut definition of a fraction, one is forced to "talk around" every possible interpretation of the many guises of fractions in daily life in an effort to overcompensate. A good example is the over-stretching of a common expression such as "a third of a group of fifteen people" into a main theme in the teaching of fractions (Moynahan 1996). Or, instead of offering *mathematical* explanations to children of why the usual algorithms are logically valid—a simple task *if* one starts from a precise definition of a fraction,—algorithms are justified through "connections among real-world experiences, concrete models and diagrams, oral language, and symbols (p. 181 of Huinker 1998; see also Lappan & Bouck 1998 and Sharp 1998). It is almost as if one makes the concession from the start: "We will offer everything but the real thing".

Let us look more closely at the way fractions are introduced in the classroom. Children are told that a fraction $\frac{c}{d}$, with positive integers c and d, is simultaneously at least five different objects (cf. Lamon 1999 and Reys et al. 1998):

> (a) parts of a whole: when an object is equally divided into d parts, then $\frac{c}{d}$
> denotes c of those d parts.
> (b) the size of a portion when an object of size c is divided into d equal portions.
> (c) the quotient of the integer c divided by d.

(d) the ratio of c to d.
(e) an operator: an instruction that carries out a process, such as "$\frac{2}{3}$ of".

It is quite mystifying to me how this glaring "crisis of confidence" in fractions among children could have been been consistently overlooked. Clearly, even those children endowed with an overabundance of faith would find it hard to believe that a concept could be so versatile as to fit all these descriptions. More importantly, such an introduction to a new topic in mathematics is contrary to every mode of mathematical exposition that is deemed acceptable by modern standards. Yet, even Hans Freudenthal, a good mathematician before he switched over to mathematics education, made no mention of this central credibility problem in his Olympian ruminations on fractions (Freudenthal 1983). Of the existence of such crisis of confidence there is no doubt. In 1996, a newsletter for teachers from the mathematics department of the University of Rhode Island devoted five pages of its January issue to "Ratios and Rational Numbers" ([3]). The editor writes:

> This is a collection of reactions and responses to the following note from a newly appointed teacher who wishes to remain anonymous:
> "On the first day of my teaching career, I defined a rational number to my eighth grade class as a number that can be expressed as a ratio of integers. A student asked me: What exactly are ratios? How do ratios differ from fractions? I gave some answers that I was not satisfied with. So I consulted some other teachers and texts. The result was confusion ..."

This is followed by the input of many teachers as well as the editor on this topic, each detailing his or her inconclusive findings after consulting existing texts and dictionaries (!). In a similar vein, Lamon (1999) writes: "As one moves from whole number into fraction, the variety and complexity of the siutation that give meaning to the symbols increases dramatically. Understanding of rational numbers involves the coordination of many different but interconnected ideas and interpretations. There are many different meanings that end up looking alike when they are written in fraction symbol" (pp. 30–31). All the while, students are told that no one single idea or interpretation is sufficiently clear to explain the "meaning" of a fraction. This is akin to telling someone how to get to a small town by car by offering fifty suggestions on what to watch for each time a fork in the road comes up and how to interpret the road signs along the way, when a single clearly drawn road map would have done a much better job. Given these facts, is it any wonder that Lappan-Bouck (1998) and Lamon (1999) would lament that students "do" fractions without any idea of what they are doing? For example, it is certainly difficult to learn how to add two "operators" in the sense of (e).

Sometimes one could "get by" a mathematical concept without a precise definition if its rules of operation are clearly explained. Conjecturally, that was how Europeans in the 14th and 15th centuries dealt with negative numbers. In the case of fractions, however, this is not true even when interpretation (b) of fractions is used. The worst case is the rule of adding two fractions. In book after book (with very few exceptions, such as Lang (1988)), $\frac{a}{b} + \frac{c}{d}$ is defined as $(pa + cq)/m$, where $m = \mathrm{lcm}\{\mathrm{b}, \mathrm{d}\}$ and $m = bp = cq$. Now at least two things are wrong with this definition. First, it turns off many students because they cannot differentiate between lcm and gcd. This definition therefore sets up an entirely unnecesary roadblock in students' path of learning. Second, from a mathematical point of view, this definition is seriously flawed because it tacitly implies that without the concept of the lcm of two integers, fractions cannot be added. If we push this reasoning another step, we would arrive at the absurd conclusion that unless an integral domain has the unique factorization property, its quotient field cannot be defined.

Informal surveys among teachers consistently reveal that many of their students simply give up learning fractions at the point of the introduction of addition. It is probably not just a matter of being confused by gcd and lcm, but more likely a feeling of bewilderment and disgust at being forced to learn a new way of doing addition that seems to bear no relation to the addition of whole numbers. This then brings us to the problem area (3) at the beginning of this article. We see, for example, that Bezuk and Cramer (1989) willingly concede that "Children must adopt new rules for fractions that often conflict with well-established ideas about whole number" (p.156). In mathematics, one of the ultimate goals is to achieve simplicity. In the context of learning, it is highly desirable, perhaps even mandatory, that we convey this message of simplicity to students. However, when we tell students that something as simple as the addition of two numbers is different for whole numbers and fractions, we are doing them a great disservice. Even when students are willing to suspend disbelief and go along on such a weird journey, they pay a dear price. Indeed, there are recurrent reports of students at the University of California at Berkeley and at Stanford University claiming in their homework and exam papers that $\frac{a}{b} + \frac{a}{c} = \frac{a}{b+c}$ and $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$.

All in all, a mathematician approaching the subject of fractions in school mathematics cannot help but be struck by the total absence of the characteristic features of mathematics: precise definitions as starting point, logical progression from topic to topic, and most importantly, explanations that accompany each step. This is not to say that the teaching of fractions in elementary school should be rigidly formal from the beginning. Fractions should be informally introduced as early as the second grade (because even second graders need to worry about drinking "half a glass" of orange juice!), and there is no harm done in allowing children to get acquainted with fractions in an intuitive manner up to, say, the fourth grade. An analogy may be helpful here. The initial exploration of fractions may be taken as the "data-collecting phase" of a working scientist: just take it all in and worry about the meaning later. In time, however, the point will be reached when the said scientist must sit down to organize and theorize about his or her data. So it is that when students reach the fifth grade ([2]) or the sixth grade ([1]), their mathematical development cannot go forward unless "miracles" such as having one object $\frac{c}{d}$ enjoying the five different properties of (a)–(e) above are fully explained, and rules such as $\frac{a}{b}/\frac{c}{d} = \frac{ad}{bc}$ justified. And it at this critical juncture of students' mathematical education that I hope to make a contribution.

The work done on the teaching of fractions thus far has come mainly from the education community. Perhaps because of the recent emphasis on situated learning, fractions tend to be discussed *at the source*, in the sense that attention is invariably focussed on the interpretation of fractions in a "real world" setting. Since fractions are used in many contexts in many ways, students are led through myriad interpretations of a fraction from the beginning in order to get some idea of what a fraction is. At the end, a fraction is never defined and so the complexities tend to confuse rather than clarify (cf. (2) at the beginning of the article). More to the point, such an approach deprives students the opportunity to learn about an essential aspect of doing mathematics: when confronted with complications, try to abstract in order to achieve understanding. Students' first serious encounter with the computation of fractions may be the right moment in the school curriculum to turn things around by emphasizing its abstract, simple component and make the abstraction the center of classroom instruction. By so doing, one would also be giving students a substantial boost in their quest for learning algebra. The ability to abstract, so essential in algebra, should be taught as early as possible in the school curriculum, which would mean during the teaching of fractions. By giving abstraction its due in teaching fractions, we would be easing students' passage to algebra as well.

It takes no insight to conclude that two things have to happen if mathematics education in K-8 is to improve: there must be textbooks that treats fractions logically, and teachers must have the requisite mathematical knowledge to guide their students through this rather sophisticated subject. I propose to take up the latter problem by writing a monograph to improve teachers' understanding of fractions.

The first and main objective of this monograph is to give a treatment of fractions and decimals for teachers of grades 5–8 which is mathematically correct in the sense that everything is explained and the explanations are sufficiently elementary to be understood by elementary school teachers. In view of what has already been said above, an analogy may further explain what this monograph hopes to accomplish. Imagine that we are mounting an exhibit of Rembrandt's paintings, and a vigorous discussion is taking place about the proper lighting to use and the kind of frames that would show off the paintings to best advantage. Good ideas are also being offered on the printing of a handsome catalogue for the exhibit and the proper way to publicize the exhibit in order to attract a wider audience. Then someone takes a closer look at the paintings and realizes that all these good ideas might go to waste because some of the paintings are fakes. So finally people see the need to focus on the most basic part of the exhibit—the paintings—before allowing the exhibit to go public. In like manner, what this monograph tries to do is to call attention to the need of putting the *mathematics* of fractions in order before lavishing the pedagogical strategies and classroom activities on the actual teaching.

An abbreviated draft of the part of the monograph on fractions is already in existence (Wu 1998). The main point of the latter can be summarized as follows.

> (i) It gives a complete, self-contained mathematical treatment of fractions that explains every step logically.
> (ii) It starts with the definition of a fraction as a number (a point on the number line, to be exact), and deduces all other common properies ascribed to fractions (cf. (a)–(e) above) from this definition alone.
> (iii) It explicitly and emphatically restores the simple and correct definition of the addition of two fractions (i.e., $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$).

(iv) The four arithmetic operations of fractions are treated as extensions of what is already known about whole numbers. This insures that learning fractions is similar to learning the mathematics of whole numbers.
(v) On the basis of this solid mathematical foundation for fractions, precise explanations of the commonly used terms such as "17 percent of", "three-fifths of", "ratio", and "proportional to" are now given.
(vi) The whole treatment is elementary and, in particular, is appropriate for grades 5–8. In other words, it eschews any gratuitous abstractions.

In the eighteen months since Wu (1998) was written, I have gotten to know more about the culture of elementary school teachers and have come to understand better their needs. I have also gotten to know, quite surprisingly, that there are objections to a logically coherent mathematical treatment of fractions in grades 5–8 by a sizable number of educators. This objection would seem to be grounded on a misunderstanding of the basic structure of mathematics. There are also some gaps in the treatment of Wu (1998). All this new information must be fully incorporated into the forthcoming monograph. More specifically, the envisioned expansion will address the following areas:

(a) Discuss in detail from the beginning the pros and cons of the usual "discrete" models of fractions, such as pies and rectangles, versus the number line. Special emphasis will be placed on the pedagogical importance of point (iv) above. Such a discussion would address the concerns of many school teachers and educators who are used to having "models" for an abstract concept and have difficulty distinguishing between the number line as a model for fraction and its use as a *definition* which underlies the complete logical development of fractions.
(b) Explain carefully that at a certain point of elementary education, a mathematical concept should be given one definition and then have all other properties must be deduced from this definition *by logical deductions*. This need is not generally recognized in the education community. For fractions, the point in question would seem to be reached in the fifth grade ([2]) or sixth

grade ([1]). Whether such logical deductions are properly taught is likely to determine whether the learning of fractions is an immense aid or obstacle to the learning of algebra later on.
(c) More clearly delineate which part of the exposition primarily addresses teachers, and which part could be directly used in the classroom. Early readers of Wu (1998) have been to known to complain that "no student in the fifth grade could understand the algebraic notation", without realizing it was a document for *teachers*.
(d) Add a treatment of negative fractions and complete the discussion of the rational number system.
(e) Add a treatment of decimals and the relationship between decimals and fractions. Emphasis will be placed on a precise definition of decimals and the logical explanations of the common properties of decimals, viz., why fractions are the same as repeating decimals, and which fractions have finite decimal representations.
(f) Add a discussion of the role of calculators. Although calculators already appear in the exercises at the end of §3 and §4 in Wu (1998), there is need of a discursive and direct discussion of this important issue.
(g) Amplify on the brief discussion of ratio and proportion in Wu (1998). Carefully explain the unfortunate historical origin of the concept of "ratio" in Euclid's *Elements* which has led to immense and unnecessary confusion about what it means. Clarify the concept of so-called "proportional thinking" which has been treated with unwarranted reverence in the literature.
(h) Add generous pedagogical comments on how to deliver such an approach to fractions in the classroom.
(j) Expand on the rather terse expository style of Wu (1998), everywhere. In particular more exercises and more examples are needed.

Thus Wu (1998) will have to be fleshed out in a substantial way, and its vision has to be sharpened. More importantly, to ensure its usefulness as a resource for professional development, it is essential that it be tested on teachers before it is finalized. Perhaps a draft of this mongraph can be put into service in some professional development activities in the coming months.

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