Solutions to the Activities *in* Understanding Numbers in Elementary School Mathematics

H. Wu*

September 14, 2018

P. 12. *Assume the usual terminology that 100 is* one hundred*, 1000 is* one thousand*, etc.* (a) *Now make believe that you are explaining to a third grader and explain why the 3 in 352 stands for 300, the 5 stands for 50, and the 2 stands for 2.* (b) *Explain to this third grader why, in ten steps of skip-counting by 100, one can go from 1000 to 2000.*

(a) If we write out all the numbers up to three digits in rows of a hundred numbers, then the first row consists of all the three-digit numbers starting with 0, i.e., this is the row of one-digit and two-digit numbers. The next row consists of the numbers starting with 1, i.e., 100, 101, . . . , 199, the next row consists of the numbers starting with 2, i.e., 200, 201, . . . , 299, and then the next row consists of the numbers starting with 3, i.e., 300, 301, . . . , 399. The number 352 is to be found in the last row. Now from the point of view of counting, one reaches 300 only after counting through the *three hundred* steps from 0, to 1, . . . , to 99, to 100, to 101, . . . , to 199, to 200, to 201, . . . , to 299, all the way to 300. So clearly the 3 of 352 signifies 300. Next, look at 352 among the numbers in this row. Because they all start with 3, we will ignore the first digit and only look at the two-digit numbers consisting of the next two digits. We count through rows of ten numbers, starting from 00, to 01, . . . , to 09, to 10, to 11, . . . , to 19, to 20, to 21, . . . , to 29, to 30, to 31, . . . , to 39, to 40,

*I want to thank Sunil Koswatta for his help in the writing of these solutions, and David Collins for his uncanny ability to detect subtle errors.

to 41, . . . , to 49, and then finally to 50, to 51, to 52 . . . , and to 59. Visibly, 52 is the second number after 50. In terms of counting, we have to count fifty steps from 0, 1, 2, . . . , 49, all the way to 50. So the 5 of 52 signifies 50, and 2 is two steps after 50.

(b) Look at all the numbers from 1000 to 1999. If we put them in rows of 100, then we have 1000, 1001, 1002, . . . , 1099, and then 1100, 1101, 1102, . . . , 1199, then 1200, 1201, 1202, . . . , 1299, then 1300, . . . , . . . , 1900, 1901, 1902, . . . , 1999. If we count every 100 steps from 1000, then we get

$$1100, 1200, 1300, \ldots, 1900$$

Another 100 steps from 1900, as we have seen, is 2000. So nine steps of 100 numbers go from 1000 to 1900, and the tenth step then takes us to 2000.

P. 15. *Count by using only the four symbols 0, 1, 2, 3, and the same idea of using different places.* (a) *Write down the first 48 numbers in this numeral system if we start counting with 0.* (b) *What is the 35th number if we start counting from 0? Can you figure this out without looking at the list in (a)?* (c) *What is the 51st number if we start counting from 0? The 70th number?*

(a) With one place, we can only write down 4 numbers: 0, 1, 2, 3. With two places, each place has 4 choices and so we get $4 \times 4 = 16$ numbers: 00, 01, . . . , . . . , 32, 33. With three places, we can write down $4 \times 4 \times 4 = 64$ numbers. So we can limit ourselves to three digits as they already contain more than 48 numbers. We write out all three-digit numbers: the first row of 16 numbers is

$$000, 001, 002, \ldots, 033$$

the second row of 16 numbers is

$$100, 101, 102, \ldots, 133$$

and 133 is the 32nd number if we start counting with 0. So when the third row of 16 numbers is written out,

$$200, 201, 202, \ldots, 233$$

the number 233 is the 48th number. Thus in the first three rows, we have the first 48 numbers.

(*b*) Since 133 is the 32nd number, and the next four numbers are 200, 201, 202, 203, we see that 202 is the 35th number.

(*c*) Since 233 is the 48th number, and the next four numbers are 300, 301, 302, 303, we see that 302 is the 51st number. As for the 70th number, the 64th number is 333. The next eight numbers are

$$1000,\ 1001,\ 1002,\ 1003,\ 1010,\ 1011,\ 1012,\ 1013$$

So the 70th number is 1011.

P. 19. *If you count in the usual way, what is the 200th number beyond 6490721? What about the 230th number? And the 236th number? The 5164th number beyond 6490721?*

The 100th number beyond 6490721 is 6490821. Another 100 steps get us to 6490921 So the 200th number beyond 6490721 is 6490921.

The 230th number beyond 6490721 is therefore the 30th number beyond 6490921, which is 6490951.

The 236th number beyond 6490721 is therefore the 6th number beyond 6490951, which is 6490957.

The 5000th number beyond 6490721 is 6495721, and another 100 steps land us on 6495821. Another 60 steps land us on 6495881, and another 4 steps land us on 6495885. So the 5164th number beyond 6490721 is 6495885.

P. 20. *What is the number if we count 500 more from 516234? 50000 more from 516234?*

516734 and 566234.

P. 21. *Write 88 and 99 in Roman numerals; contrast the Hindu-Arabic numeral system with the Roman numeral system in this case. Do the same with the numbers 420 and 920.*

88 is LXXXVIII; 90 is XC and therefore 99 is XCIX. It is difficult to "see" how XCIX could be 10 steps *beyond* LXXXVIII because XCIX "looks" smaller than LXXXVIII. Since 400 is CD, 420 is CDXX. Because 900 is CM, 920 is CMXX. In this case, we can see that 920 is 500 steps beyond 420 by inspecting the leftmost digit in each number, but CDXX and CMXX differ only in the *second symbol!*

P. 27. *Can you explain to a third grader why 7001 is greater than 5897? Don't forget, you are not supposed to use subtraction or any kind of computation. You have to first explain what "greater than" means, and then use that to explain* $7001 > 5897$.

To show 7001 is greater than 5897, *by definition*, we have to show that if we count from 0 then we get to 5897 first before we get to 7001. One way is to first count 3 steps from 5897 to get to 5900, and another 100 steps to get to 6000. So by definition, 6000 is greater than 5897. However, if we skip-count by 1000 from 0, then we get to 6000 after six steps before getting to 7000. So 7000 is greater than 6000. Since we have to take another step from 7000 to get to 7001, we see that 7001 is greater than 6000 which, as we have just seen, is greater than 5897. Altogether, 7001 is greater than 5897.

P. 28. *Consider the following introduction to* multiplication *taken from a third-grade textbook* (*the text has the goal of making sure that at the end of the third grade, students know the multiplication table of numbers up to 10*)*:*

> `Look at the 3 strips of stickers shown on the right` (there is a picture of three strips of stickers)`.  There are 5 stickers on each strip.  How can you find the number of stickers there are in all?`
>
> `You can find the total number in different ways.`

<table>
<tr>
<td>You can write an ADDITION sentence.
$5+5+5=15$

THINK: 3 groups of 5 $=$ 15.</td>
<td>You can write a MULTIPLICATION sentence
$3\times 5=15$
READ: Three times 5 equals 15.</td>
</tr>
</table>

Answer: 15 stickers.

Do you think this is an ideal way to convey to third graders what multiplication *means? Explain.*

It is not clear that the meaning of multiplication has been explained. More precisely, there are at least two errors:

Error 1: It says there are "different" ways to find the total number, so one expects the different ways to be explained in terms of what students already know. Instead, the way on the right makes use of multiplication, which has yet to be defined.

Error 2: If it is claimed that there are different ways to do the computation, the book should explain why the different ways give the same answer. This is not done, and students are left guessing as to why $5+5+5=3\times 5$. Is this something they are supposed to know already?

P. 31. *As an exercise in the use of exponential notation, write out the expanded form of the following numbers:* 14600418, 500007009, 94009400940094.

$$\begin{aligned}
14600418 &= (1\times 10^7)+(4\times 10^6)+(6\times 10^5)+(4\times 10^2)+(1\times 10)+8\\
500007009 &= (5\times 10^8)+(7\times 10^3)+9\\
94009400940094 &= (9\times 10^{13})+(4\times 10^{12})+(9\times 10^9)+(4\times 10^8)\\
&\quad +(9\times 10^5)+(4\times 10^4)+(9\times 10)+4
\end{aligned}$$

P. 32. *In the expanded form of a number, the term with the highest power of 10 is called the* **leading term**. (*Thus* 7×10^6 *is the leading term of* $(7\times 10^6)+(2\times 10^3)+(4\times 10^2)+(1\times 10^0)$, *which is the expanded form of 7002401.*) *Explain why the leading term of the expanded form of any number is always larger than the sum of all the other terms in the expanded form.*

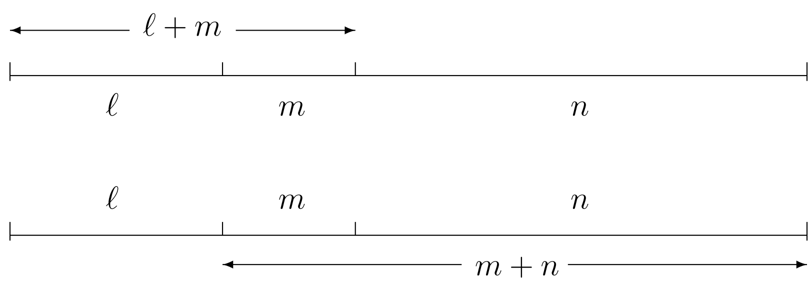
We will look at a special case, and then it will be clear that the reasoning there is perfectly general. Suppose we have a ten-digit number N that begins with the digit n, so $n \geq 1$ and the leading term of the expanded form of N is $n \times 10^9$. Suppose the sum of all the other terms in the expanded form of this ten-digit number, other than $n \times 10^9$, is m. Thus m is a 9-digit number. We have to show that $n \times 10^9 > m$. In terms of counting, since the last 9-digit number is 999,999,999, we see that $999{,}999{,}999 \geq m$. Of course $1{,}000{,}000{,}000 > 999{,}999{,}999$. Together, we have $10^9 > m$. Since $n \geq 1$, we finally get

$$n \times 10^9 \;\geq\; 1 \times 10^9 \;=\; 10^9 > m$$

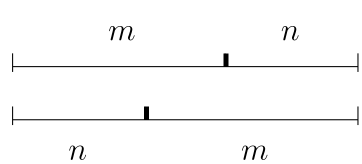
P. 42. *Interpret equations* (2.2) *and* (2.3) *in terms of the concatenation of segments.*

(2.2) states $(\ell + m) + n = \ell + (m + n)$, while (2.3) states $m + n = n + m$ for any three whole numbers ℓ, m, and n.

(2.2) can be visualized from the following two pictures.



Similarly, (2.3) can be visualized from the next two pictures:



P. 43. $37 + 189 + 163 = \,?$ $\quad 275 + 892 + 225 + 4211 + 108 = \,?$

By Theorem 2.1 on page 41,

$$37 + 189 + 163 = (37 + 163) + 189 = 200 + 189 = 389.$$

By the same theorem:

$$275+892+225+4211+108 \;=\; (275+225)+(892+108)+4211 \;=\; 500+1000+4211,$$

and the latter is easily seen to be 5711.

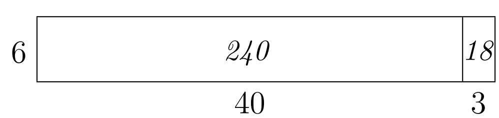
P. 43. $666{,}666{,}667 + 788{,}646{,}851{,}086 + 333{,}333{,}333 = \,?$

By Theorem 2.1 on page 41 again, the sum is equal to

$$(666{,}666{,}667 + 333{,}333{,}333) + 788{,}646{,}851{,}086 \;=\; 1{,}000{,}000{,}000 + 788{,}646{,}851{,}086,$$

which is equal to $789{,}646{,}851{,}086$.

P. 46. *One can use the distributive law to multiply a two-digit number by a one-digit number using mental math. For example, to compute* 43×6*, we break up 43 into* $(40 + 3)$ *so that* $43 \times 6 = (40 + 3) \times 6 = (40 \times 6) + (3 \times 6)$*, and the last is just* $240 + 18 = 258$.



Therefore, $43 \times 6 = 258$*. Now follow this example and use mental math to compute:* (a) 24×8, (b) 53×7, (c) 39×6, (d) 79×5, (e) 94×9, (f) 47×8.

(a) $24 \times 8 = (20 + 4) \times 8 = 160 + 32 = 192$.
(b) $53 \times 7 = (50 + 3) \times 7 = 350 + 21 = 371$.
(c) $39 \times 6 = (30 + 9) \times 6 = 180 + 54 = 234$.
(d) $79 \times 5 = (70 + 9) \times 5 = 350 + 45 = 395$.

(*e*) $94 \times 9 = (90 + 4) \times 9 = 810 + 36 = 846$.
(*f*) $47 \times 8 = (40 + 7) \times 8 = 320 + 56 = 376$.

P. 48. *If a, b, c, d are whole numbers such that $a + c = b + d = 11$, what is $ab + bc + da + dc$?*

$ab + bc + da + dc = b(a + c) + d(a + c) = (b + d)(a + c) = 11 \times 11 = 121$.

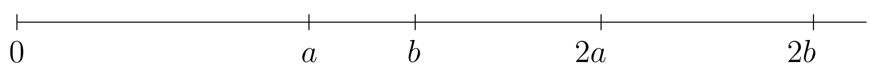
P. 50. *Assume the truth of the inequalities* (2.7). *Which is bigger?* $86427895 \times 172945678$, *or* $86427963 \times 173000001$*?*

By (ii) on page 27, 86,427,895 $<$ 86,427,963 and 172,945,678 $<$ 173,000,001. Hence by the third inequality of (2.7),

$$86427895 \times 172945678 \;<\; 86427963 \times 173000001$$

P. 50. *Without using* (2.7), *convince yourself that $2a < 2b$ is equivalent to $a < b$ by using the number line.*

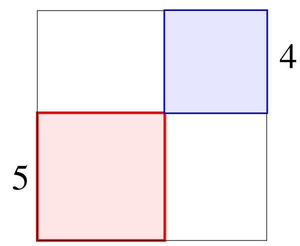
First, the fact that if $a < b$ then $2a < 2b$ is clear from the picture:



Conversely, if $2a < 2b$, then half the distance from 0 to $2a$ is less than half the distance from 0 to $2b$. In other words, $a < b$.

P. 51. *Recall the notation of the* **square** *of a number:* $9^2 = 9 \times 9$, $4^2 = 4 \times 4$, *etc.* (*Of course the reason for calling, e.g.,* 9^2 *the* square *of 9 is that it is the area of the* square *with all four sides equal to 9.*) *Compare* 9^2 *and* $4^2 + 5^2$. *Which is bigger? Do the same with:* (*i*) 12^2 *and* $5^2 + 7^2$. (*ii*) 18^2 *and* $12^2 + 6^2$. (*iii*) 23^2 *and* $15^2 + 8^2$. *Now if a and b are any two positive whole numbers, compare* $(a + b)^2$ *and* $a^2 + b^2$ *and explain why your answers to* (*i*)–(*iii*) *are correct.*

Clearly $4^2 + 5^2 < (4 + 5)^2 = 9^2$ because of the following picture:



The same picture also disposes of (i)-(iii). Replacing 5 in the picture by a and 4 by b, we see that $a^2 + b^2 < (a+b)^2$.

P. 62. *Suppose you want to buy twenty-five 39-cent stamps, twenty 24-cent stamps, ten 63-cent airmail stamps, and thirty 84-cent airmail stamps.* (*a*) *Forget about the addition algorithm and think of the normal way you would compute the total amount of money it would take to buy this many stamps. How would you do this computation?* (*b*) *Now that you have done it, reflect on your computation and draw a parallel between it and the addition algorithm.*

(a) I would add up all the 39's first (25×39), then add up all the 24's (20×24), then add up all the 63's (10×63), and finally add up all the 84's (30×84). Finally I add these numbers together: $975 + 480 + 630 + 2520 = 4605$ cents, which is 46 dollars and 5 cents.

(b) This is analogous to adding, say, $152 + 348 + 227$. The addition algorithm says we first add up all the hundreds $100+300+200$, then add up all the tens, $50+40+20$, and then add up all the ones, $2 + 8 + 7$. Finally we add these numbers together at the end: $600 + 110 + 17 = 727$.

P. 63. *Compute* $4502 + 273$ *by the algorithm, and give an explanation to your neighbor of why it is correct.*

First, the algorithm:

$$\begin{array}{rcccc} & 4 & 5 & 0 & 2 \\ + & & 2 & 7 & 3 \\ \hline & 4 & 7 & 7 & 5 \end{array}$$

The explanation:

$$\begin{aligned} 4502 &= 4000 + 500 + 2 \\ 273 &= 200 + 70 + 3 \end{aligned}$$

Therefore, using Theorem 2.1 on page 41, we have:

$$4502 + 273 = 4000 + (500 + 200) + 70 + (2 + 3)$$

which is the algorithm.

P. 65. *Explain to your neighbor why the addition algorithm for* $95 + 46$ *works.*

$$\begin{array}{rrrr} & & 9 & 5 \\ & & 4 & 6 \\ + & {\scriptstyle 1} & {\scriptstyle 1} & \\ \hline & 1 & 4 & 1 \end{array}$$

Here is the explanation:

$$\begin{aligned} 95 &= 90 + 5 \\ 46 &= 40 + 6 \end{aligned}$$

By Theorem 2.1 on page 41, we have

$$95 + 46 = (90 + 40) + (5 + 6) = (90 + 40) + (10 + 1) = (90 + 40 + 10) + 1,$$

which is equal to $100 + 40 + 1$. That is what is recorded in the algorithm.

P. 67. *Verify the preceding computation, and give an explanation by following the explanations of addition examples* (4.1) *and* (4.2)*.*

The "preceding computation" is $165 + 27 + 83 + 829$, and the algorithm gives:

$$\begin{array}{rrrrr} & & 1 & 6 & 5 \\ & & & 2 & 7 \\ & & & 8 & 3 \\ & & 8 & 2 & 9 \\ + & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 2} & \\ \hline & 1 & 1 & 0 & 4 \end{array}$$

Here is the explanation:

$$\begin{aligned}
165 &= 100+60+5 \\
27 &= 20+7 \\
83 &= 80+3 \\
829 &= 800+20+9
\end{aligned}$$

As usual, we make repeated use of Theorem 2.1 on page 41 to conclude:

$$\begin{aligned}
165+27+83+829 &= (100+800)+(60+20+80+20)+(5+7+3+9) \\
&= (100+800)+(60+20+80+20)+(20+4) \\
&= (100+800)+(60+20+80+20+20)+4 \\
&= (100+800+200)+0+4 \\
&= 1000+100+0+4
\end{aligned}$$

The last three lines are what is recorded in the algorithm.

P. 72. *Use the formal definition of subtraction to compute and explain:* $1200-500$*;* $580{,}000{,}000 - 500{,}000{,}000$*;* $580{,}000{,}000 - 20{,}000{,}000$*;* $15\times 10^6 - 7\times 10^6$.

Since $700+500 = 1200$, we see that, by definition, $1200-500 = 700$. Similarly, $80{,}000{,}000 + 500{,}000{,}000 = 580{,}000{,}000$ implies that $580{,}000{,}000 - 500{,}000{,}000 = 80{,}000{,}000$; $560{,}000{,}000 + 20{,}000{,}000 = 580{,}000{,}000$ implies that $580{,}000{,}000 - 20{,}000{,}000 = 560{,}000{,}000$, and finally $8\times 10^6 + 7\times 10^6 = (8+7)\times 10^6 = 15\times 10^6$ implies that $15\times 10^6 - 7\times 10^6 = 8\times 10^6$.

P. 74. *Use mental math to compute* $493{,}625 - 273{,}514$ *and* $57{,}328{,}694 - 4{,}017{,}382$.

Since $4-2=2$, $9-7=2$, $3-3=0$, $6-5=1$, $2-1=1$, $5-4=1$, we see that $493{,}625 - 273{,}514 = 220{,}111$. Similarly, $57{,}328{,}694 - 4{,}017{,}382 = 53{,}311{,}312$.

P. 75. *Use the subtraction algorithm to compute* $2345 - 687$*.*

$$\begin{array}{rrrrr} & \scriptstyle 1 & \scriptstyle 12 & \scriptstyle 13 & \scriptstyle 15 \\ & \cancel{2} & \cancel{3} & \cancel{4} & 5 \\ - & & 6 & 8 & 7 \\ \hline & 1 & 6 & 5 & 8 \end{array}$$

P. 75. *Use the subtraction algorithm to compute* $300207 - 14629$*.*

$$\begin{array}{rrrrrrr} & \scriptstyle 2 & \scriptstyle 9 & \scriptstyle 9 & \scriptstyle 11 & \scriptstyle 9 & \scriptstyle 17 \\ & \cancel{3} & \cancel{0} & \cancel{0} & \cancel{2} & \cancel{0} & 7 \\ - & & 1 & 4 & 6 & 2 & 9 \\ \hline & 2 & 8 & 5 & 5 & 7 & 8 \end{array}$$

P. 77. *Explain the subtraction algorithm for* $315 - 82$*, first schematically, and then by the conventional method using* (5.5)*.*

$$\begin{array}{rrrr} & \scriptstyle 2 & \scriptstyle 11 & \\ & \cancel{3} & \cancel{1} & 5 \\ - & & 8 & 2 \\ \hline & 2 & 3 & 3 \end{array}$$

Schematically:

$$\begin{array}{rrrr} & 3 & 1 & 5 \\ - & & 8 & 2 \\ \hline & ? & ? & ? \end{array} \iff \begin{array}{rrrrrr} & 300 & + & 10 & + & 5 \\ - & & & 80 & + & 2 \\ \hline & & & ? & & \end{array}$$

$$\iff \begin{array}{rrrrrr} & 200 & + & 110 & + & 5 \\ - & & & 80 & + & 2 \\ \hline & & & ? & & \end{array} \overset{(5.5)}{\iff} \begin{array}{rrrrrr} & 200 & + & 110 & + & 5 \\ - & & & 80 & + & 2 \\ \hline & 200 & + & 30 & + & 3 \end{array}$$

$$\iff \begin{array}{rrrr} & 3 & 1 & 5 \\ - & & 8 & 2 \\ \hline & 2 & 3 & 3 \end{array}$$

Now, the conventional method:

$$\begin{aligned} 315 - 82 &= (300 + 10 + 5) - (0 + 80 + 2) \\ &= (200 + 110 + 5) - (0 + 80 + 2) \qquad \text{(associative law of +)} \\ &= (200 - 0) + (110 - 80) + (5 - 2) \qquad \text{(by (5.5))} \\ &= 200 + 30 + 3 \;=\; 233 \end{aligned}$$

P. 78. *Do the preceding subtraction* $50003 - 465$ *from left to right, and compare with the computation from right to left.*

Recall: the following is the computation from right to left:

$$\begin{array}{rccccc}
 & {\scriptstyle 4} & {\scriptstyle 9} & {\scriptstyle 9} & {\scriptstyle 9} & {\scriptstyle 13} \\
 & \cancel{5} & \cancel{0} & \cancel{0} & \cancel{0} & 3 \\
- & & & 4 & 6 & 5 \\
\hline
 & 4 & 9 & 5 & 3 & 8
\end{array}$$

Now if we do the computation from left to right, we have the left two digits of the answer right away:

$$\begin{array}{rccccc}
 & 5 & 0 & 0 & 0 & 3 \\
- & & & 4 & 6 & 5 \\
\hline
 & 5 & 0 & & &
\end{array}$$

The subtraction in the next column to the right is $0-4$, which is, for us, not defined. So we have to trade:

$$\begin{array}{rccccc}
 & {\scriptstyle 4} & {\scriptstyle 9} & {\scriptstyle 10} & & \\
 & \cancel{5} & \cancel{0} & 0 & 0 & 3 \\
- & & & 4 & 6 & 5 \\
\hline
 & 4 & 9 & 6 & &
\end{array}$$

Then the subtraction in the next column is $0-6$, which is again undefined. We trade again:

$$\begin{array}{rccccc}
 & {\scriptstyle 4} & {\scriptstyle 9} & {\scriptstyle 9} & {\scriptstyle 10} & \\
 & \cancel{5} & \cancel{0} & \cancel{0} & 0 & 3 \\
- & & & 4 & 6 & 5 \\
\hline
 & 4 & 9 & 5 & 4 &
\end{array}$$

Now the subtraction in the last column is $3-5$, which is also not defined, so we trade one more time:

$$\begin{array}{rccccc}
 & {\scriptstyle 4} & {\scriptstyle 9} & {\scriptstyle 9} & {\scriptstyle 9} & {\scriptstyle 13} \\
 & \cancel{5} & \cancel{0} & \cancel{0} & \cancel{0} & 3 \\
- & & & 4 & 6 & 5 \\
\hline
 & 4 & 9 & 5 & 3 & 8
\end{array}$$

This is the answer.

It is clear that the insistence on doing it from left to right is possible, but just not worth the trouble.

P. 82. *Compute* $1004 - 758$ *and* $60005 - 12348$ *by mental math.*

$1004 - 758 = (999+5) - (758+0) = (999-758) + (5-0)$, using (5.5). The answer is $241 + 5 = 246$. Next, we will be more brief:

$$60005 - 12348 \;=\; 59999 - 12348 + 6 \;=\; 47651 + 6 \;=\; 47657$$

P. 86. *Revisit Exercise 6 on page 69 by comparing what you get from the addition algorithm in* $7826+7826+7826+7826+7826$ *and what you get from the multiplication algorithm in* 7826×5.

If we add according to the addition algorithm, we have:

$$\begin{array}{cccccc}
 & & 7 & 8 & 2 & 6 \\
 & & 7 & 8 & 2 & 6 \\
 & & 7 & 8 & 2 & 6 \\
 & & 7 & 8 & 2 & 6 \\
 & & 7 & 8 & 2 & 6 \\
+ & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 1} & {\scriptstyle 3} & \\
\hline
 & 3 & 9 & 1 & 3 & 0
\end{array}$$

If we now do it as multiplication, we have

$$\begin{array}{cccccc}
 & & 7 & 8 & 2 & 6 \\
 & & & & & 5 \\
\times & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 1} & {\scriptstyle 3} & \\
\hline
 & 3 & 9 & 1 & 3 & 0
\end{array}$$

Notice the similarity in the lines containing the entries of the carrying.

P. 87. *Do the following computation using the multiplication algorithm:*

$$\begin{array}{ccccccc}
 & & & & 5 & 2 & 7 \\
\times & & & & 3 & 6 & 4 \\
\hline
 & & & 2 & 1 & 0 & 8 \\
 & & 3 & 1 & 6 & 2 & \\
+ & 1 & 5 & 8 & 1 & & \\
\hline
 & 1 & 9 & 1 & 8 & 2 & 8
\end{array}$$

P. 88. *Give an analogous explanation of* (6.2).

(6.2) is the multiplication

$$\begin{array}{rcccc} & & 8 & 5 & 2 \\ & & & & 7 \\ \times & {\scriptstyle 5} & {\scriptstyle 3} & {\scriptstyle 1} & \\ \hline & 5 & 9 & 6 & 4 \end{array}$$

So

$$\begin{aligned}
852 \times 7 &= [(8 \times 10^2) + (5 \times 10) + 2] \times 7 \\
&= (56 \times 10^2) + (35 \times 10) + 14 \qquad \text{(dist. law)} \\
&= [(50+6) \times 10^2] + [(30+5) \times 10] + [(10+4)] \\
&= (5 \times 10^3) + (6 \times 10^2) + (3 \times 10^2) + (5 \times 10) + (1 \times 10) + 4 \\
&= (5 \times 10^3) + [(6+3) \times 10^2] + [(5+1) \times 10] + 4
\end{aligned}$$

P. 89. *Compute* 73×852 *in the form:*

$$\begin{array}{rccc} & & 7 & 3 \\ \times & 8 & 5 & 2 \\ \hline & & ? & \end{array}$$

and give an explanation. (*Note: By the commutativity of multiplication, you know in advance that the answer is 62196* (= 852×73). *So the point is to see that the multiplication algorithm does give the same answer.*)

As in the preceding Activity, we have:

$$\begin{aligned}
73 \times 2 &= 146 \\
73 \times 5 &= 365 \\
73 \times 8 &= 584
\end{aligned}$$

Therefore,

$$\begin{aligned}
73 \times 852 &= 73 \times [(8 \times 10^2) + (5 \times 10) + 2] \\
&= [(73 \times 8) \times 10^2] + [(73 \times 5) \times 10] + [73 \times 2] \qquad \text{(dist. law)} \\
&= (584 \times 10^2) + (365 \times 10) + 146 \\
&= [(5 \times 10^4) + (8 \times 10^3) + (4 \times 10^2)] + \\
&\quad\; [(3 \times 10^3) + (6 \times 10^2) + (5 \times 10)] + [(1 \times 10^2) + (4 \times 10) + 6] \\
&= (5 \times 10^4) + [(8 + 3) \times 10^3] + [(4 + 6 + 1) \times 10^2] + [(5 + 4) \times 10] + 6 \\
&= [(5 + 1) \times 10^4] + [(1 + 1) \times 10^3)] + [(1 \times 10^2)] + [(5 + 4) \times 10] + 6 \\
&= (6 \times 10^4) + (2 \times 10^3) + (1 \times 10^2) + (9 \times 10) + 6
\end{aligned}$$

This computation justifies the following algorithmic description:

$$\begin{array}{cccccc}
 & & & & 7 & 3 \\
\times & & & 8 & 5 & 2 \\
\hline
 & & & 1 & 4 & 6 \\
 & & 3 & 6 & 5 & 0 \\
+ & 5 & 8 & 4 & 0 & 0 \\
\hline
 & 6 & 2 & 1 & 9 & 6
\end{array}$$

P. 90. *Suppose you are presented with the following computation:*

$$\begin{array}{ccccccc}
 & & & & 5 & 2 & 7 \\
\times & & & 3 & 0 & 0 & 4 \\
\hline
 & & & 2 & 1 & 0 & 8 \\
+ & 1 & 5 & 8 & 1 & & \\
\hline
 & 1 & 6 & 0 & 2 & 0 & 8
\end{array}$$

Is this correct? Why?

It is not correct, because if we apply the multiplication algorithm exactly, then we would get:

$$\begin{array}{cccccccc}
 & & & & & 5 & 2 & 7 \\
\times & & & & 3 & 0 & 0 & 4 \\
\hline
 & & & & 2 & 1 & 0 & 8 \\
 & & & 0 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 0 & & \\
+ & 1 & 5 & 8 & 1 & & & \\
\hline
 & 1 & 5 & 8 & 3 & 1 & 0 & 8
\end{array}$$

P. 90. *Give a precise description as well as explanation of the preceding algorithm.*

The "preceding algorithm" in question is the following:

$$\begin{array}{cccccc} & & 6 & 7 & 1 & 8 \\ \times & & & & & 5 \\ \hline & 3 & 0 & & & \\ & & 3 & 5 & & \\ & & & & 5 & \\ + & & & & 4 & 0 \\ \hline & 3 & 3 & 5 & 9 & 0 \end{array}$$

The explanation:

$$\begin{aligned} 6718 \times 5 &= [(6 \times 10^3) + (7 \times 10^2) + (1 \times 10) + 8)] \times 5 \\ &= [(6 \times 5) \times 10^3] + [(7 \times 5) \times 10^2] + [(1 \times 5) \times 10] + [8 \times 5] \quad \text{(dist. law)} \\ &= (30 \times 10^3) + (35 \times 10^2) + (5 \times 10) + 40 \\ &= (3 \times 10^4) + (35 \times 10^2) + 50 + 40 \\ &= 33590 \end{aligned}$$

When this computation is written out in algorithmic form, it is the same as the above:

$$\begin{array}{cccccc} & & 6 & 7 & 1 & 8 \\ \times & & & & & 5 \\ \hline & 3 & 0 & 0 & 0 & 0 \\ & & 3 & 5 & 0 & 0 \\ & & & & 5 & 0 \\ + & & & & 4 & 0 \\ \hline & 3 & 3 & 5 & 9 & 0 \end{array}$$

P. 100. *In a third grade textbook, division is introduced as follows:*

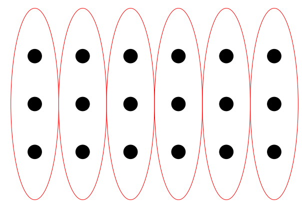
`You can use counters to show two ways to think about dividing.`

`(A) Suppose you have 18 counters and you want to make 6 equal groups. You can DIVIDE to find how many to put into each group.`

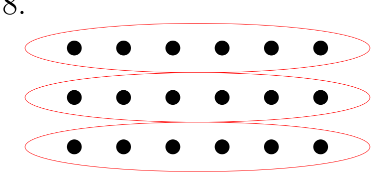
(B) Suppose you have 18 counters and you want to put them into equal groups, with 6 counters in each group. You can also DIVIDE to find how many groups there are altogether.

Although you can see easily in this special case that the answer to both problems is 3, discuss which of these two divisions uses the measurement interpretation and which uses the partitive interpretation. Use rectangular rows of dots to illustrate your answer.

Part (A) uses the partitive division of $18 \div 6$, because it divides 18 objects into 6 equal groups.



On the other hand, (B) uses the measurement division of $18 \div 6$ because it measures how many 6's there are in 18.



P. 103. *If a rectangle has area equal to 84 square units and a vertical side equal to 7 units, what is the length of the horizontal side?*

The length is $84 \div 7 = 12$ units.

P. 106. *Without looking at the statement of Theorem 7.1 above, explain to your neighbor the meaning of "the division-with-remainder of 115 by 17", first algebraically (i.e., write down the analogue of equation* (7.3) *for this case and explain what the numbers mean), and then by using a number line to describe the quotient and the remainder.*

$115 = (6 \times 17) + 13$, so the division-with-remainder has quotient 6 and remainder 13. Thus the 6th multiple of 17 ($= 102$) is less than 115, and misses 115 by 13, but the 7th multiple of 17 (= 119) is bigger than 115. On the number line, we mark off the multiples of 17 one by one, and we see how the 6th is to the left of 115, but the 7th is to its right:

P. 106. (*a*) *What is the remainder of 116 divided by 6? 118 divided by 7?* (*b*) *Find the quotient and remainder of 577 divided by 23 by inspecting the multiples of 23.*

(*a*) $116 = (19 \times 6) + 2$, so the remainder of 116 divided by 6 is 2. $118 = (16 \times 7) + 6$, so the remainder of 118 divided by 7 is 6.

(*b*) The 20th multiple of 23 is 460, and the 25th is $= 460 + (5 \times 23) = 575$, so it is clear that the division-with-remainder of 577 by 23 has quotient 25 and remainder 2.

P. 107. *Consider a typical presentation of these concepts in standard textbooks:*

> *division* An operation on two numbers that tells how many groups or how many in each group.
> *quotient* The answer in division.
> *remainder* The number that is left over after dividing.

How many mathematical errors can you find in the preceding passage? (*There are at least three.*)

"An operation on two numbers that tells how many groups or how many in each group." This makes no sense: if the division is between 15 and 17, what does it mean to say *how many groups of 15 there are in 17* or is it *how many groups of 17 there are in 15*?

Since "division" has not be defined clearly, what could be "the answer"?

"The number that is left over after dividing." If we divide 15 by 4, there is 1 group of 4 in 15 and 11 is left over. Is 11 the remainder of 15 divided by 4?

P. 110. *Find the quotient and remainder of 78645 divided by 119* strictly *in accordance with the long division algorithm.*

$$
\begin{array}{rccccccc}
 & & 0 & 0 & 6 & 6 & 0 \\
\cline{2-7}
119 &) & 7 & 8 & 6 & 4 & 5 \\
 & & 0 & & & & \\
\cline{3-4}
 & & 7 & 8 & & & \\
 & & 0 & 0 & & & \\
\cline{3-5}
 & & 7 & 8 & 6 & & \\
 & & 7 & 1 & 4 & & \\
\cline{3-6}
 & & & 7 & 2 & 4 & \\
 & & & 7 & 1 & 4 & \\
\cline{4-7}
 & & & & 1 & 0 & 5 \\
 & & & & 0 & 0 & 0 \\
\cline{5-7}
 & & & & 1 & 0 & 5
\end{array}
$$

P. 115. *Write out the long division of 235 by 4, and interpret it in terms of money.*

$$
\begin{array}{rcccc}
 & & 0 & 5 & 8 \\
\cline{2-5}
4 &) & 2 & 3 & 5 \\
 & & 0 & & \\
\cline{3-5}
 & & 2 & 3 & \\
 & & 2 & 0 & \\
\cline{3-5}
 & & & 3 & 5 \\
 & & & 3 & 2 \\
\cline{3-5}
 & & & & 3
\end{array}
$$

Suppose we have 2 hundred-dollar bills, 3 ten-dollar bills, and 5 one-dollar bills, and we want to divide these bills equally into 4 stacks. Since there are only 2 one-hundred bills, we can't do anything with those and so exchange them for 20 ten-dollar bills. Coupled with the original 3 ten-dollar bills, we now have 23 ten-dollar bills. Putting **5** in each stack, we have 3 left over and we exchange them for 30 one-dollar bills. But we had 5 one-dollar bills to begin with, so we now have 35 one-dollar bills. After putting **8** one-dollar bills in each stack, we have 3 one-dollar bills left over. So in each stack, we have 5 ten-dollar bills and 8 one-dollar bills; that is the quotient: 58.

The remaining 3 one-dollar bills indicate that the remainder is 3.

P. 119. *Perform the long division algorithm for the division-with-remainder of 1162 by 19, and then give the explanation of why it is correct in the way we explained why* (7.6) *is correct.*

$$
\begin{array}{rl}
 & \;\; 0\;\;0\;\;6\;\;1 \\ \cline{2-2}
19 &)\;\; 1\;\;1\;\;6\;\;2 \\
 & \;\; 0 \\ \cline{2-2}
 & \;\; 1\;\;1 \\
 & \;\; 0\;\;0 \\ \cline{2-2}
 & \;\; 1\;\;1\;\;6 \\
 & \;\; 1\;\;1\;\;4 \\ \cline{2-2}
 & \;\; \;\;2\;\;2 \\
 & \;\; \;\;1\;\;9 \\ \cline{2-2}
 & \;\; \;\;3
\end{array}
$$

The explanation goes as follows:

$$
\begin{aligned}
1 &= (0 \times 19) + 1 \\
11 &= (0 \times 19) + 11 \\
116 &= (6 \times 19) + 2 \qquad (1) \\
22 &= (1 \times 19) + 3 \qquad (2)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
1162 &= (116 \times 10) + 2 \\
&= ([(6 \times 19) + 2] \times 10) + 2 && \text{(equation (1))} \\
&= (60 \times 19) + 20 + 2 \\
&= (60 \times 19) + 22 \\
&= (60 \times 19) + [(1 \times 19) + 3] && \text{(equation (2))} \\
&= (61 \times 19) + 3 && \text{(dist. law)}
\end{aligned}
$$

So we have $1162 = (61 \times 19) + 3$, i.e., the division-with-remainder of 1162 by 19 has quotient 61 and remainder 3.

P. 130. *Each of the following addition and subtraction statements is illegal, but can you make sense of them anyway?* $9 - 2 = 1$*,* $8 + 16 = 2$*, and* $206 + 82 = 2$*.* (Hint*, in no particular order: think of counting in terms of dozens, measuring area, and counting the number of days.*)

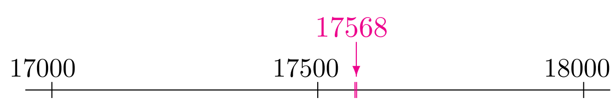
9 days − 2 days = 1 week
8 eggs + 16 eggs = 2 dozen eggs
206 sq. in + 82 sq. in = 2 sq. ft.

P. 141. *Explain to the person sitting nearest you why this algorithm is correct.*

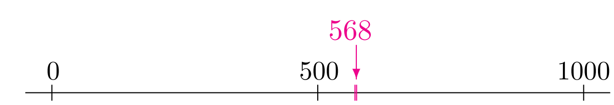
The algorithm in question is about rounding a whole number n to the nearest thousand, as follows:

> Write n as $N + \overline{n}$, where $\overline{n}$ is the three-digit number equal to the last three digits (to the right) of n. Then rounding n to the nearest thousand yields the number which is equal to N if the left digit of $\overline{n}$ is < 5, and equal to $N + 1000$ if the left digit of $\overline{n}$ is ≥ 5.

To explain this, it is best to use specific numbers; first we try 17568. To round it to the nearest thousand, first look at the multiples of 1000 around 17568, and they are 17000 and 18000. Clearly 17568 is closer to 18000 than 17000 because the midpoint between 17000 and 18000 is 17500 and 17568 is between 17500 and 18000.



So the rounded number of 17568 is 18000. Now we can express this procedure differently. If we write 17568 as $17000 + 568$, what we are doing is to ignore 17000 and look only at 568 and decide whether we want to change it to either 1000 or 0, these being the two multiples of 1000 closest to 568.



Is it equal to or greater than the midpoint 500 or is it < 500? If the former, then we change 568 to 1000 in the sum $17000 + 568$, but if the latter, then we change 568 to 0 in the sum $17000 + 568$. But this is exactly what the above algorithm says with $N = 17000$ and $\overline{n} = 568$.

There is an extra bit of subtlety if the number is 1,999,568. Then the neighboring multiples of 1000 are now 1,999,000 and 2,000,000, and the rounded number will be 2,000,000. In other words, one does not round up in this case by simply changing the four digits on the right (as in $17568 \to 1\mathit{8000}$) but that, in this case, by going up to the next 1000 after 1,999,000, *all* the digits get changed.

P. 142. *Round 20,245,386 to the nearest ten, nearest hundred, nearest thousand, nearest ten-thousand, and nearest million, respectively. Do the same to 59,399,248.*

In succession: 20,245,390, 20,245,400, 20,245,000, 20,250,000, and 20,000,000. For 59,399,248, the answers are, in succession: 59,399,250, 59,399,200, 59,399,000, 59,400,000, 59,000,000.

P. 142. *Show that this procedure is correct.*

The procedure in question is the following.

> To round a decimal m to the nearest hundredth, write it as $m = M + \overline{m}$, where M is the decimal whose digits (starting from the left) agree with those of m up to and including the second decimal digit, and are 0 elsewhere. Then the rounded decimal is M if the third decimal digit of $\overline{m}$ is < 5, and is equal to $M + 0.01$ if the third decimal digit of $\overline{m}$ is ≥ 5.

Again it is easier to explain by looking at a specific decimal and discuss its rounding. So consider 1.7568. The hundredths digit is 5, of course, and the nearby (whole number) multiples of 0.01 are 1.75 and 1.76. The best way to see this is to go back to the definition of a decimal as a fraction. We are considering 1.7568, and multiples of 0.01, so we begin by putting them on equal footing using FFFP. Consider then 1.7568 and multiples of 0.0100, i.e.,

$$\frac{17568}{10000} \quad \text{and multiples of} \quad \frac{100}{10000}.$$

Then clearly the multiples next to 17568/10000 are

$$\frac{17500}{10000} \quad \text{and} \quad \frac{17600}{10000},$$

and a look at the numerators (and using what we know from the Activity on page 141, for example) will show that 17568/10000 rounds up to 17600/1000), i.e., 1.7568 rounds to 1.76.

Let us express this reasoning directly in terms of decimal notation. Write $1.7568 = 1.75 + 0.0068$. Then because $6 \geq 5$, the rounded decimal is $1.75 + 0.01 = 1.76$. If the decimal had been 1.7529, then we would have $1.75 + 0.0029$ and the rounded decimal would be $1.75 + 0 = 1.75$. This is of course the above algorithm with $M = 1.75$ and $\overline{m} = 0.0068$.

P. 143. *Round 1.70995 to the nearest* 10^{-k}*, for k = 1, 2, 3, 4.*

In succession: 1.7, 1.71, 1.710, and 1.7100.

P. 147. *The area of the U.S.A. is 9,629,091 square kilometers. Round it to the nearest thousand, ten-thousand, hundred-thousand, and million. In everyday conversation, which of these rounded figures do you think would be most useful? What is the relative error in that case?*

The desired roundings, in succession, are: 9,629,000, 9,630,000, 9,600,000, 10,000,000.

In everyday life, one would say *ten million sq. km.* The relative error is

$$\frac{10{,}000{,}000 - 9{,}629{,}091}{9{,}629{,}091} = 0.038\ldots = 4\% \text{ approx.},$$

which shows why *ten million sq. km.* is a good figure to use.

P. 157. (a) *Write out in base 2, 5, and 7 the first 20 numbers starting with 1* (*rather than 0*)*.* (b) *What is the base 3 expansion of 79?*

(a) We list the first 20 numbers of base 2, base 5, and base 7 in succession:

	1	10	11	100	101	110	111
1000	1001	1010	1011	1100	1101	1110	1111
10000	10001	10010	10011	10100			

	1	2	3	4
10	11	12	13	14
20	21	22	23	24
30	31	32	33	34
40				

	1	2	3	4	5	6
10	11	12	13	14	15	16
20	21	22	23	24	25	26

(b) Since $3^4 = 81 > 79$, there will be no need for using 3^4 to represent 79. So we do it step-by-step:

$$79 \;=\; 2\cdot(3^3)+25 \;=\; 2\cdot(3^3)+2\cdot(3^2)+7 \;=\; 2\cdot(3^3)+2\cdot(3^2)+2(3^1)+1$$

Thus $79 = (2221)_3$.

P. 157. (a) *What are the following numbers in our decimal numeral system?* $(6507)_8$, $(101110101)_2$, $(\underline{58}\ \underline{20}\,48\,\underline{16})_{60}$. ($b$) *What is the base 5 expansion of 98?*

(a) $(6507)_8 = 6\cdot(8^3)+5\cdot(8^2)+7 = 3399$.

$(101110101)_2 = 2^8+2^6+2^5+2^4+2^2+1 = 373$.

$(\underline{58}\ \underline{20}\,48\,\underline{16})_{60} = 58\cdot(60^4)+20\cdot(60^3)+4\cdot(60^2)+8\cdot(60^1)+16 = 756{,}014{,}896$.

(b) $98 = 3\cdot(5^2)+23 = 3\cdot(5^2)+4\cdot(5^1)+3$. So $98 = (343)_5$.

P. 159. (a) *Find the base 7 expansion of 280.* (b) *Find the binary expansion of 67.*

(a) $280 = 5\cdot(7^2)+35 = 5\cdot(7^2)+5\cdot(7^1) = (550)_7$.

(b) $67 = 1\cdot(2^6)+3 = 1\cdot(2^6)+1\cdot(2^1)+1 = (1000011)_2$.

P. 160. *Derive the base 3 representation of 279.*

$279 = 3 \cdot (3^4) + 36 = 3 \cdot (3^4) + 4 \cdot (3^2) = (30400)_3$.

P. 162. *Verify that the preceding table is correct.*

The table in question is the addition in base 7 of $a + b$ for $a, b = 0, 1, 2, \ldots, 6$. This is a routine computation. For example, $5 + 6 = 11 = 1 \cdot (7^1) + 4 = (14)_7$, and $3 + 6 = 9 = 1 \cdot (7^1) + 2 = (12)_7$.

P. 162. $(66)_7 + (1)_7 = ?$ $(2666)_7 + (1)_7 = ?$ $(266660)_7 + (10)_7 = ?$

$(66)_7 + (1)_7 = (100)_7$, $(2666)_7 + (1)_7 = (3000)_7$ and $(266660)_7 + (10)_7 = (300000)_7$.

P. 162. *Give an explanation of the preceding subtraction.*

The subtraction in question is the following in base 7:

$$\begin{array}{cccc} & {\scriptstyle 4} & {\scriptstyle 6} & \\ & \cancel{5} & \cancel{0} & 2 \\ - & 2 & 1 & 3 \\ \hline & 2 & 5 & 6 \end{array}$$

This is because:

$$\begin{aligned} 502 - 213 &= (5 \cdot 7^2 + 0 \cdot 7^1 + 2) - (2 \cdot 7^2 + 1 \cdot 7^1 + 3) \\ &= (4 \cdot 7^2 + 7 \cdot 7^1 + 2) - (2 \cdot 7^2 + 1 \cdot 7^1 + 3) \\ &= (4 \cdot 7^2 + 6 \cdot 7^1 + (7 + 2)) - (2 \cdot 7^2 + 1 \cdot 7^1 + 3) \\ &= (4 \cdot 7^2 - 2 \cdot 7^2) + (6 \cdot 7^1 - 1 \cdot 7^1) + ((7 + 2) - 3) \quad \text{((5.5) on page 75)} \\ &= 2 \cdot 7^2 + 5 \cdot 7^1 + 6 \\ &= 256 \end{aligned}$$

P. 163. *Check that each entry in the multiplication table is correct.*

Again, this is routine. For example, $6 \times 4 = 24 = 3 \cdot (7^1) + 3 = 33$ and $6 \times 6 = 36 = 5 \cdot 7^1 + 1 = 51$.

P. 163. $(345)_7 \times (10)_7 = ?$ $(345)_7 \times (100)_7 = ?$ $(345)_7 \times (10000)_7 = ?$

$(345)_7 \times (10)_7 = (3450)_7$, $(345)_7 \times (100)_7 = (34500)_7$, and $(345)_7 \times (10000)_7 = (3450000)_7$. For example:

$$
\begin{array}{rccccc}
 & & & 3 & 4 & 5 \\
\times & & & 1 & 0 & 0 \\
\hline
 & & & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & \\
+ & 3 & 4 & 5 & & \\
\hline
 & 3 & 4 & 5 & 0 & 0
\end{array}
$$

P. 164. *Use the multiplication algorithm for base 7 to directly compute* $(540)_7 \times (26)_7$. *Check the result by converting both numbers in base 7 to decimal numbers, multiply them, and then convert the result back to base 7.*

We will have to use the multiplication table on page 163:

$$
\begin{array}{rccccc}
 & & & 5 & 4 & 0 \\
\times & & & & 2 & 6 \\
\hline
 & & 4 & 5 & 3 & 0 \\
+ & 1 & 4 & 1 & 0 & \\
\hline
 & 2 & 1 & 6 & 3 & 0
\end{array}
$$

Now $(540)_7 = 5(7^2) + 4(7) = 273$ and $(26)_7 = 2(7) + 6 = 20$, so $(540)_7 \times (26)_7 = 273 \times 20 = 5460$. On the other hand, $5460 = 2(7^4) + 1(7^3) + 6(7^2) + 3(7) = (21630)_7$, so we have independently checked that

$$(540)_7 \times (26)_7 \;=\; (21630)_7$$

P. 165. *Verify that these are indeed the first 20 numbers.*

See the first Activity on page 157. Note the difference: on page 157, we had to write down the first 20 numbers beginning with 1, but here we begin with 0.

P. 166. *Do the addition* $(11)_2 + (11)_2 + (11)_2 + (11)_2$ *vertically, as in ordinary addition, and use the addition algorithm. What do you notice about carrying?*

The addition algorithm says:

$$\begin{array}{rcccc} & & & 1 & 1 \\ & & & 1 & 1 \\ & & & 1 & 1 \\ & & & 1 & 1 \\ + & {\scriptstyle 1} & {\scriptstyle 1} & & \\ \hline & 1 & 1 & 0 & 0 \end{array}$$

So each carry is *two* columns to the left.

P. 166. *Can you think of an analogue of the equation* $(11)_2+(11)_2+(11)_2+(11)_2 = (1100)_2$ *in base 10?*

The addition $(11)_2 + (11)_2 + (11)_2 + (11)_2$ is that of adding $(11)_2$ to itself $(100)_2$ times (4 times). Since multiplication is repeated addition, we see that

$$(11)_2 + (11)_2 + (11)_2 + (11)_2 \;=\; (100)_2 \times (11)_2.$$

If we recall the phenomenon of the second Activity on page 163, the product $(100)_2 \times (11)_2$ is equal to $(1100)_2$. From this point of view, we know what the analogue in base 10 should be: take any two-digit number, say 37, and add it to itself 100 times, then the result will be 3700, because this addition is equal to 100×37, which is equal to 3700.

P. 184. *Describe in words—without looking at the definition—what* $\frac{7}{5}$ *is, what* $\frac{12}{35}$ *is, what* $\frac{33}{17}$ *is, and what* $\frac{127}{63}$ *is.*

We will use the language of the *multiple* of a point as defined on page 99.

$\frac{7}{5}$: Divide the unit segment into 5 parts of equal length. Call the first division point to the right of 0 the fraction $\frac{1}{5}$. Then the 7th multiple of $\frac{1}{5}$ is the fraction $\frac{7}{5}$.

$\frac{12}{35}$: Divide the unit segment into 35 parts of equal length. Call the first division point to the right of 0 the fraction $\frac{1}{35}$. Then the 12th multiple of $\frac{1}{35}$ is the fraction $\frac{12}{35}$.

$\frac{33}{17}$: Divide the unit segment into 17 parts of equal length. Call the first division point to the right of 0 the fraction $\frac{1}{17}$. Then the 33rd multiple of $\frac{1}{17}$ is the fraction $\frac{33}{17}$.

$\frac{127}{63}$: Divide the unit segment into 63 parts of equal length. Call the first division point to the right of 0 the fraction $\frac{1}{63}$. Then the 127th multiple of $\frac{1}{63}$ is the fraction $\frac{127}{63}$.

P. 186. *Using the preceding examples as models, describe in words where each of the fractions is on the number line and also draw a rough picture to show its location.* (*a*) $\frac{7}{9}$. (*b*) $\frac{6}{11}$. (*c*) $\frac{9}{4}$. (*d*) $\frac{17}{5}$. (*e*) $\frac{17}{3}$. (*f*) $\frac{k}{5}$*, where* k *is a whole number satisfying* $11 \leq k \leq 14$*.* (*g*) $\frac{k}{6}$*, where* k *is a whole number satisfying* $25 \leq k \leq 29$*.*

(*a*) $\frac{7}{9}$ is 7 copies of $\frac{1}{9}$. Since 9 copies of $\frac{1}{9}$ is 1 and since 5 copies of $\frac{1}{9}$ is already beyond the midpoint between 0 and 1, $\frac{7}{9}$ is to the left of 1, but to the right of the midpoint between 0 and 1.

(*b*) $\frac{6}{11}$ is 6 copies of $\frac{1}{11}$. Now 6 copies of $\frac{1}{12}$ would be the midpoint between 0 and 1, and since it is intuitively clear that $\frac{1}{11}$ is to the right of $\frac{1}{12}$, we see that $\frac{6}{11}$ is near the midpoint between 0 and 1 but also to the right of it.

(*c*) $\frac{9}{4}$ is 9 copies of $\frac{1}{4}$. Now 8 copies of $\frac{1}{4}$ is 2, so $\frac{9}{4}$ is to the right of 2, and is a quarter of the way to 3.

(*d*) and (*e*). Briefly, $\frac{17}{5}$ is beyond 3 and almost halfway between 3 and 4; $\frac{17}{3}$ is to the left of 6 and to the right of the midpoint between 5 and 6.

(*f*). $\frac{11}{5}$ is a little to the right of 2 ($= \frac{10}{5}$), while $\frac{14}{5}$ is a little to the left of 3 ($= \frac{15}{5}$). Therefore, $\frac{k}{5}$, where k is a whole number satisfying $11 \leq k \leq 14$, consists of 4 points trapped in a segment lying strictly inside the segment between 2 and 3.

(g). $\frac{25}{6}$ is a little to the right of 4 $(= \frac{24}{6})$ while $\frac{29}{6}$ is a little to the left of 5 $(= \frac{30}{6})$. Thus, $\frac{k}{6}$, where k is a whole number satisfying $25 \leq k \leq 29$, consists of 5 points trapped in an interval lying strictly inside the segment between 4 and 5.

P. 188. *(a) Express* $\frac{163079}{10^8}$ *and* $\frac{230000}{10^2}$ *in decimal notation. (b) Express* 10000.2001 *and* 0.000000071008000 *as fractions.*

(a) 0.00163079 and 2300.00.

(b) $\dfrac{100002001}{10^4}$ and $\dfrac{71008000}{10^{15}}$.

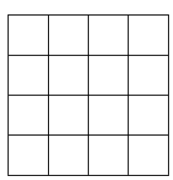
P. 190. *Suppose the unit 1 represents (the value of) a dollar. What numbers would represent a penny, a nickel, a dime, and a quarter? If on the other hand, the unit 1 represents (the value of) a quarter, what numbers would represent a penny, a nickel, a dime, and a dollar? Still with a quarter as your unit, how many dollars would* $\frac{13}{5}$ *represent?*

If 1 = dollar: Penny $= \frac{1}{100}$. Nickel $= \frac{5}{100}$. Dime $= \frac{10}{100}$. Quarter $= \frac{25}{100}$.

If 1 = quarter: Penny $= \frac{1}{25}$. Nickel $= \frac{5}{25}$. Dime $= \frac{10}{25}$. Dollar $= \frac{100}{25}$.

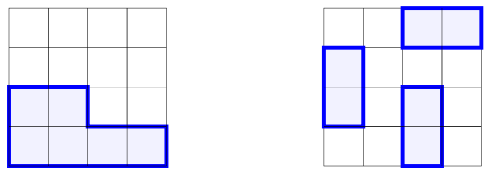
If 1 = quarter, then $\frac{1}{5}$ is a nickel (quarter = 5 nickels). Therefore $\frac{13}{5}$, being 13 copies of a nickel, is 65 cents.

P. 193. *In the following picture, suppose the unit 1 is the area of the whole square.*

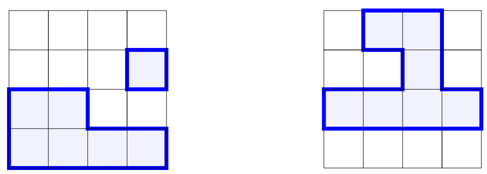


Shade two different regions so that each represents $\frac{3}{8}$*; do the same for* $\frac{7}{16}$*. Can you tell by visual inspection which of* $\frac{3}{8}$ *and* $\frac{7}{16}$ *has more area?*

Representations of $\frac{3}{8}$:



Representations of $\frac{7}{16}$:



$\frac{7}{16}$ has more area because it encloses 7 rather than 6 small squares.

P. 196. *Give the approximate locations of the following fractions on the number line.* (a) $\frac{29}{100}$. (b) $\frac{255}{101}$. (c) $\frac{1234}{2467}$. (d) $\frac{49}{5}$. (e) $\frac{73}{12}$.

(a) $\dfrac{29}{100} = \dfrac{25+4}{100}$. Since $\frac{25}{100}$ is $\frac{1}{4}$, we see that $\frac{29}{100}$ is a little to the right of $\frac{1}{4}$.

(b) $\dfrac{255}{101} = \dfrac{(2 \times 101) + 53}{101}$. Thus $\frac{255}{101}$ is 53 copies of $\frac{1}{101}$ beyond 2. Now 53 copies of $\frac{1}{101}$ are clearly the same as 106 copies of $\frac{1}{202}$, and since 101 copies of $\frac{1}{202}$ is $\frac{1}{2}$, we see that 53 copies of $\frac{1}{101}$ is a little bit more than $\frac{1}{2}$. It follows that $\frac{255}{101}$ is a little beyond the midpoint between 2 and 3.

(c) Similarly, $\frac{1234}{2467}$ is a trifle beyond $\frac{1}{2}$.

(d) $\frac{49}{5}$ is 49 copies of $\frac{1}{5}$. Since 50 copies of $\frac{1}{5}$ is 10, $\frac{49}{5}$ is a little bit to the left of 10.

(e) Similarly $\frac{73}{12}$ is a little bit to the right of 6.

P. 205. *Prove* $\frac{1651}{762} = \frac{13}{6}$.

Observe that $1651 \div 13 = 127$, and that is reason enough to try $762 \div 127$, which is indeed equal to 6. So by the cancellation law (equation (13.1) on page 204),

$$\frac{1651}{762} \;=\; \frac{13 \times 127}{6 \times 127} \;=\; \frac{13}{6}$$

P. 213. *Rewrite* $\frac{15}{13}$ *and* $\frac{2}{17}$ *as two fractions with equal denominators. Do the same for* $\frac{8}{9}$ *and* $\frac{17}{11}$, *also for* $\frac{15}{4}$ *and* $\frac{13}{25}$.

$$\frac{15}{13} \;=\; \frac{17 \times 15}{17 \times 13} \;=\; \frac{255}{221} \quad \text{and} \quad \frac{2}{17} \;=\; \frac{13 \times 2}{13 \times 17} \;=\; \frac{26}{221}$$

Similarly,

$$\frac{8}{9} \;=\; \frac{11 \times 8}{11 \times 9} \;=\; \frac{88}{99} \quad \text{and} \quad \frac{17}{11} \;=\; \frac{9 \times 17}{9 \times 11} \;=\; \frac{153}{99},$$

$$\frac{15}{4} \;=\; \frac{25 \times 15}{25 \times 4} \;=\; \frac{375}{100} \quad \text{and} \quad \frac{13}{25} \;=\; \frac{4 \times 13}{4 \times 25} \;=\; \frac{52}{100}$$

P. 226. *Convert each of the following improper fractions to a mixed number, and vice versa:* $77\frac{5}{6}$, $4\frac{5}{7}$, $6\frac{1}{7}$, $13\frac{4}{5}$, $\frac{32}{7}$, $\frac{148}{9}$, $\frac{166}{15}$.

$$77\frac{5}{6} \;=\; 77 + \frac{5}{6} \;=\; \frac{6 \times 77}{6 \times 1} + \frac{5}{6} \;=\; \frac{(77 \times 6) + 5}{6} \;=\; \frac{467}{6}$$

Similarly,

$$4\frac{5}{7} \;=\; \frac{33}{7}, \quad 6\frac{1}{7} \;=\; \frac{43}{7}, \quad 13\frac{4}{5} \;=\; \frac{69}{5}\,.$$

Next,

$$\frac{32}{7} \;=\; \frac{(4 \times 7) + 4}{7} \;=\; 4\frac{4}{7}$$

Similarly,

$$\frac{148}{9} \;=\; 16\frac{4}{9}, \quad \frac{166}{15} \;=\; 11\frac{1}{15}\,.$$

P. 226. *Verify that the two answers above,* $32\frac{35}{221}$ *and* $\frac{7107}{221}$*, are indeed the same.*

$$32\frac{35}{221} \;=\; \frac{221 \times 32}{221 \times 1} + \frac{35}{221} \;=\; \frac{7107}{221}$$

P. 229. *Check that the equality* $\frac{1955}{159239} = \frac{115}{9367}$ *is correct.*

We can use the cross-multiplication algorithm to check that $1955 \times 9367 = 18312485 = 159239 \times 115$, or observe that

$$\frac{1955}{159239} \;=\; \frac{1955 \div 17}{159239 \div 17} \;=\; \frac{115}{9367}$$

P. 238. *I bought 4 lbs. of ice cream, and I have to distribute it equally to 25 children. How much ice cream does each child get? Explain.*

On the number line where the unit 1 stands for 1 lb. of ice cream, the problem can now be reformulated as follows: "On this number line, if I divide the segment $[0, 4]$ into 25 equal parts, how long is one part?" According to equation (15.1) on page 236, the answer is $\frac{4}{25}$. Since the unit is 1 lb. of ice cream, this means each child gets $\frac{4}{25}$ lbs. of ice cream.

P. 242. *Compare the following pairs of fractions (you may use a four-function calculator for the last two pairs):*

$$\frac{5}{6} \textit{ and } \frac{4}{5};\;\; \frac{6}{7} \textit{ and } \frac{8}{9};\;\; \frac{9}{51} \textit{ and } \frac{51}{289};\;\; \frac{49}{448} \textit{ and } \frac{56}{512}.$$

$\frac{5}{6}$ and $\frac{4}{5}$: since $4 \times 6 < 5 \times 5$, CMA (cross-multiplication algorithm) implies that $\frac{4}{5} < \frac{5}{6}$.

$\frac{49}{448}$ and $\frac{56}{512}$: since $49 \times 512 = 25088 = 448 \times 56$, CMA implies that $\frac{49}{448} = \frac{56}{512}$

Similarly, $\frac{6}{7} < \frac{8}{9}$ and $\frac{9}{51} = \frac{51}{289}$.

P. 244. *Use the following theorem to verify that no luck was involved when both methods in Example 5 led to the same answer.*

The theorem in question is Theorem 15.2.

In the first approach to the problem, we compared $\frac{18}{248}$ and $\frac{12}{172}$, and in the second approach, we compared $\frac{18}{230}$ and $\frac{12}{160}$. Now observe that

$$\frac{18}{248} = \frac{18}{230+18} \quad \text{while} \quad \frac{12}{272} = \frac{12}{160+12}$$

But by the equivalence of (a) and (b) in Theorem 15.2, we know that

$$\frac{12}{160+12} < \frac{18}{230+18} \quad \text{if and only if} \quad \frac{12}{160} < \frac{18}{230}\,.$$

This is why the two approaches yield the same answer.

P. 246. Without looking at the preceding definition, *explain to your neighbor the meaning of: (a) $\frac{4}{5}$ of $21\frac{1}{3}$. (b) $3\frac{3}{7}$ of $1\frac{1}{2}$. (c) $\frac{7}{3}$ of $\frac{3}{4}$. In the last item, locate the point which is $\frac{7}{3}$ of $\frac{3}{4}$.*

(a) $\frac{4}{5}$ of $21\frac{1}{3}$ means the length of 4 concatenated parts when $[0, 21\frac{1}{3}]$ is divided into 5 equal parts.

(b) Note that $3\frac{3}{7} = \frac{24}{7}$. Thus $3\frac{3}{7}$ of $1\frac{1}{2}$ means the length of 24 concatenated parts when $[0, 1\frac{1}{2}]$ is divided into 7 equal parts. We can do better: we know that 21 concatenated parts when $[0, 1\frac{1}{2}]$ is divided into 7 equal parts are equal to 3 copies of $1\frac{1}{2}$, and of course $24 = 21 + 3$. So $3\frac{3}{7}$ of $1\frac{1}{2}$ is the total length of 3 copies of $1\frac{1}{2}$ plus the length of 3 concatenated parts when $[0, 1\frac{1}{2}]$ is divided into 7 equal parts. This directly relates the answer to the mixed number $3\frac{3}{7}$.

(c) $\frac{7}{3}$ of $\frac{3}{4}$ means the length of 7 concatenated parts when $\frac{3}{4}$ is divided into 3 equal parts. However, $\frac{3}{4}$ is 3 copies of $\frac{1}{4}$, so if $\frac{3}{4}$ is divided into 3 equal parts, then one part is just $\frac{1}{4}$. Hence, we see that $\frac{7}{3}$ of $\frac{3}{4}$ means the length of 7 copies of $\frac{1}{4}$, i.e., $\frac{7}{4}$.

P. 247. *What is $\frac{4}{5}$ of $\frac{45}{7}$?*

$\frac{45}{7} = \frac{5\times 45}{5\times 7} = \frac{5\times 45}{35}$. Therefore $\frac{4}{5}$ of $\frac{45}{7}$ is 4 copies of $\frac{45}{35}$, which is equal to $\frac{4\times 45}{35} = \frac{180}{35}$.

P. 255. (*a*) *Compute $4\frac{1}{10} - 2\frac{3}{4}$ both ways.* (*b*) *For a simple subtraction such as $2\frac{1}{6} - 1\frac{1}{3}$, it may be a good idea to do it via pictures as well as direct computations. Do both.*

(*a*) First,

$$\begin{aligned} 4\frac{1}{10} - 2\frac{3}{4} &= \left(3 + 1\frac{1}{10}\right) - \left(2 + \frac{3}{4}\right) \\ &= (3-2) + \left(1\frac{1}{10} - \frac{3}{4}\right) \\ &= 1 + \left(\frac{11}{10} - \frac{3}{4}\right) \\ &= 1 + \frac{14}{40} \;=\; 1\frac{7}{20} \end{aligned}$$

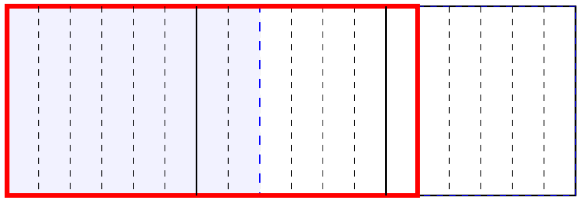
Second method:

$$4\frac{1}{10} - 2\frac{3}{4} \;=\; \frac{41}{10} - \frac{11}{4} \;=\; \frac{164-110}{40} \;=\; \frac{54}{40} \;=\; \frac{27}{20}$$

and the two answers are easily seen to be equal.

(*b*) $2\frac{1}{6} - 1\frac{1}{3} \;=\; \frac{13}{6} - \frac{4}{3} \;=\; \frac{5}{6}$.

We can also do it by pictures. Let 1 be represented by the area of a unit square, and 3 such squares are shown below side-by-side. The red rectangle in the picture below then represents $2\frac{1}{6}$, and the shaded area in pale blue represents $1\frac{1}{3} = 1\frac{2}{6}$. The subtraction $2\frac{1}{6} - 1\frac{1}{3}$ is therefore the area outside the pale blue region inside the red rectangle. One can count that there are 5 thin rectangular strips, each representing $\frac{1}{6}$, and so the answer is $\frac{5}{6}$.



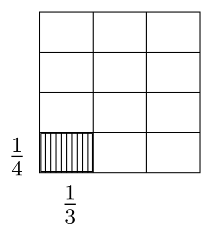
P. 256. $0.402 - 0.0725 = $ *?* $3.14 - 1\frac{5}{8} = $ *?*

Naturally, the decimal subtraction can be done using the subtraction algorithm, but we will do it using the definition of a decimal as a fraction:

$$0.402 - 0.0725 \;=\; 0.4020 - 0.0725 \;=\; \frac{4020}{10^4} - \frac{725}{10^4} \;=\; \frac{3295}{10^4} \;=\; 0.3295$$

$$3.14 - 1\frac{5}{8} \;=\; \frac{314}{100} - \frac{13}{8} \;=\; \frac{628}{200} - \frac{325}{200} \;=\; \frac{303}{200}$$

P. 266. *Compute* $\frac{1}{4} \times \frac{1}{3}$.

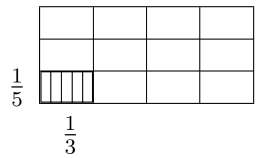


As usual, the horizontal lines divide the unit square into equal fourths (in terms of area) and the vertical lines divide it into equal thirds. The shaded rectangle is therefore one of 4×3 congruent rectangles that divide up the unit square (which has area equal to 1). Hence the area of the shaded rectangle is $\frac{1}{4\times 3}$. Since the sides of the shaded rectangle are of lengths $\frac{1}{4}$ and $\frac{1}{3}$, by the definition of fraction multiplication on page 263,

$$\frac{1}{4} \times \frac{1}{3} \;=\; \frac{1}{4\times 3}$$

P. 267. *Compute* $\frac{3}{5} \times \frac{4}{3}$.

We stack 3 rows and 4 columns of rectangles of dimensions $\frac{1}{5}$ and $\frac{1}{3}$:



The big rectangle has dimensions $\frac{3}{5}$ and $\frac{4}{3}$, and its area is therefore $\frac{3}{5} \times \frac{4}{3}$, by definition. But its area is the sum of the areas of the 3×4 congruent rectangles shown in the picture, each having area equal to $\frac{1}{5 \times 3}$ (see Activity on page 266). Therefore,

$$\frac{3}{5} \times \frac{4}{3} \;=\; \underbrace{\frac{1}{5 \times 3} + \cdots + \frac{1}{5 \times 3}}_{3 \times 4} \;=\; \frac{3 \times 4}{5 \times 3}$$

P. 290. (*a*) *Rewrite each of the following as a division:*

$$\frac{14}{27} \;=\; \frac{2}{3} \times \frac{7}{9}, \quad 4\frac{8}{15} \;= 5\;\frac{2}{3} \times \frac{4}{5}, \quad \frac{a}{b} \times \frac{c}{d} \;=\; \frac{ac}{bd},$$

where a, ..., d are nonzero whole numbers.

(*b*) *Rewrite each of the following as a multiplication:*

$$\frac{\frac{45}{7}}{3} \;=\; \frac{15}{7}, \quad \frac{3\frac{5}{8}}{\frac{6}{5}} \;=\; \frac{x}{y}, \quad \frac{87}{\frac{x}{y}} \;=\; \frac{3}{22},$$

where x, y are nonzero whole numbers.

(*c*) *What is the fraction A in each of the following?*

$$\frac{A}{\frac{6}{7}} \;=\; \frac{5}{14}, \quad \frac{1\frac{7}{8}}{A} \;=\; \frac{5}{2}, \quad \frac{A}{2\frac{4}{5}} \;=\; 2\frac{7}{9}.$$

(*a*) In succession, we have:

$$\frac{\frac{14}{27}}{\frac{7}{9}} \;=\; \frac{2}{3}, \qquad \frac{4\,\frac{8}{15}}{\frac{4}{5}} \;=\; 5\,\frac{2}{3}, \qquad \frac{\frac{ac}{bd}}{\frac{c}{d}} \;=\; \frac{a}{b}.$$

(*b*) $\dfrac{45}{7} = \dfrac{15}{7} \times 3$, $3\dfrac{5}{8} = \dfrac{x}{y} \times \dfrac{6}{5}$, and $87 = \frac{3}{22} \times \frac{x}{y}$.

(c) Given

$$\frac{A}{\frac{6}{7}} \;=\; \frac{5}{14}\,,$$

we get

$$A \;=\; \frac{5}{14} \times \frac{6}{7} \;=\; \frac{30}{98}\,.$$

Given

$$\frac{1\frac{7}{8}}{A} \;=\; \frac{5}{2}\,,$$

we have

$$1\frac{7}{8} \;=\; \frac{5}{2} \times A\,,$$

which is equivalent to

$$\frac{15}{8} \;=\; \frac{5}{2} \times A\,.$$

By inspection, $A = \frac{3}{4}$. Finally, if

$$\frac{A}{2\frac{4}{5}} \;=\; 2\frac{7}{9}\,,$$

then

$$A \;=\; 2\frac{7}{9} \times 2\frac{4}{5} \;=\; \frac{25}{9} \times \frac{14}{5} \;=\; \frac{350}{45}\,.$$

P. 292. *A rod* $15\frac{5}{7}$ *meters long is cut into short pieces which are* $2\frac{1}{8}$ *meters long. How many short pieces are there?*

Let there be A short pieces in the rod, then

$$15\frac{5}{7} \;=\; A \times 2\,\frac{1}{8}\,.$$

Therefore, by definition of division,

$$A \;=\; \frac{15\frac{5}{7}}{2\frac{1}{8}} \;=\; \frac{\frac{110}{7}}{\frac{17}{8}} \;=\; \frac{110}{7} \times \frac{8}{17} \;=\; \frac{880}{119} \;=\; 7\,\frac{47}{119}\,.$$

Thus, the rod can be cut into 7 and $\frac{47}{119}$ of the short pieces (see (b) of the Activity on p. 246). In other words, there are 7 short pieces with one really short piece left over which is $\frac{47}{119}$ of the length of the short piece.

P. 296. *A train running at constant speed takes* $2\frac{1}{3}$ *hours to go 125 miles. At the same speed, how long would it take to go 180 miles? (You do not need to "set up a proportion" to do this problem. If you insist on setting up a proportion, then you will be setting yourself up for the impossible task of explaining what it means.)*

Suppose it takes t hours for the train to go 180 miles. Then by the definition of constant speed,

$$\frac{180}{t} = \frac{125}{2\frac{1}{3}}$$

In other words,

$$\frac{180}{t} = \frac{125}{\frac{7}{3}} = \frac{375}{7}.$$

Therefore, by the definition of division,

$$180 = \frac{375}{7} \times t = t \times \frac{375}{7}.$$

Again by the definition of division,

$$t = \frac{180}{\frac{375}{7}} = 180 \times \frac{7}{375} = \frac{1260}{375} = \frac{84}{25} = 3\frac{9}{25}.$$

So it takes the train 3 and $\frac{9}{25}$ hours to go 180 miles.

Caution. From

$$\frac{180}{t} = \frac{375}{7},$$

it is tempting to say that, *by the cross-multiplication algorithm*, we have $180 \times 7 = 375 \times t$. However, the cross-multiplication algorithm is, up to this point, only valid for fractions, so unless we know t is a whole number, we cannot apply it to this situation *yet*. In Chapter 19, item (c) on page 310, we will prove that this conclusion is legitimate.

P. 298. (*a*) *Why could we* not *give this explanation back in Chapter 13?* (*b*) *Discuss what would happen to the preceding conversion of* $\frac{3}{8} = 0.375$ *if the number* k

(the power of 10) had been chosen to be 2, or 4, or 5.

(a) The explanation on this page depends on the product formula (or, what is the same, the *cancellation phenomenon* on page 271), and back in Chapter 13, this fact was not available.

(b) Thus if k is a whole number, we want to know if the following product depends on k for $k = 2$, 3, 4, or 5:

$$\left(\frac{3 \times 10^k}{8}\right) \times \frac{1}{10^k}$$

The answer is that it doesn't. For example, if $k = 5$, then the cancellation phenomenon on page 271 implies that it is equal to the same product with $k = 2$:

$$\left(\frac{3 \times 10^5}{8}\right) \times \frac{1}{10^5} \;=\; \left(\frac{3 \times 10^2 \times 10^3}{8}\right) \times \frac{1}{10^2 \times 10^3} \;=\; \left(\frac{3 \times 10^2}{8}\right) \times \frac{1}{10^2}$$

P. 321. *Verify that the preceding statement about* $\frac{3}{28} = \frac{1\frac{17}{28}}{15}$ *is correct.*

$$\frac{1\frac{17}{28}}{15} \;=\; \frac{\frac{45}{28}}{15} \;=\; \frac{45}{28 \times 15} \;=\; \frac{3}{28}\,.$$

P. 324. *(a) Express* $\frac{1}{85}$ *as a percent. (b) What is* 28% *of 45? (c) 17 is 35 percent of what number? (d) 48 is what percent of 35?*

(a) We want a fraction N so that

$$\frac{1}{85} \;=\; \frac{N}{100}$$

Thus $85N = 100$ (by (c) of page 310), and $N = \frac{100}{85}$. So $\frac{1}{85} = \frac{100}{85}\%$.

(b) By (17.7) on page 273, 28% of 45 is equal to

$$\frac{28}{100} \times 45 \;=\; 12.6.$$

(c) Let n be the number so that 35 percent of n is equal to 17. Again by (17.7) of page 273, $35\% \times n = 17$, so that

$$n \;=\; \frac{17}{35\%} \;=\; \frac{1700}{35} \;=\; 48\frac{20}{35}.$$

(d) Let us say 48 is $N\%$ of 35. Then $48 = N\% \times 35$, and therefore $N\% = \frac{48}{35}$. So

$$N = \frac{4800}{35} = 137\frac{5}{35}.$$

So 48 is $137\frac{5}{35}\%$ of 35.

P. 324. *A bed costs \$200. Because of unprecedented demand, the price went up 15% overnight. Then came recession and the price tumbled down by the same 15% from its higher price. Does it get back to \$200?*

When the price of the 200-dollar bed went up 15%, it cost

$$200 + \left(\frac{15}{100} \times 200\right) = 230 \text{ dollars.}$$

If at this price, we deduct 15%, then the new price will be

$$230 - \left(\frac{15}{100} \times 230\right) = 230 - 34.5 = 195.5 \text{ dollars.}$$

Therefore it does not come back down to \$200.

P. 327. *Verify that the preceding computation resulting in* $1\frac{19}{81}\%$ *is correct.*

$$\frac{\frac{5}{6}\%}{67\frac{1}{2}\%} = \frac{\frac{5}{6}}{100} \times \frac{100}{67\frac{1}{2}} = \frac{\frac{5}{6}}{67\frac{1}{2}} = \frac{\frac{5}{6}}{\frac{135}{2}} = \frac{5 \times 2}{6 \times 135} = \frac{1}{81}$$

To express $\frac{1}{81}$ as a percent, let N be the fraction so that $\frac{1}{81} = N\%$, i.e.,

$$\frac{1}{81} = \frac{N}{100}.$$

Then $81N = 100$ and therefore $N = \frac{100}{81} = 1\frac{19}{81}$.

P. 342. *If A, B, C, are in the ratio $3 : 8 : 5$ and $B = 20$, what are A and C?*

We are given that

$$\frac{A}{3} = \frac{B}{8} = \frac{C}{5}.$$

If $B = 20$, then

$$\frac{A}{3} = \frac{20}{8} \quad \text{implies} \quad A = \frac{20}{8} \times 3 = 7\frac{1}{2}.$$
$$\frac{C}{5} = \frac{20}{8} \quad \text{implies} \quad C = \frac{20}{8} \times 5 = 12\frac{1}{2}.$$

P. 342. *A school district has a teacher-student ratio of* $1:30$*. If the number of students is 450, how many more teachers does the district need to hire in order to improve the ratio to* $1:25$*?*

Let the number of students be S and let the original number of teachers be T. We are given that $\frac{T}{S} = \frac{1}{30}$ and $S = 450$. Therefore

$$T = \frac{1}{30} \times S = \frac{1}{30} \times 450 = 15.$$

If n more teachers are hired, then there will be $15 + n$ teachers. We want

$$\frac{15+n}{450} = \frac{1}{25}, \quad \text{so that} \quad 15 + n = \frac{1}{25} \times 450 = 18.$$

Hence $n = 3$, which means hiring 3 more teachers will improve the teacher-student ratio to $1:25$.

P. 346. *Paul rides a bicycle at constant speed. It takes him 25 minutes to go* $3\frac{1}{2}$ *miles. At the same speed, how long would it take him to go* $11\frac{1}{2}$ *miles?*

Let t be the time, in minutes, it takes Paul to go $11\frac{1}{2}$ miles. Also, assume that Paul's constant speed is v miles per minute. Then we know:

$$\frac{3\frac{1}{2}}{25} = v.$$

Because the speed is constant, we also know that

$$\frac{11\frac{1}{2}}{t} = v.$$

By inspecting these two equations, we get:

$$\frac{3\frac{1}{2}}{25} = \frac{11\frac{1}{2}}{t}.$$

By (c) of page 310,

$$3\frac{1}{2} \times t \;=\; 11\frac{1}{2} \times 25.$$

That is, $\frac{7}{2}t = \frac{575}{2}$, or $7t = 575$. Therefore, $t = 82\frac{1}{7}$. So it takes $82\frac{1}{7}$ minutes for Paul to go $11\frac{1}{2}$ miles.

P. 348. *Sunil mows lawns at a constant rate of* r *sq. ft. per minute (this means, if in a time interval of* t *minutes he mows* A *sq. ft. of lawn, then the quotient* $\frac{A}{t}$ *equals* r *for any* t*). He mows a certain lawn in 15 minutes. If he reduces his rate to* $85\%\ r$ *sq. ft. per minute, how long would it take him to mow the same lawn?*

Suppose Sunil mows lawns at a constant rate of r sq. ft. per minute. Suppose the area of the given lawn is A sq. ft. Then

$$\frac{A}{15} = r, \quad \text{so that} \quad A \;=\; r \times 15.$$

Suppose it takes t minutes for him to mow the same lawn at a rate of $85\%\ r$. Then

$$\frac{A}{t} \;=\; 85\%\ r, \quad \text{so that} \quad A \;=\; 85\%\ r\,t$$

where we write $r\,t$ in place of $r \times t$ as usual. Equating the two expressions of A gives

$$15\,r \;=\; 85\%\ r\,t,$$

which implies, upon multiplying both sides by $\frac{1}{r}$, that

$$15 \;=\; 85\%\ t.$$

Multiply both side by $\frac{100}{85}$ to get

$$t \;=\; \frac{100}{85} \times 15 \;=\; \frac{1500}{85} \;=\; 17\frac{11}{17}.$$

Hence, it would take $17\frac{11}{17}$ minutes for Sunil to mow the same lawn at the new rate.

P. 353. *Two water pipes drain into a tank. It takes the first pipe 10 hours to fill the tank, but it takes the second pipe 12 hours. If both water pipes drain into the tank at the same time, how long does it take them to fill the tank?* (*Remember, we are assuming the rate of water flow to be constant.*)

Suppose the volume of the tank is V in some volume unit (say, gallons). Then the constant rate of the first pipe is $\frac{V}{10}$ gal per hour and the constant rate of the second pipe is $\frac{V}{12}$ gal per hour. Let t be a positive number. After t hours, suppose the first pipe drains v gal into the tank, and we are going to find out what v is. Since the rate is constant ($= \frac{V}{10}$ gal per hour),

$$\frac{v}{t} = \frac{V}{10}, \quad \text{so that} \quad v = \frac{V}{10}\, t \text{ gal.}$$

Similarly, the second pipe drains $\frac{V}{12}t$ gal into the tank in the same time interval. Thus in a given time interval t, both pipes drain

$$\left(\frac{V}{10} + \frac{V}{12}\right) t = \left(\frac{1}{10} + \frac{1}{12}\right) V\, t \;\text{ gallons.}$$

Now assume it takes T hours to fill the tank with both pipes open and operating at the respective constant rates. Then in T hours, the two pipes together drain V gallons into the tank. But we have just computed that in T hours, both pipes drain $(\frac{1}{10} + \frac{1}{12})\, V\, T$ gallons into the tank. It follows that

$$\left(\frac{1}{10} + \frac{1}{12}\right) V\, T = V.$$

Multiplying both sides by $\frac{1}{V}$, we get

$$\left(\frac{1}{10} + \frac{1}{12}\right) T = 1,$$

which implies

$$\frac{11}{60}\, T = 1.$$

Hence,

$$T = \frac{60}{11} = 5\frac{5}{11} \;\text{ hours.}$$

Therefore, it takes $5\frac{5}{11}$ hours to fill the tank when both pipes are open.

P. 353. *Two shuttle trains go between cities A and B. It takes the first train 10 hours to make the trip, but it takes the second train 12 hours. Suppose now the first train is at city A and the second train is at city B and they take off at the same time on parallel tracks. How long will it be before they meet?*

Suppose the distance between the two cities is L in some length unit (say, miles). Then the constant speed of the first train is $\frac{L}{10}$ mph (miles per hour) and the constant speed of the second train is $\frac{L}{12}$ mph. We first find out how many miles the first train travels in t hours, where t is some positive number. Suppose it travels d miles in t hours, then the constancy of the speed implies that

$$\frac{d}{t} \;=\; \text{the constant speed} \;=\; \frac{L}{10} \;\text{ mph}$$

So $d = \frac{1}{10}L\,t$ mi. Similarly, the second train travels $\frac{1}{12}L\,t$ mi. in t hours.

Now assume that the first train leaves city A and the second train leaves city B at the same time and they meet in T hours. Then both trains together will have traveled L miles in T hours. On the other hand, the first train will have traveled a distance of $\frac{1}{10}L\,T$ mi. in T hours and the second train will have traveled a distance of $\frac{1}{12}L\,T$ mi. at the same time. So the two trains together will have traveled a total distance of $(\frac{1}{10}+\frac{1}{12})LT$ miles. It follows that

$$\left(\frac{1}{10}+\frac{1}{12}\right) L\,T \;=\; L.$$

Multiplying both sides by $\frac{1}{L}$, we get

$$\left(\frac{1}{10}+\frac{1}{12}\right) T \;=\; 1,$$

which implies

$$\frac{11}{60}\,T \;=\; 1.$$

Hence,

$$T \;=\; \frac{60}{11} \;=\; 5\frac{5}{11} \;\text{ hours.}$$

Therefore, it takes $5\frac{5}{11}$ hours before the trains meet.

Note the similarity between this solution and the solution of the preceding Activity.

P. 358. *Verify that the answer* $16\frac{4}{11}$ *is correct.*

We compute:

$$\frac{80}{\frac{40}{15}+\frac{40}{18}} = \frac{80}{40\left(\frac{1}{15}+\frac{1}{18}\right)} = \frac{2}{\left(\frac{1}{15}+\frac{1}{18}\right)} = \frac{2}{\frac{22}{180}} = \frac{360}{22} = 16\frac{4}{11}\,.$$

P. 358. *Do Problem 1 by assuming in addition that the distance between Paul's hometown and Lanterntown is 65 miles.*

Recall: Paul rode his motorbike to Lanterntown by maintaining a constant speed of 15 miles per hour. On the way back, he decided to increase his (still constant) speed to 18 miles per hour.

The time it took Paul to travel 65 miles at a constant speed of 15 mph is $\frac{65}{15}$ hours. The time it took Paul to travel 65 miles at a constant speed of 18 mph is $\frac{65}{18}$ hours. Therefore, it took $\left(\frac{65}{15}+\frac{65}{18}\right)$ hours for Paul to travel 130 miles. Therefore, his average speed for the entire trip is $\dfrac{130}{\left(\frac{65}{15}+\frac{65}{18}\right)}$ mph. That means his average speed is:

$$\frac{130}{65\left(\frac{1}{15}+\frac{1}{18}\right)} = \frac{130}{65\left(\frac{11}{90}\right)} = \frac{2}{\left(\frac{11}{90}\right)} = \frac{180}{11} = 16\tfrac{4}{11} \text{ mph.}$$

P. 383. *Check the last assertion, i.e., "Conversely, if a special vector has y as its endpoint, then it is equal to* $\overrightarrow{y}$*."*

Consider the special vector with the end point at y. Then, this vector has the starting point at 0 and the end point at y. But, by definition, the vector with the starting point at 0 and the ending point at y is $\overrightarrow{y}$. Therefore, the special vector with the end point at y is $\overrightarrow{y}$.

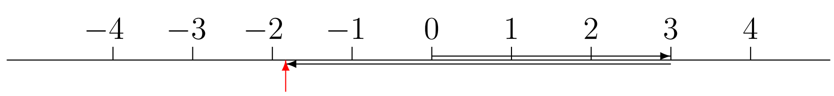
P. 384. *Check this!* $\overrightarrow{x} + \overrightarrow{x^*} = \overrightarrow{0}$.

First suppose x is a number to the right of 0. Then the special vector $\overrightarrow{x}$ is right-pointing with the length x. The special vector $\overrightarrow{x^*}$ is left-pointing with the length x so that, when we slide the vector $\overrightarrow{x^*}$ along the number line (to the right) so that its starting point is at x, its new ending point is at 0. Therefore, the resulting vector has the starting point at 0 and the ending point also at 0. This is the special vector $\overrightarrow{0}$. By the definition of the addition of special vectors, we have just showed why $\overrightarrow{x} + \overrightarrow{x^*} = \overrightarrow{0}$.

Now suppose x is a number to the left of 0. Then $\overrightarrow{x}$ is left-pointing with the length x^* and $\overrightarrow{x^*}$ is right-pointing with the length x^*. Therefore, when we slide the vector $\overrightarrow{x^*}$ along the number line (to the left) so that its starting point is at x, then its new ending point is at 0. Then by the definition of the addition, $\overrightarrow{x} + \overrightarrow{x^*} = \overrightarrow{0}$.

Finally, if $x = 0$, then $x^* = 0$. Since the vector $\overrightarrow{0}$ has length 0, the assertion is trivially true.

P. 389. *Compute* $3 + (4\frac{5}{6})^*$ *by drawing the vectors and by direct computation using the Key Observation.*



$$
\begin{aligned}
3 + \left(4\frac{5}{6}\right)^* &= \left(\left(4\frac{5}{6}\right) - 3\right)^* \\
&= \left(1\frac{5}{6}\right)^*
\end{aligned}
$$

P. 390. *Compute* $\frac{7}{8} + (\frac{1}{3})^*$, $(7\ \frac{2}{3})^* + 9\frac{1}{4}$, $401 + 193.7^*$.

$$\frac{7}{8} + \left(\frac{1}{3}\right)^* = \frac{7}{8} - \frac{1}{3} = \frac{13}{24}$$

$$\begin{aligned} \left(7\frac{2}{3}\right)^* + 9\frac{1}{4} &= 9\frac{1}{4} + \left(7\frac{2}{3}\right)^* \\ &= 9\frac{1}{4} - 7\frac{2}{3} = 1\frac{7}{12} \end{aligned}$$

$$401 + 193.7^* = 401 - 193.7 = 207.3$$

P. 390. *Compute:* $12\frac{1}{3} + 57^*$, $\frac{11}{12} + (2\frac{1}{2})^*$, $(71\frac{1}{3})^* + 68\frac{1}{5}$.

$$12\frac{1}{3} + 57^* = \left(57 - 12\frac{1}{3}\right)^* = 44\,\frac{2}{3}$$

$$\frac{11}{12} + \left(2\frac{1}{2}\right)^* = \left(2\frac{1}{2} - \frac{11}{12}\right)^* = \left(1\,\frac{7}{12}\right)^*$$

$$\left(71\frac{1}{3}\right)^* + 68\frac{1}{5} = \left(71\frac{1}{3} - 68\frac{1}{5}\right)^* = \left(3\,\frac{2}{15}\right)^*$$

P. 391. *Use the definition of rational number subtraction* to calculate: (a) $2.1-7$, (b) $7^* - 3$, (c) $7.5^* - 12.6$, (d) $5^* - 2^*$, (e) $(\frac{2}{5})^* - (\frac{8}{7})^*$.

(a) $2.1 - 7 = 2.1 + 7^* = 7^* + 2.1 = (7 - 2.1)^* = 4.9^*$.

(b) $7^* - 3 = 7^* + 3^* = (7+3)^* = (10)^*$.

(c) $7.5^* - 12.6 = 7.5^* + 12.6^* = (7.5 + 12.6)^* = (20.1)^*$.

$(d)\quad 5^* - 2^* \;=\; 5^* + (2^*)^* \;=\; 5^* + 2 \;=\; 2 + 5^* \;=\; (5-2)^* \;=\; 3^*.$

$$(e)\quad \left(\frac{2}{5}\right)^* - \left(\frac{8}{7}\right)^* \;=\; \left(\frac{2}{5}\right)^* + \left(\left(\frac{8}{7}\right)^*\right)^* \;=\; \left(\frac{2}{5}\right)^* + \frac{8}{7}$$

$$= \frac{8}{7} + \left(\frac{2}{5}\right)^* \;=\; \frac{8}{7} - \frac{2}{5} \;=\; \frac{26}{35}.$$

P. 392. *Compute directly, and also compute by using* (27.6) *and* (27.7)*:* $5-(\frac{1}{2}-8)$, $1-(-\frac{1}{3}+\frac{5}{4})$, $\frac{2}{3}-(5-\frac{4}{7})$.

For $5-(\frac{1}{2}-8)$, first, the direct computation:

$$\begin{aligned} 5 - \left(\frac{1}{2} - 8\right) \;&=\; 5 + \left(\frac{1}{2} - 8\right)^* \\ &=\; 5 + \left(\frac{1}{2} + 8^*\right)^* \\ &=\; 5 + \left(\left(\frac{1}{2}\right)^* + 8^{**}\right) = 5 + \left(8 + \left(\frac{1}{2}\right)^*\right) \\ &=\; 5 + \left(8 - \frac{1}{2}\right) \;=\; 12\frac{1}{2} \end{aligned}$$

Next we use (27.6) and (27.7):

$$5 - \left(\frac{1}{2} - 8\right) \;=\; 5 + \left(-\frac{1}{2} + 8\right) \;=\; 5 + 7\frac{1}{2} \;=\; 12\frac{1}{2}$$

For $1-(-\frac{1}{3}+\frac{5}{4})$, first, the direct computation:

$$1 - \left(-\frac{1}{3} + \frac{5}{4}\right) \;=\; 1 - \left(\frac{5}{4} - \frac{1}{3}\right) \;=\; 1 - \left(\frac{11}{12}\right) \;=\; \frac{1}{12}.$$

Next we use (27.6) and (27.7):

$$1 - \left(-\frac{1}{3} + \frac{5}{4}\right) \;=\; 1 + \left(-\left(-\frac{1}{3}\right)\right) - \frac{5}{4} \;=\; 1 + \frac{1}{3} - \frac{5}{4} \;=\; \frac{4}{3} - \frac{5}{4} \;=\; \frac{1}{12}$$

Finally, for $\frac{2}{3}-(5-\frac{4}{7})$, first the direct computation:

$$\frac{2}{3} - \left(5 - \frac{4}{7}\right) \;=\; \frac{2}{3} - \frac{31}{7} \;=\; -\left(\frac{31}{7} - \frac{2}{3}\right) \;=\; -\left(\frac{79}{21}\right)$$

Next we use (27.6) and (27.7):

$$\frac{2}{3} - \left(5 - \frac{4}{7}\right) \;=\; \frac{2}{3} - 5 + \frac{4}{7} \;=\; \frac{26}{21} - 5 \;=\; -\left(5 - \frac{26}{21}\right) \;=\; -\frac{79}{21}$$

P. 398. *Try proving the following special case before approaching the general case: if y satisfies $67 + y = 67$, then $y = 0$. Don't forget, you are supposed to make use of only* (A1)–(A3).

Since $67 + y$ is, by assumption, the same rational number as 67, if we add 67^* to it, then we should get a unique rational number, by fundamental assumption (A1). That is,

$$(67 + y) + 67^* \;=\; 67 + 67^*, \quad \text{and the latter is 0, by (A2).}$$

This means $(67 + y) + 67^* = 0$. But $(67 + y) + 67^*$ is also equal to y because, since addition is associative and commutative,

$$(67 + y) + 67^* \;=\; (y + 67) + 67^* \;=\; y + (67 + 67^*) \;=\; y + 0 \;=\; y.$$

Therefore, $y = 0$.

P. 399. *Compute:* $(2\frac{6}{7})^* + (3\frac{2}{5})^*$, $(\frac{24}{11})^* + (156\frac{1}{2})^*$.

$$\left(2\frac{6}{7}\right)^* + \left(3\frac{2}{5}\right)^* \;=\; \left(2\frac{6}{7} + 3\frac{2}{5}\right)^* \;=\; \left(6\frac{9}{35}\right)^*$$

$$\left(\frac{24}{11}\right)^* + \left(156\frac{1}{2}\right)^* \;=\; \left(\frac{24}{11} + 156\frac{1}{2}\right)^* \;=\; \left(2\frac{2}{11} + 156\frac{1}{2}\right)^* \;=\; \left(158\frac{15}{22}\right)^*$$

P. 407. *Practice explaining as if to a seventh grader, in two different ways, why $(-1)(-1) = 1$ by using your neighbor as a stand-in for the seventh grader.* (*Theorem 29.1 is so basic to understanding rational number multiplication that this Activity is strongly recommended.*)

If we can show that $(-1)(-1) + (-1)$ is equal to 0 then $(-1)(-1) = 1$ (basic Fact 2 on page 398). So our goal is to see if $(-1)(-1) + (-1)$ is actually equal to 0. We have $(-1)(-1) + (-1) = (-1)(-1) + 1 \cdot (-1)$, because any integer times 1 is the same integer (by (M2) on page 404). Now we use the distributive law to get $(-1)(-1) + 1 \cdot (-1) = (-1 + 1)(-1)$. In other words,

$$(-1)(-1) + (-1) \;=\; (-1 + 1)(-1).$$

But we know that, $-1 + 1 = 0$. Therefore,

$$(-1)(-1) + (-1) \;=\; 0 \cdot (-1) \;=\; 0$$

because 0 times any rational number is 0 ((M3) on page 404). As we noted above, this means $(-1)(-1) = 1$.

A second way begins with $1 + (-1) = 0$. Multiplying both sides by (-1), we get $(-1)(1 + (-1)) = (-1)0$. Since the right-hand side is 0 ((M3) again), we get $(-1)(1 + (-1)) \;=\; 0$. We will use the distributive law on the left-hand side. Then

$$(-1)(1 + (-1)) \;=\; (-1) \cdot 1 + (-1)(-1) \;=\; (-1) + (-1)(-1)$$

where the last equality is because $a \cdot 1 = a$ for any rational number a. Therefore, $(-1) + (-1)(-1) = 0$. By Basic Fact 2 on page 398, we have $(-1)(-1) = -(-1) = 1$.

P. 441. *Do you have a quick way to see why* 123,456,789 *is divisible by* 3*, and why* 67,814,235 *is divisible by* 3*?*

Consider 123,456,789. First, we can quickly eliminate (in our head) any digit which is a multiple of 3. Therefore, we eliminate 3, 6, and 9. Next, a quick (mental) check reveals that 1+2, 4+5, and 7+8 are multiples of 3 as well. Therefore, the sum of the digits of 123,456,789 is divisible by 3 (Lemma 32.1 is implicitly at work), and therefore 123,456,789 is also divisible by 3, by the divisibility rule on page 440.

For 67,814,235, as before, we can (mentally) eliminate 6 and 3 from the digits of $67, 814, 235$. $5 + 4$, $7 + 8$, and $1 + 2$ are all divisible by 3, and therefore, 67,814,235 is divisible by 3.

P. 448. *Is* 337 *a prime? Is* 373 *a prime?*

$\sqrt{337} \approx 18.4$. Since 337 is odd, it is not divisible by 2 or any proper multiple of 2 (part (b) of Lemma 32.2). By the divisibility rules for 3 and 5, 337 is not divisible by any multiple of 3 or 5. 337 is not divisible by 11, by the divisibility rule for 11. Therefore, among whole numbers between 2 and 18, we have to check and see if 337 is divisible by 7, 13 and 17 only. A quick check reveals that 337 is not divisible by any of these numbers. Therefore, by Theorem 33.1, 337 is a prime number.

$\sqrt{373} \approx 19.3$ and 373 is odd and therefore not divisible by any even number. 373 is not divisible by 3, 5 or 11, and therefore we only have to check and see if 373 is divisible by 7, 13 or 17. A quick check reveals that 373 is not divisible by any of these numbers. Therefore, by Theorem 33.1, 373 is a prime number.

P. 459. *Find the prime decompositions of* 252 *and* 1119.

252 is divisible by 2. Therefore, $252 = 2 \times 126$. 126 is also divisible by 2 and $126 = 2 \times 63$. Therefore, $252 = 2 \times\ 2 \times 63$. 63 is divisible by 3 and $63 = 3 \times 21$. 21 is also divisible by 3 and $21 = 3 \times 7$. Therefore, the prime decomposition of 252 is $2 \times 2 \times 3 \times 3 \times 7$.

1119 is divisible by 3. Therefore, $1119 = 3 \times 373$. We have proved before that 373 is a prime. (See the Activity on page 448). Therefore, the prime decomposition of 1119 is 3×373.

P. 460. *Prove the uniqueness of the prime decomposition of* 841 ($841 = 29 \times 29$).

Suppose 841 is divisible by a prime number $p \neq 29$. Then $p < 29$. (The trichotomy law for real numbers offers only one other possibility, namely, $p > 29$. This cannot happen, as any prime divisor of 841 must be less than or equal to $\sqrt{841} = 29$.) p cannot be 2 as 841 is odd. By divisibility tests, p cannot be 3, 5, or 11. A quick check (long division algorithm or calculator) reveals that it is not divisible by 7, 13, 17 and 23. Therefore, our assumption is false and the prime factorization of 841, $(841 = 29 \times 29)$ is unique.

P. 464. *Use only one division-with-remainder to determine the gcd of* 665 *and* 7353.

$7353 = (11 \times 665) + 38$. Therefore, by Lemma 32.1, $\gcd(7353, 665)$ must divide 38. Now 38 has only two proper divisors: 2 and 19 because the prime factorization of 38 is 2×19. Since both 7353 and 665 are odd, $\gcd(7353, 665)$ cannot be 2. Could it be 19? A quick check reveals that 19 is a common divisor of 7353 and 665. Therefore, $\gcd(7353, 665) = 19$.

P. 468. *Express the gcd of* 14 *and* 82 *as an integral linear combination of* 14 *and* 82.

By using the Euclidean algorithm repeatedly as shown below, we will show that $\gcd(82, 14) = 2$.

$$\begin{aligned} 82 &= 5 \times 14 + 12 \\ 14 &= 1 \times 12 + 2 \\ 12 &= 6 \times 2 + 0 \end{aligned}$$

Thus $\gcd(82, 14) = 2$. Now,

$$\begin{aligned} 2 &= 14 - 1 \times 12 \\ 2 &= 14 - 1 \times (82 - 5 \times 14) \\ 2 &= 14 + (-1) \times 82 + 5 \times 14 \end{aligned}$$

Therefore, $2 = 6 \times 14 + (-1) \times 82$.

P. 477. *Suppose we are given the following prime decompositions,*

$$\begin{aligned} 26460 &= 2^2 \times 3^3 \times 5 \times 7^2 \\ 15225 &= 3 \times 5^2 \times 7 \times 29 \end{aligned}$$

Find the gcd and lcm of these two numbers.

$$\gcd(26460, 15225) = 3 \times 5 \times 7$$

$$\mathrm{lcm}(26460, 15225) = 2^2 \times 3^3 \times 5^2 \times 7^2 \times 29.$$

P. 488. (a) *Check that the triple* 9, 12, 15 *forms a Pythagorean triple.* (b) *Check that there are no whole numbers* m *and* n *so that* 9, 12, 15 *are given by* $\{m^2 - n^2, 2mn, m^2 + n^2\}$. ($c$) *Does the triple* 9, 12, 15 *contradict Theorem 37.1?*

Repeat (a), (b), *and* (c) *for the triple* 15, 36, 39.

(a) $\{9\ , 12\ , 15\}$ is a Pythagorean triple, because $9^2 + 12^2 = 15^2$.

(b) Suppose there are positive integers m and n such that the three numbers $\{9, 12, 15\}$ are equal to the three numbers $\{m^2 - n^2, 2mn, m^2 + n^2\}$ *in some order.* Clearly, $m^2 + n^2 > m^2 - n^2$. Also, $m^2 + n^2 > 2mn$. (The argument goes as follows: $(m-n)^2 > 0$. Therefore, $m^2 - 2mn + n^2 > 0$, and $m^2 + n^2 > 2mn$.) Therefore among the three numbers $\{m^2 - n^2, 2mn, m^2 + n^2\}$, $m^2 + n^2$ is the largest. It follows that whatever m and n may be, $15 = m^2 + n^2$. Since $2mn$ is even and 9 is odd, we cannot have $2mn$ being 9, so $2mn$ has to be 12. Thus also $m^2 - n^2 = 9$, i.e., m and n are positive integers such that

$$m^2 + n^2 = 15, \qquad 2mn = 12, \qquad m^2 - n^2 = 9$$

By adding the first and the last equations we get, $2m^2 = 24$, and therefore, $m^2 = 12$. But there is no whole number m so that $m^2 = 12$, as 12 is not a perfect square. Therefore, there are no such m, n.

(c) The observation in (b) above, does not contradict Theorem 37.1 because $\{9\ , 12\ , 15\}$ is not a primitive triple. (3 is a common divisor of 9, 12 and 15.)

(a) The triple $\{15\ , 36,\ 39\}$ is a Pythagorean triple, because $15^2 + 36^2 = 39^2$.

(b) By a similar argument (as in the previous case), if there are positive integers m and n (with $m > n$) so that $\{15\ ,36,\ 39\}$ is given by $\{m^2 - n^2, 2mn, m^2 + n^2\}$, then $m^2 + n^2 = 39$ and $m^2 - n^2 = 15$. Adding them leads to $2m^2 = 54$ and therefore $m^2 = 27$. As there is no positive integer with a square equal to 27, such m and n do not exist.

(c) The triple $\{15\ ,36,\ 39\}$ is not primitive as 3 is a common divisor of 15, 36 and 39.

P. 509. *Which is bigger:* 1.92×10^{-7} *or* 2.004×10^{-7}*?* 1.92×10^{-6} *or* 2.004×10^{-7}*?* 1.92×10^6 *or* 2.004×10^7*?*

2.004×10^{-7} is bigger than 1.92×10^{-7}, since $2.004 > 1.92$.

1.92×10^{-6} is bigger than 2.004×10^{-7} since $-6 > -7$.

In greater detail, $1.92 \times 10^{-6} = 1.92 \times 10 \times 10^{-7} = 19.2 \times 10^{-7}$. Since it is obvious that $19.2 > 2.004$, we have $19.2 \times 10^{-7} > 2.004 \times 10^{-7}$, which is to say, $1.92 \times 10^{-6} > 2.004 \times 10^{-7}$

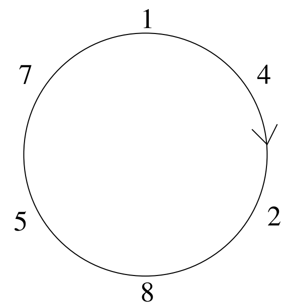
2.004×10^7 is bigger than 1.92×10^6, since $7 > 6$.

In greater detail, $2.004 \times 10^7 = 2.004 \times 10 \times 10^6 = 20.04 \times 10^6$. Now it is also obvious that $20.04 > 1.92$, so that $20.04\times10^6 > 1.92\times10^6$, i.e., $2.004\times10^7 > 1.92\times10^6$.

P. 521. (a) *We have just come across the whole number* 142857. *Can you guess what fraction is equal to* $0.\overline{142857}$*?* (b) *Multiply* 142857 *successively by* 2, 3, 4, 5, 6 *and examine carefully the numbers you get. What do you notice about these numbers?* (c) *How are the numbers in* (b) *related to* $\frac{2}{7}, \frac{3}{7}, \ldots, \frac{6}{7}$*?*

(a) One would guess that $0.\overline{142857}$ is $\frac{1}{7}$, because $2\times142857 = 285714$. This would correspond to $2 \times \frac{1}{7} = \frac{2}{7}$.

(b) The numbers are 285714, 428571, 571428, 714285, 857142 respectively. They all have the same 6 digits. We can understand them better if we arrange the six digits of 142857 around a circle in a clockwise direction, as shown below. Then if we start with any digit and go around the circle in the clockwise direction, we will get one of these six numbers. For example, if we take the digit 8 and go clockwise around the circle, we get 857142.



(c) $0.\overline{285714} = \frac{2}{7}$, $0.\overline{428571} = \frac{3}{7}$, $0.\overline{571428} = \frac{4}{7}$, $0.\overline{714285} = \frac{5}{7}$ and $0.\overline{857142} = \frac{6}{7}$.

P. 526. *Make use of Theorems 41.1 and 42.1 to show that there are irrational numbers.* (*This argument complements Theorem 36.5 in Section 36.3.*) (The solution of this Activity requires a theorem not proven in the book.)

We define an infinite sequence of single digits $\{a_n\}$ as shown:

$$1,\ \underbrace{0,\ 1,}\ \underbrace{0,\ 0,\ 1,}\ \underbrace{0,\ 0,\ 0,\ 1,}\ \underbrace{0,\ 0,\ 0,\ 0,\ 1,}\ \underbrace{0,\ 0,\ 0,\ 0,\ 0,\ 1,}\cdots\cdots$$

This sequence can be defined as follows. It begins with a single digit 1, followed by the two digits 0 and 1, then by the three digits 0, 0, and 1, followed by the four digits 0, 0, 0, and 1, ..., and at the n-th step, it will be a string of $n-1$ consecutive 0's followed by 1. The resulting infinite decimal, defined by this sequence $\{a_n\}$ according to Theorem 41.1 on page 514, has no repeating block because of the way we make sure that the consecutive string of 0's increases as we go to the right (strictly speaking, this argument has to be made more precise, but it is sufficiently intuitive that we can let it go at this point). Now Theorem 41.1 guarantees that this infinite decimal defines a number γ. If γ were rational, it would be a fraction and therefore, by Theorem 42.1, it would be equal to a finite or repeating decimal. Therefore a decimal without a repeating block is equal to a finite or repeating decimal, a contradiction (this contradiction requires an unproven uniqueness theorem for decimals mentioned above; see Theorem 3.7 of my book, *Pre-Calculus, Calculus, and Beyond*, forthcoming).