Review of the Interactive Mathematics Program (IMP)

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Note added March 25, 2000: *This review has been circulated since March of 1992 under the title,* Review of College Preparatory Mathematics (CPM) at Berkeley High School. *In its original form, it was a report commissioned in January of 1992 by the Berkeley Unified School District on a new curriculum (called* CPM *at the time) being introduced in a local high school. From the beginning, however, I intended the report to be a review of the mathematics of* CPM *with only a minimal reference to the local high school. For this reason, it has been met with some interest through the years. The version of this report being offered below*

[0] I am very much indebted to my friends whose help in various forms back in 1992 made the writing of this review possible: Professors P. Chernoff, T.Y. Lam, B. Parlett, M.H. Protter, J. Sethian, and P. Vojta of UC Berkeley, Dr. Ian Brown of Lawrence Berkeley Laboratory, and Professors S.Y. Cheng and R.E. Greene of UCLA. I should also record the list of the people whom I have consulted in the writing of this review and thank them all for their courtesy, without implying in any way that they are responsible for the factual accuracy of what follows: Ms. Lynne Alper, Professor Dan Fendel, Ms. Sherry Fraser, Ms. Heidi Boley, and Mr. Harvey Garn. Finally, special thanks are also due Mr. Todd Boley and Mr. Fred Dunn-Ruiz for their critical comments which led to significant improvements in the exposition.

dates back essentially to July of 1997, and it differs from the original only in the omission of all references to Berkeley High School, changing the erstwhile CPM *to* IMP *(which is how this curriculum now chooses to identify itself), clarifying the meaning of the word "tracking" as used in the original version, and adding a few footnotes to bring some of the original comments up-to-date. In early 1997, the first IMP text finally appeared in print (Interactive Mathematics Program,* Year 1*, Key Curriculum Press, Berkeley, CA, 1997), and by now all four volumes have appeared. Other than the expected pedagogical and expository refinements, the virtues and defects — such as I perceived them and discussed in the report — of the preliminary 1991 version made available to me back in 1992 have in the main survived in the published version. In particular, the reservations against IMP detailed in §§III and IV below regarding its lack of precision and its inattention to mathematical closure apply equally well to the 1997 text. Thus I believe this review still serves a purpose. My recommendation against the use of IMP for future college students in science, engineering and (of course) mathematics is in my view as valid now as before. On the other hand, two additional comments must be made. The first is that one virtue of IMP which I failed to notice in 1992 and which is becoming increasingly apparent to me is its ability to put students at ease in working on word problems, no matter how long. The ability to read ordinary English in the context of mathematics is, paradoxically, a quality sorely lacking among college students. It is much to be regretted that IMP could not capitalize on this achievement to launch students into substantive mathematics. The other comment concerns whether, eight years after I wrote this review, hindsight has changed my initial judgment on the suitability of IMP for use by students who either do not go to college or will not pursue scientific studies in college. Having read quite a few high school texts in the intervening years of both the reform and the traditional varieties, I find it difficult to give a simple answer. Without a doubt, IMP is mathematically flawed even for these students. It is not without merits, however (see §§III), and as of March 2000, I know of no high school mathematics textbook series that is clearly superior in every way, and many are substantially worse. With the* *new mathematics textbook adoption in California looming in the horizon, there is reason for optimism that at least some algebra I texts will be better in terms of precision, rigor, and mathematical closure. Because this adoption has no control over textbooks beyond the algebra I level, there will be a big area of advanced school mathematics still unaccounted for. With this in mind, each teacher must consider the trade-offs carefully in terms of his or her needs before making a decision concerning IMP. My conservative recommendation is that, for students who do not go to college or do not intend to pursue scientific studies in college, all teachers would do well to consult IMP often for supplementary materials to be used in the classroom.*

Contents

I. Prologue

During his campaign for the presidency (in 1992), President Bush stated that he wanted to be remembered as the education president and, in the same breath, that he wanted the high school students of this country to be first in science and mathematics. Perhaps without realizing it, he had in fact set two separate goals for his presidency: (1) that all high school graduates should be literate and scientifically informed, and (2) that the high schools should also produce a corps of potentially top flight scientists and engineers. While it is obvious that these two goals in no way contradict each other, it is not usually realized that the methods of achieving them are necessarily distinct. An educational program designed strictly to achieve the first objective would most likely produce students that are too deficient in skill and knowledge to do serious science, while one designed exclusively for the second would be preoccupied with the kind of technical information that is *neither necessary for the intellectual growth of the non-science students nor of any real interest to them.* Thus a program designed exclusively for either cannot, and should not be applied to both. An analogy may be made in musical terms: the difference between these two goals is that of trying to nurture a concert-going public on the one hand and trying to produce great performers and composers on the other. It should not be difficult to see that the strategy used to train a violinist (say) would not be suitable for the education of the average music lover.

Let us now turn from President Bush to something much less imposing, namely, my task at hand, which is to give an informal review[1] from the perspective of a professional mathematician of the Interactive Mathematics Program (IMP) for high school. The complexities of mathematics education in high school would seem to stem from the confluence of two facts: (1) at the high school level, the mathematics is beginning to get sophisticated enough so that one can no longer expect every student to be comfortable with the kind of technical materials to be found in a full-blown mathematics course, (contrast this with the situation in, say, English or History), and (2) students are faced with genuine choices for the first time in their academic lives: should they go on to a four-year college, and if so would they want to get into a science-related subject? Given this reality as well as the diversity of the students, especially in California, it is no longer possible to speak of one

[1] Conducted from January to March of 1992.

kind of mathematical education for all high school students. Therefore common sense dictates that, allowing for a bit of oversimplification, mathematics education in high school must eventually bifurcate into separate curricula for

> **Group 1**: those who will not go to college as well as those who will, but do not plan to pursue the study of any of the exact sciences (mathematics,[2] astronomy, physics, chemistry), engineering, economics, or biology, and
> **Group 2**: those who plan to pursue the study of one of the exact sciences, engineering, economics or biology,[3] and those who entertain such a possiblity.

We will try to address the effectiveness of IMP for each of these two groups separately.

It should be firmly stated at the outset that I am not advocating the "tracking" of mathematics classes in the usual sense of having the school authorities dictate who should be assigned to which track. What I have in mind is a system whereby the high school students get the free choice of enrolling in either track, and are allowed to switch between tracks later on. In other words, they should be allowed to choose their mathematics classes the same way college students do. One may call this *tracking-by-choice*. The issue of when this choice should be first offered to the students may be left to another discussion (although the 10th grade would seem to be a natural starting point).

Within the context of mathematics education, what Group 1 needs is a broad understanding of the fundamentals of logical reasoning, one that would meet their needs either in their daily life or in their intellectual pursuits. But the needs of Group 2 go beyond the fundamentals; these students must be given a firm technical foundation which is needed for their next step in college, where the pace will be fast and furious by comparison with high school. It is certainly no accident that, by an independent route, we have arrived at a mirror image of the two goals of President Bush's campaign promise. Indeed, I would like to keep the discussions in the following pages, which at times can get technical, in the broad social context of a national aspiration as

[2] We shall understand mathematics in the broad sense and use it to include both computer science and statistics.

[3] On the research level, both biology and economics have increasingly turn to very sophisticated mathematics.

enunciated by its leader. It is hoped that with this vision clearly in sight, this review can steer clear of acrimonious debates should it (unavoidably) get into some controversial territories.

According to the statistical figures of CPEC (as of 1992), only 17% of all California 9th graders go on to a four-year college[4]. Even if half of them major in engineering or one of the exact sciences, taken literally, these figures would suggest that Group 2 comprises at most 10% of the high school students in California and therefore Group 1 represents the remaining 90%. In reality, however, the percentage of 9th graders who *think*, or *whose parents think* they might study science or engineering in college must be quite large. These students too would want to have access to a rigorous mathematical training. Therefore the de facto percentage of the students in Group 2 may be as high as 20%, but to be conservative, let us settle for 15%. Group 1 would then be 85%. For the purpose of the ensuing discussion, however, two additional caveats in connection with these figures are in order. The first one is that this 85% of the students (Group 1) is far from being a homogeneous unit; for example, it would include future actuaries and accountants (who would be working in the periphery of the mathematical world) as well as artists of various kinds (who would seem to need little mathematics). Nevertheless, a modest review such as this will have to treat Group 1 as a single entity.

There is some justification in taking such a broad view of IMP in that it is a pilot project that has nation-wide aspirations.[5] Therefore a discussion of the intrinsic merits of IMP, independent of its implementations in a school or district, may be of general interest.

The above division of high school students acording to their mathematical needs is reflected in the calculus curriculum of most universities, in particular, the University of California at Berkeley (UCB). For the students in the physical sciences and engineering, we offer a two-year sequence 1A-1B-53-54. We also offer, at least most of the time, an honors section of the same sequence, though this is strictly for those students unusually talented in mathematics and its enrollment is about 1/40 of the normal sequence. For the students in the social and life sciences, we offer a one-year sequence 16A-16B. Needless to say, 16A-16B covers less material and is less demanding in terms of manipulative skills than even the one year sequence 1A-1B

[4] I am grateful to Lynne Alper for supplying these figures

[5] As of April 1997, the official figure is that over two hundred schools across the nation have adopted IMP. In October of 2000, IMP was chosen by the Department of Education to be one of five Exemplary Curricula.

alone. While no one claims that the calculus teaching at UCB is perfect, at least there have never been any complaints that such a division into different sequences is inherently discriminatory. It is simply accepted as a fact that such a division would better serve the needs of the different segments of the student population at UCB.

Before proceeding to the detailed discussion of IMP itself, the reader may well ask what purpose it serves to have a professional mathematician look over a high school curriculum. I will have to deal with this issue in greater detail when I come to the summary of this review (§VI). Here, I will simply say in general terms that *if* that mathematician is interested and knowledgeable about the teaching of mathematics, he may be able to bring a fresh viewpoint to high school mathematics education. My experiences in the trenches, so to speak, of having taught all three kinds of calculus (see preceding paragraph) for so long have given me some insight into the mathematical weaknesses of the incoming freshmen; they have also given me a good idea of what a high school graduate must know if he or she hopes to survive, mathematically, at UCB. This kind of knowledge is perhaps not usually part of an educator's intellectual arsenal. Too often it is forgotten that there are two distinct components to mathematical education: *mathematics* and education. There is undoubtedly a sociological side to education that, by and large, the mathematicians should leave to the educators. This is no more than proper. Yet in the heat of the argument about the sociological implications, the mathematical voice too often gets muffled or even lost. So without a doubt, the mathematician has an important rôle to play. In case the reader wonders why I brought up something so utterly obvious, that mathematics is a basic component of mathematical education, it is because an average educator may (and dare I say, often does) overlook the fact that in the teaching of mathematics, the most difficult, and also the most important part is to be able to get the *technical* subject matter across to the students.[6] No amount of wonderful pedagogy can redeem a mathematical education if the technical aspect is not sound, and unfortunately, the technical soundness of a curriculum or a teacher cannot at all be taken for granted. In the Instructor's Manual for the TA Workshop which I wrote up for my department[7] (see §IV of [5] in the

[6] Needless to say, the same can be said of any subject. But it is especially crucial in mathematics.

[7] This is the workshop of the mathematics department at UCB where we teach the teaching assistants how to teach calculus. I have been teaching it at least every other year from 1976 to 1997.

references at the end), I spared no pain in bringing up this point time and again because I consider the mathematical component to be crucial to any success in the teaching of mathematics. To a large extent, this mathematical voice is what I hope to bring to this review.

II. IMP: an overview

The IMP curiculum[8] differs from the traditional one in three essential aspects. First, the text is *problem-oriented* rather than *theory-oriented*. In greater detail, the presentation of mathematics in IMP revolves around *concrete* real world problems, whereas the traditional curriculum proceeds from concept to concept, thereby emphasizing more the internal structure of mathematics. Second, classes are conducted in such a way that active participation is *demanded* of each student. Third, there is no neat division of mathematical topics into algebra, geometry, trigonometry, etc. in the IMP curriculum. The text is built around five or six major concrete problems per year. Each time such a problem comes up, new tools and concepts (regardless of whether they belong to algebra or geometry) are introduced until the problem can be solved. By contrast, the traditional curriculum runs as follows:

algebra I (1 year)

geometry (1 year)

algebra II (1 year)

trigonometry (1 semester); pre-calculus (1 semester)

calculus (1 year, optional)

Erudite arguments pro and con have been advanced regarding a problem-oriented curriculum. To someone not involved in this fierce battle, such debates bears some resemblance to those over whether English or French is a better language for great literature. In point of fact, however, theory and application in mathematics go hand-in-hand, and any basic text should ideally present both. Either one could be emphasized over the other at any given point, but what ultimately makes or breaks a mathematics text is not the amount of emphasis either way but the quality of the writing. Leaving a more detailed discussion to §§III & IV, in general terms, the IMP text is

[8] This review was based on the units of IMP (dated 1991) made available to me in January of 1992.

single-minded in its attempt to present mathematics as a series of responses to the needs of real life situations. The logical inter-relationships of the concepts that arise from different situations are not seriously pursued. There is ample discussion of the motivation behind almost every concept and every technique that come up in the text. On the other hand, some topics that are usually expected to be discussed in such a curriculum may fail to make their appearance (see §IV). The writing of the text is lively, much more so than most of the texts at this level that I have seen. To most students, this would be an attractive feature. The exposition is more reasonable than the traditional texts that I have seen[9] in the sense that it flows better and seems more accessible. The problems are also more interesting on the whole than the traditional ones (but see §§III & IV for more detailed comments in this regard).

This curriculum encourages, and even demands, group activities in the classroom. Students are asked to discover for themselves each new idea whenever possible, again preferably in a group. The advantage of this method, which is reminiscent of the famous R.L. Moore method[10] still used in some colleges, is obvious. The disadvantage, perhaps intentionally ignored, is that much less material can be covered in a given period of time. IMP also makes a point of encouraging students to write about their ideas and their thought-processes. The enforced talking and writing about mathematics no doubt succeeds in *humanizing* what is to most people an arcane subject. What may be lost in this atmosphere of compulsory socializing is the need for private contemplation in order to really learn mathematics.

In the following two sections, I shall examine the effectiveness of the IMP curriculum separately for Groups 1 and 2.

III. IMP as a mathematics curriculum for Group 1

IMP represents a new way of teaching mathematics to high school students and, as such, it will arouse controversy. There are certain obvious deficiencies in this new curriculum (even in the context of Group 1) that must be addressed before it can be considered for adoption on a large scale; these will be discussed presently. Nevertheless, I believe that, for students not motivated in mathematics, the approach adopted by IMP may be supe-

[9] Obviously I have read only a handful of such texts.

[10] Although R.L. Moore forbade the use of all books, especially textbooks.

rior to the traditional curriculum. Since Group 1 includes such students, this must count as a significant achievement.[11]

It is not difficult to put one's finger on those features where IMP scores over the traditional curriculum:

(1) The mathematics treated in the text grows out of concrete problems that the students can relate to. It galvanizes their interest at the outset. The choice of these problems is often inspired; for example, the use of honeycombs to introduce the discussion of area, perimeter and volume, or the idea of using the growth of trees to motivate similarity, perpendicular bisector, congruence, etc.

(2) By design, the students are forced to participate actively in the classroom to achieve the eventual solution of those problems. While it may be possible to sleepwalk through the group activities and learn nothing, this is less likely than in the traditional setting with any teacher that is at least half-way reasonable.

(3) The explanation of the mathematics is usually more down-to-earth, less abstract and less formal than the usual approach. Consequently, it is more palatable.

(4) The pace is less intense than the traditional one, so even mathphobics can probably keep up.

(5) The presentation of the mathematical topics is never less than interesting.

(6) There are almost no boring exercises. These used to be the bane of every reluctant mathematics student in the past.

(7) There is a much greater emphasis on mathematical *reasoning* than on the piling up of mathematical facts. This enhances the usefulness of such a mathematics curriculum as a training in logical thinking.

(8) Last but not least, the inclusion of probability, statistics and linear programming is a very welcome break from tradition. These are certainly basic topics that deserve to be made known to even the average high school student.

On the debit side, several aspects of the IMP curriculum are disturbing, and I will discuss them in some detail. These should be addressed in future revisions if IMP hopes to achieve any success on a large scale. Moreover, if we anticipate the discussion in the next section (§IV), then these proposed revisions acquire additional significance: to prepare for the eventuality that

[11] **Added July 31, 1997**: But see the comment on pp. 1–2.

some students in Group 1 may wish to transfer to a curriculum designed for Group 2, one would make sure that the gap between these two curricula is not astronomical. The revisions would have the effect of closing this gap. Now, onto the perceived defects of IMP as a curriculum for Group 1:

(a) *The almost total absence of drills.*[12]

> The IMP curriculum seems to be promoting the novel concept that learning is based entirely on "understanding" and not at all on memory or the acquisition of technical fluency. Were this the case, none of us would remember the 26 letters of the alphabet, and we would all be diligently consulting our city maps each time we try to go home.[13] Not to belabor an obvious point, but it needs to be said that an important aspect of mathematics is that it is the (nearly) universal language for science. To master a language, certain skills must be performed correctly without conscious thought, i.e., must become automatic, and this is the purpose of having drills. Therefore, I would urge in the strongest terms possible that a moderate number of simple drills be incorporated into the text after the introduction of each new concept or technique, and be labelled as such.

(b) *The inability of the IMP text to* follow through *in its presentation of new ideas.*

> This requires a careful explanation. There is a tendency in the text to touch on a new topic, talk around it profusely, but stop short of coming to the main point (perhaps out of fear for over-challenging the students?). There are exceptions of course,[14] but overall, this would seem to be a persistent problem. Maybe two concrete examples would suffice to pin it down. In the first one, the discussion of quadratic equations is limited to graphing

[12] **Added July 31, 1997**: There are some drills in the 1997 published version of the IMP text.

[13] Coincidentally, there are two ready examples at hand to show how technical fluency plays a crucial rôle in the complicated process of understanding mathematics: see equations (2) and (3) in Appendix 1 (§VI), especially the discussions following the equations.

[14] The explanation of the meaning of $\log_a b$ for arbitrary a and b, for example, is particularly well done.

and approximating the real roots by the use of calculators. Consequently it shies away from the climax of any such discussion, which is the presentation of the *quadratic formula*. In Appendix 1 (§VI), one can find a technical discussion of why this formula is important. It must also be pointed out that, contrary to the common belief that the presentation of this formula is discouraged by the **NCTM Curriculum and Evaluation Standards** (see reference [2] at the end of this review), such a presentation is in fact *taken for granted* there (see line 19 from the bottom on p.153 of [2]). A second example is the discussion of the derivative of a function in the unit **Leave Room for Me!**. This unit spends three days talking around the concept of a derivative (Days 13-15) without ever settling down to a precise definition. Even when it does advertise "Derivatives in functional notation" (p.81, line 12 from bottom), it manages *never* to write down the definition in the functional notation. Surely, writing it down just once to erase all that vagueness will do the student no harm. An exposition such as this gives a misrepresentation of mathematics as a vague and descriptive subject, whereas mathematics is nothing if not precise. To push this a little further, on Day 22 of the same unit, the topic of "Derivatives of exponential functions" is explicitly brought up (p. 126) and, incredibly, the simple formula of such a derivative is never written down in symbolic form anywhere, then or later.

If the belief is that formulas are not important, then this belief is totally erroneous. If the belief is that the students are incapable of understanding such a profound formula, then the task would obviously be one of explaining things better until they are. *But make sure that such a formula is written down.*

(c) *The misrepresentation of mathematics through the abuse of "open-ended problems" and the de-emphasis of correct answers.*[15]

As a branch of knowledge, mathematics is *deterministic* in the sense that every mathematical statement falls under one and

[15] **Added July 31, 1997**: A more thorough discussion of this circle of ideas can be found in the author's paper: The role of open-ended problems in mathematics education, *J. Math. Behavior* 3(1994), 115-128.

only one of three headings: true, false, or as-yet-unknown.[16] The last are the open problems of mathematical research, and these are not likely to turn up on the high school level. Moreover, with conditions clearly stated, every problem in mathematics has one and only one correct answer, *even if that answer consists of a complete list of all the possibilities.* The presence of numerous open-ended problems, plus the attitude of the IMP curriculum that the ideas of solving a problem are always more important than the correct solution itself, raise a legitimate concern in this regard. All these open-ended problems certainly admit complete solutions and, as was emphasized above, there are no loose ends and there is a total absence of subjectivity in these solutions. I consider these problems desirable in a high school curriculum because they are stimulating and they challenge the students to become more involved with the mathematics. However, unless they are formulated correctly,[17] unless the teacher is extremely conscientious and competent, and unless the teacher has an ample supply of time and patience, these problems can easily lead to unforeseen adverse effects. For example, the typically incomplete solutions of the students may not get the kind of inspection or comments they require and deserve. Such would indeed seem to be the case if the small number of students' portfolios made available by the IMP group are to be trusted. Since no absolute standard is clearly set for the benefit of the students[18], more likely than not, the latter end up not knowing whether what they did is acceptable and, if not, why not. Consequently, in practice, the students would come away feeling that there is plenty of room in mathematics for sloppy and incomplete answers. For beginners in mathematics, such an attitude should not be encouraged.

(d) *The presentation of mathematical puzzles* (also known as brain-teasers) *as straight mathematics.*

A judicious use of mathematical puzzles has its place in a mathematics curriculum as a tool for training mental agility.

[16] We will bypass the subtleties of Gödel's theorem here.

[17] There are some comments on this particular point in (B) of Appendix 2 (§VII).

[18] Or, as some would say, for the teacher as well.

However, in view of the fact that the IMP curriculum gives a consistent impression of teaching mathematics that is truly basic and relevant, the not-infrequent appearances of such puzzles in POW's and homework problems without any preamble can only reinforce the popular (and unfortunate) misconception that mathematics is nothing but a bag of cute tricks. It would be far better if each puzzle is prefaced by a disclaimer to the effect that "This is a test of your ingenuity". Incidentally, I would hesitate to recommend making puzzles part of an examination (e.g., Sally's party in the IMP Final). An examination should test only whether a student has learned well, not whether she is inspired at the particular moment of exam-taking. To most of us, solving a puzzle does require inspiration.

(e) *the refusal to acknowledge that mathematics could be inspired by abstract considerations.*

It is a fact that even at the elementary level, not all mathematics was inspired by real world problems. For example, negative numbers came about, not because of cookies or bees or orchard hideouts, but because people wanted to solve the equation $a + x = b$ for x regardless what a or b might be. As another example, the hugh amount of mathematics inspired by the search for the exact value of π as well as for the understanding of this number has little or no connection with practical problems; if it had, the approximate value of 22/7 found by the Greeks more than two thousand years ago would have sufficed and the whole subject would have been long forgotten. Thus the total refusal of the IMP text to show that mathematics often arises from abstract considerations gives the students a very biased view of mathematics. Note that it does not take much to correct this problem; a discussion of two to three days' worth each year, together with some good examples of abstractly-conceived mathematics would get the job done. One cannot imagine that any responsible educator would want a typical student in Group 1 to leave mathematics for good with such a misconception, namely, that mathematics is just a collection of solutions to some practical problems.

Finally, I wish to make a plea for complex numbers to be introduced into the high school mathematics curriculum as early as possible and to be used as much as possible. This plea is not just addressed to IMP but to the traditional curriculum as well. In this day and age, no one can claim to be scientifically informed if he or she has never heard of complex numbers.

Incidentally, by allowing for abstract considerations as motivation of mathematics, IMP would be able to introduce complex numbers as mathematicians' response to the need of solving the equation $x^2 + 1 = 0$.

IV. IMP as a mathematics curriculum for Group 2[19].

We now come to the evaluation of IMP as a curriculum addressed to a true minority of the high school population, Group 2, which represents about 15%, as previously noted. One would have to assume that this group is already motivated to learn. In addition, one must take into account the fact that their technical skill must be sufficiently developed in order to meet the challenge they face in college. Thus the mathematics curriculum for this group can minimize the sweet-talk and at the same time be more exacting. When viewed from this perspective, the IMP curriculum falls far short of the ideal. In making this judgment, the detailed justification of which will occupy the rest of this section, I should hasten to make explicit a few facts to avoid misunderstanding. First, I am definitely not using the traditional curriculum as an absolute standard to base my judgment. In fact, almost all working mathematicians I know of have very serious misgivings about both the traditional curriculum and the traditional texts presently in use in high school. Second, in pointing out what I perceive to be defects in the IMP text or the IMP methodology, I am not by any means condemning the so-called problem- based approach to high school mathematics instruction. Rather, my suggestions are along the line of improvements, not abolition. It is my belief that either the traditional approach or the problem-based approach is perfectly capable of producing a curriculum that can well serve the needs of Group 2; it is the detailed execution that makes the difference. Finally, my frustrations in teaching calculus to inadequately prepared incoming students for more than twenty years at Berkeley have gradually crystallized into a clear mental picture of "what every high school student should know". This mental picture is ultimately the standard I use to measure the IMP curriculum. With these understood then, I will now divide my comments into four broad areas.

[19] This section overlaps in a few instances the discussion in the preceding one. Since the emphasis and the context are quite different, I have decided that instead of awkward back-references, I will simply make some harmless repetitions here in order to make this section self-contained.

(A) *Lack of depth and breadth in the topics covered.*

The IMP text presents mathematics almost exclusively as a tool for solving the concrete problems discussed in the various units. At the outset, let us note that indeed all branches of science must start with the presentation of raw data and the immediate analysis thereof, and mathematics is no exception. But this is just the first step. A second step, which to most is also the most important step, is the abstraction from these disparate facts and the organization of them into a coherent entity. Thus no one would think of writing a textbook on biology which presents only the data on evolution and heredity without also discussing the theories of Darwin and Mendel. By the same token, calculus is not just the method of computing slopes of tangents to curves and the approximation of areas by rectangles, any more than algebra is just the graphing of polynomials by a few well-chosen values and obtaining the approximate locations of the roots by a judicious use of the calculator. The last thing one wants to do to a beginner in mathematics is to give her the distorted view that this subject is no more than a disjointed collection of solutions to real life problems, and that this is an area where the imagination never soars and the human intellect has no place to roam freely. By refusing to adequately discuss the inter-connections of the different concepts and techniques that arise from the different problems, the IMP text is denying the student the opportunity to see how abstract reasoning can develop a life of its own, thereby also denying them the opportunity to learn about a basic characteristic of all branches of modern science.

Of course the idea of tying mathematics tightly down to the problems which inspired it is not new; it was the prevailing dogma during the Cultural Revolution in the People's Republic of China (1966-76). When the mathematics delegation from the National Academy of Sciences visited that country in 1976 and took a firsthand look at the consequences of such a policy, what it had to say is unkind (see [1] in the References at the end of this review). Beyond the expected criticisms of the resulting research in pure mathematics, here is what the *applied* mathematicians of the delegation had to say about the *applied* mathematics shaped by such a policy:

> ... Indeed [the Chinese work] seems to aim for the particular solutions of particular problems ... This gives much of the work the character of engineering analysis rather than that of applied mathematics ... The gap between activities in mathematics per se and routine applications of mathematics to real-world problems,

> however cleverly carried out, should not be allowed to remain ([1], p.22).

It is precisely these "activities in mathematics" that are found wanting in the IMP curriculum. Take the example of the treatment of linear and quadratic equations. The IMP text discusses the solution of the former by the usual means, and the approximate location of the roots of the latter by the use of calculators. But it makes no mention of (1) quadratic equations without real roots, (2) the quadratic formula, and (3) roots of polynomials of higher degree. This points to several problems. The first one is that of a lack of thoroughness: are we developing the correct scientific attitude in the young when we fail to address even the most obvious questions such as these? The second one is the matter of failing to provide the students with even a "minimal survival kit" in their tentative first step of scientific exploration. I will leave the rather technical reasons to Appendix 1 (§VI), except to mention here that by not doing (1), IMP has missed the excellent opportunity of introducing complex numbers as a matter of necessity, and that by not doing (2) and (3), it has likewise let slip an opportunity to give the students a glimpse of the historical development of algebra that culminated in the work of Abel and Galois, which in turn ushered in the modern era. As another example, take the case of *similarity* and *congruence* in Euclidean geometry. Similarity is first discussed in Grade 9, and by the time the text gets around to discussing (very briefly) congruence in Grade 11, there is no mention of the fact that the theory of similarity ("the glory of Greek mathematics") is in fact firmly based on the theory of congruence. Even two simple examples of the case of two triangles, once where one triangle is twice the size of the other and a second time where one is two-and-a-half times the size of the other, would have sufficed to pin down the main ideas as well as to hint at the difficulty of incommensurability that plagued the Greek geometers for so long and was subsequently so brilliantly overcome by Eudoxus and others. This would have also given the students some idea of the complicated nature of the real number system. Instead, the student comes away with only a flimsy notion of two topics misrepresented as being logically related only in the most superficial way.

Let me illustrate this point one last time with a minor example. In connection with the problem of constructing an orchard hideout, the student is exposed to the concepts of the circumcenter, the incenter and the excenters of a triangle. However, no attempt is made to discuss the orthocenter and the centroid, and then to pull all these together in an overview of the geometry of the triangle. Are we to assume that the students are so intellectually constipated as not to even wonder whether the altitudes or the medians meet at a point?

The potential danger of a problem-based approach to mathematics text-writing is that, if not properly executed, it leaves too many lose ends dangling. This is not unlike a presentation of human history, not in the chronological order, but in terms of broad topics such as *wars, sea-faring expeditions, technological advances, commercial activities, types of governments*, etc. The chronological approach is undoubtedly boring, but if a student is confronted with a discussion of the Peloponnesian War alongside the Hundred Years War without a detailed explanation of the respective social and cultural backgrounds as well as the time frame, she is hardly to blame if she feels totally disoriented and harbors a gross misconception of Athens and France as a consequence. Clearly, there needs to be some sorting out of the events to put them in historical perspective. In the same vein, once mathematical ideas and tools have been presented as the natural products of the solutions to concrete problems, there has to be a sorting out of the ideas and the affiliated tools in order that they be put in the proper mathematical perspective. The few examples above are meant to point out the need throughout the whole IMP curriculum for more mathematical discussions of this kind.

When I first looked through the IMP text, I must admit to having been mildly shocked by the many obvious omissions and the superficial character of the mathematics. After talking to the designers of the curriculum, I slowly came to an understanding of their objectives and their accomplishments. They aim this curriculum squarely at the students of Group 1, seeing as how the traditional curriculum has failed in this regard so miserably. IMP's success is therefore based on its ability to reach out to the typical student in this group and to hold his or her interest in the subject. The price of this success is that, in order not to lose this interest, IMP has to be careful not to overtax the students. This constrains IMP's ability to get into a prolonged discussion of a technical nature. Unfortunately, the kind of mathematics that can be taught given this constraint may not be the kind of mathematics that would enable President Bush to proclaim with pride to the world that "We are now number one in mathematics and science!".[20] Thus we are faced with the constant reminder that the teaching of mathematics to Groups 1 and 2

[20] See §I.

should be kept reasonably distinct.

The question remains as to whether it is possible to cram more materials into the existing IMP curriculum. For Group 2, the answer is an emphatic yes. To make room, it is sufficient to prune away some of the sidelights that are supposed to make mathematics “fun” and to reduce the large amount of informal discussions meant to motivate the various concepts. For Group 2, the “fun” part is clearly dispensable. As to the motivational material, one can say without hesitation that while this is a good thing, too much of a good thing is often counterproductive. First of all, it is not true that *everything* must be motivated before we can teach it to the students; sometimes motivation has to come later, e.g., learning how to read. In addition, after the first semester of the 9th grade, it would not be unrealistic if the motivation is merely sketched and the student is asked to think it through by himself. I believe that the amount of time thus saved would make the discussion of substantial mathematics possible.

(B) *Insufficient emphasis on technical drills.*

A colleague of mine once told me the following experience in teaching freshmen calculus at Berkeley. One day he wanted to discuss the rate at which the water falls in a container of the shape of a leaking circular cylinder lying on its side. To do that, he had to compute the volume of the water in term of its height in the cylindrical container. He had expected to spend a quick fifteen minutes to compute this volume and then to devote the rest of the hour to a discussion of the main points of the problem (the so-called *related rates*). Instead, he ended up spending the whole hour coaxing out the formula for the volume because the freshmen found the computation altogether too formidable.

I want to use this story to make the point that one of the horrors in the teaching of freshmen calculus is the students' lack of technical skills in recent years. Mathematics is a highly technical subject that can be mastered only within a firm conceptual framework, but the framework alone is not enough. Technique and understanding are the twin pillars of the subject, and neither can be slighted with impunity. In the traditional curriculum, there is often too much stress on technique, with the attendant practice of forcing the students to learn by rote. Perhaps as a reaction to this, the IMP curriculum has gone to the other extreme of overstressing understanding at the expense of technical fluency. I use the word “fluent” deliberately because IMP seems to have overlooked the rôle of mathematics as the *language* of the sciences. The horrid idea, first propounded by the New Math of the preceding generation, that understanding *per se* is devoutly to be wished seems to have been taken over lock, stock and barrel by IMP. Consequently, there are almost no drills in the homework assignments[21] and there is an unmistakable avoidance of hard computations. One can imagine that, faced with students coming out of IMP, my colleague would have to spend not one, but two hours to explain the preceding volume computation.

There must be plenty of drills. How else can one learn a language? What is more, the condescending attitude of the IMP curriculum towards the computational component of mathematics should be minimized, if not entirely obliterated. At present, this curriculum will produce students who have only a superficial understanding of the basic ideas and concepts (which it has striven so hard to explain and often so well) because this understanding is not anchored by a thorough *working knowledge* of these concepts through practice and repetition. Drills in mathematics may not be always exciting, but then neither are the drills that football players go through in training camps. Would anyone dream of trying to build a good football program just by putting the players through a daily routine of play-by-play analysis on paper? What good would the best game strategy do if the quarterback cannot pass, the wide-receivers cannot catch, and the defense unit does not know how to tackle?

(C) *Insufficient emphasis on precision.*

Mathematics, like all exact sciences, is precise. In the context of teaching mathematics to non-scientists, this characteristic of precision, while desirable, is arguably not crucial. There, perhaps the grand sweep of the overriding ideas and the qualitative aspects of the rational process take precedence. But in the context of educating prospective professional scientists, any compromise in getting this message of precision across would be a travesty of the very concept of such an education.

The IMP curriculum, the way it stands, compromises this characteristic of precision on three fronts. (1) The exposition overextends itself in the direction of chattiness and informality. This leads to sloppiness. Precise definitions are not always offered, and when they are it is often done with almost an apology. One finds examples of both kinds, for instance, all over the unit "**Leave Room For Me!**", but of course in every unit as well. In

[21] **Added July 31, 1997**: There are some drills in the 1997 published version of the IMP text.

order to make clear this point, I have listed a few of these in (A) of Appendix 2 (§VII). (2) The IMP curriculum does not make any serious, concentrated attempts at teaching students what a mathematical *proof* is all about. True, the IMP text has a few discussions of the meaning of counterexamples; this is rare and should be vigorously applauded. Nice but brief discussions of what constitutes a proof are also scattered in five or six places among the five units that I have read, and this too is a laudable feature. However, students do not learn about proofs from a few leisurely discussions[22]. The only way they learn is by prolonged exposure to good models of proofs[23] and by repeated trials and errors. In this regard, I firmly believe that the omission of (even a few weeks of) the Euclidean axioms and the classic two-column proofs is a grave error. Of course one does not wish to inflict a whole semester of two-column proofs (a là SAS, ASA, etc.) on the students; there is nothing to gain by carrying a good habit to excess. Yet a few weeks of this would be invaluable because: (1) the students need to be exposed to the fountainhead of all scientific methods at least once in their lives,[24] and (2) the two-column format makes clear to them, as nothing else would, what a proof is all about, namely, a statement backed up by a reason, followed by a statement backed up by a reason, followed by another statement backed by a reason . . . Once the students have gotten used to the two-column format and the concept of a formal proof has taken root, then they would be in a position to learn the normal way of writing down a proof in the everyday narrative style. They should be told that this switch is made, not because there is anything wrong or bad about the two- column format, but because it takes too long.[25] Too often students do not know the difference between precise, rigorous arguments and pure guesswork; the two-column format would be an excellent way to help them overcome this difficulty. In the traditional curriculum, the two-column format is badly presented as something cut off from the rest of mathematics. IMP has an unequalled opportunity in rectifying this error by following up the geometric proofs with a few weeks of transitional materials: not only

[22] Nor, may I add, from extra lectures given by their professors explicitly for that purpose.

[23] Such models are few and far between in the IMP text.

[24] I dare say that anyone who has seen Euclidean geometry at work would read with much greater understanding the popular accounts about how Einstein started off his theory of relativity with the *axiom* that the speed of light is constant.

[25] By analogy with learning how to ride a bicycle, the two- column format corresponds to the training wheels; while it is possible to teach a kid how to ride a bicycle without using training wheels, it is certainly foolish to deny access to them as a matter of policy.

the transition from two-column proofs to proofs-in-narrative-style, but also the transition from proofs in geometry to proofs in algebra. This would restore Euclidean geometry to its former exalted position in everybody's mathematical education.[26]

It should be pointed out that the **NCTM Curriculum and Evaluation Standards** (see [2] in the References) *in no way advocates the abolition of two-column proofs*. The common misconception that it does stigmatizes the author of [2] with a degree of professional incompetence that they do not deserve. What is recommended in [2] is that two-column proofs should "receive decreased attention" ([2], p.127, and especially p.159, lines 14 to 18), a position that is no different from the one above. It is, however, an oversight on the part of [2] that it misses the possibility of integrating Euclid with the rest of the high school curriculum.

(3) This particular aspect in which the IMP curriculum contributes to an erosion of the standard of precision is more difficult to encapsulate in a single phrase or sentence. It is more of an attitude, pervasive and ever present, that encourages excessive discursiveness and informality, and this attitude is of course amplified by the excessive discursiveness and informality of the IMP text itself. Since this is part of the deliberate effort by the designers of IMP to promote mathematics to the student population at large, they are already aware of this fact and it would need no further elucidation. For the benefit of the other readers of this review, let me use one or two concrete illustrations. The first one is the presence of numerous open-ended problems throughout the text. These are meant to stimulate the student's creativity and, as such, are an excellent antidote to the dull, routine drills. In practice, however, unless the problems are very carefully phrased and the teacher is exceptionally gifted and demanding, these problems probably end up misleading students into believing that incomplete guesswork is an acceptable part of their mathematical training. An indication of this can been seen from the students' portfolios made available by IMP. Since I try not to enter into technicalities in this review, I have relegated the discussion of two such problems and their suggested remedies to (B) of Appendix 2

[26] In making this recommendation of the two-column proof format as a good introduction to mathematical proofs, I do not wish to imply that it is the only valid introduction. But I do believe that it would work as well as any. (**Added July 31, 1997**: A more thorough discussion of the role of geometry in school mathematics can be found in the author's paper: The role of Euclidean geometry in high school, *J. Math. Behavior* 15(1996), 221-237.)

(§VII). A second example is the presence of numerous essay-type problems (e.g., "Describe in words those lines that can be used to cut off a small triangle which is similar to the larger one"), together with the implicit and explicit exhortations throughout the text (to the students as well as to the teachers) to write down all the ideas that ever come up. IMP prides itself on its success in getting students to write copiously about mathematics, and there is no denying the fact that the students must develop their writing skills. Therefore, to avoid any misunderstanding about what I am going to say, I will, with some misgivings, digress a little by describing my standard policy on examinations. For the past fifteen years or so, I have been passing out the following information sheet to all my students before the first midterm of each semester:

GRADING POLICY OF MIDTERMS AND FINALS
H. Wu

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I.  You are expected to write in a way that is intelligible by
    normal standards, and the hand-writing should also be legible
    by normal standards. Points will be deducted otherwise.

II  Points will also be deducted in either of the following cases:
      (a) the correct solution has to be picked out from the many
    you wrote down for that problem;

      (b) the final answer to a problem is not clearly indicated as
    such.

III If your personal assistance is needed in order to decipher
    your solution, then your solution will be assumed to be
    wrong and will be graded as such (even if it turns out to be
    100% correct).
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Let me dispose of an obvious point right away: this is my grading policy in the university, and is not meant to serve as a model for the same purpose in high school. Moreover, I am quoting this document verbatim only for the sake of authenticity; but in the process of doing this, some irrelevant elements unfortunately also creep in (e.g., the reference to hand-writing). The latter should be ignored. What I do hope to show is that I too care about good writing in mathematics, but my care extends beyond just getting the students to write a lot of complete sentences. Experience tells me it is necessary to also insist that they write in a way that is precise and to the point. Items II and III above are the distillation of many painful lessons learned from too many years of teaching undergraduates. As a rule, students have a habit of writing down whatever comes to their minds (especially when they are stuck), and unfortunately, they also expect to get credit for anything that happens to be correct. In other words, they believe in the blanket-bombing approach to learning. The idea that, as a method of solving a mathematical problem, one could write down ten guesses and be rewarded for one that happens to work, is truly frightening and is not one to encourage in the exact sciences. After all, would anyone want to cross the Golden Gate Bridge if it was built by an engineer with this kind of training? Given the objective conditions in the high school classroom, there is every reason to fear that the IMP's over-emphasis on writing everything down would lead to this kind of abuse. Granted, one must first be able get the students to write before one can discuss how best to correct their writing. With this in mind, a reasonable approach may be to leave things the way they are in the first semester of the 9th grade, but to gradually insist on a certain standard of precision and focus from that point on. Moreover, to set a good example for the students, there is nothing better than for IMP to clean up the exposition in its own official textbook.

(D) *Over-emphasis on group activities.*

Whereas the requirement of active participation in group activities may be beneficial to students in Group 1, it should be kept within bounds for students aspiring to be professionals in the exact sciences. I emphasize that this is not a plea for the *abolition* of all group activities in the IMP curriculum, only that it should receive much less attention. Indeed, every student should learn to discuss mathematics with his or her peers; this is at times even a necessity as it forces the student to think in more intuitive terms instead of being shackled forever by the rigid formalism so often unduly stressed in some texts. Yet the understanding of anything worthwhile (in science or mathematics) is on the whole an individual experience. It must come from within. In addition to group activities, the students should as well be encouraged to learn to ponder by themselves, to develop their own individuality, and to learn new materials by reading alone.[27] If a camel is the horse designed by a committee, what then is the kind of mathematics learned exclusively from compulsory group activities?[28] Is it really necessary to elevate mathematical gregariousness to a virtue? Perhaps I can do no better in making myself clear than to relate the following incident. In a weekly luncheon attended by several members of the mathematics department at Berkeley, the topic of doing homework in groups among the calculus students came up. The general consensus was that while group effort should be encouraged, it must be done in moderation. There was some strong reaction to the notion, currently very popular on campus, that makes studying in groups the end-all and be-all. However, one colleague who is a world famous topologist objected and claimed that he got through college by never reading anything and relying solely on talking to his friends to get the necessary information. His incredulous colleagues naturally pressed for the exact details. He thought hard for a few moments and had to admit that actually, *he always stayed by himself and thought through the professor's lectures first* before emerging from his private contemplation to ask his friends about all the things he did not understand.

V. Summary

I believe that the IMP curriculum, with the refinements and improvements that would inevitably come with future revisions, will make a good mathematics curriculum for Group 1[29] However, improvements along the line detailed in §III should be in place before it is put in use on a large scale. This curriculum makes mathematics more relevant and interesting to the average student in Group 1 than does the traditional one. Its inclusion of probability, statistics and linear programming is among its main selling points.

The IMP curriculum, as it stands, does not meet the needs of Group 2. The point-by-point critique in §IV is, I hope, sufficiently factual to be persuasive. As explained in §IV, the revision recommended there would not

[27] At least in 1992, the students in IMP had no text to read at home.

[28] Perhaps this judgement is harsher than is justified, but there is no other way for me to convey the impression (formed by long hours of reading through the IMP text.) that IMP is over-promoting group activities.

[29] In making this judgment, I intentionally leave out of my consideration the nontrivial problem of the proper way to reach out to the small group of students who are academically deficient and are unwilling to learn. I plead incompetence. (**Added July 31, 1997**: Having read the 1997 published version of the IMP text, I must admit that my optimism of 1992, as expressed in this footnote, is as yet unrealized. See the comments on pp. 1–2.)

be a matter of an occasional supplement here and there to add substance to the curriculum. By implication, I would hesitate to recommend a curriculum for Group 2 which consists of no more than that from IMP together with additional materials. Indeed, the revision should be one from the ground up, which effects a change in emphasis, a clearer division between precise mathematics and heuristic comments, a tightening of the exposition, and a much greater elaboration on the technical component of mathematics. I believe that such a revision, laborious as it may be, *can* be carried out without doing damage to IMP 's original vision.

This may be the right place to deal with the issue first raised at the end of §I, namely, is it not presumptuous for a university mathematics professor to pass judgment over a high school curriculum? This question was implicitly or explicitly raised by many people during the course my work on this review, and I believe it deserves an answer. Since I have little contact with the students in Group 1, before, now or later, (for the very good reason that they are essentially outside of mathematics), my comments in §III are automatically suspect. I am afraid those comments will have to speak for themselves. As far as my comments on the mathematics curriculum for Group 2 (§IV) are concerned, I ask the reader to take note of the fact that my position on that subject has been consistently that of a "quality control inspector at the end of the production line". It can be said without exaggeration that I have spent my whole professional life sampling the quality of the freshmen and sophomores that come through my classrooms. There may be many good reasons why one would want to question the inspector's judgment, but his not being in the production line himself should not be one of them. Therefore I hope that the comments I made in §IV can be judged on their own merits without any reference to my ignorance of the day-to-day operations in high school. In fact, I would go so far as to say that it is precisely here that a professional mathematician knowledgeable about the teaching of calculus can make a contribution to high school mathematics education. If such quality control inspectors had been used properly in the 1960's, could the New Math debacle have been averted?

At present,there is a palpable unhappiness on the part of some parents and students concerning the IMP curriculum. After talking to a few of them, I sensed that this unhappiness is principally due to their perception that (1) the IMP pretends to be an adequate curriculum for Group 2, and (2) the IMP is "soft", for the reasons given in (a) to (c) of §III. This is unfortunate because this unhappiness can easily blind them to the fact that the students in Group 1 would benefit more from the IMP curriculum, *when it is properly taught by mathematically competent teachers*, than from the traditional one.[30] I hope that both objections will disappear with time, or at least as soon as some consolidation by IMP of its aims and its scope has been achieved. When that happens, we will have an excellent core mathematics curriculum for the majority of the high school students.

If there is any unshakeable conviction to come out of this review, it must be that

the mathematics instruction in high school
must be separated into Groups 1 and 2.

(Again, I emphasize that the enrollment of the individual students into these groups should be determined by the students themselves and not by the school authorities.) One cannot fail to discern that a large part of the perceived failings of the IMP curriculum is due to its attempt to satisfy the quite different demands of these two groups. Any such attempt would, I believe, short-change both groups in the end. Part of the reluctance to recognize the necessity of this separation in 1992[31] may be due to the stigma of inferiority affixed to a curriculum aimed squarely at Group 1, which in turn is mostly likely caused by the absence of a solid curriculum for this very group. Naturally, I am not so naive as not to recognize the sensitive sociological, and perhaps even racial overtones that some people would read into this recommendation. However, let us consider the alternatives. Since the universities are not going to lower their standards any too soon, the high schools will have to continue teaching the kind of mathematics that we may refer to as "the Group 2 curriculum". If we insist on not making the above separation, then we will have to be prepared to force ALL the students to swallow the

[30] **Added March, 2000**: This statement was made on the assumption that the mathematical refinements and improvements as described in §III would eventually be made. Such has not turned out to be the case, but see the comments on pp. 1–2. Moreover, the actual implementation of a curriculum ultimately rests on the competence of the teachers in charge, especially their mathematical competence. For the first time in its history, California is heavily investing in the professional development of its mathematics teachers, with special emphasis on improving their content knowledge of mathematics. Perhaps ten years from now, we can expect the average teacher to have the necessary mathematical knowledge to handle IMP adequately.

[31] It is well to note that back in 1985, the need of this separation was officially registered in [4] (see the references at the end of this review). Thus do we progress.

Group 2 curriculum. Undoubtedly, improvements will be made on the existing curriculum to accommodate this change, and who knows, maybe such a drastic educational policy turns out not be a disaster afterall. But is this good education? Using the musical analogy of the opening paragraph of this review, we can rephrase this question as: do we really want to train the average music lover by the same method we use to train a prospective concert violinist? Instead of trying to answer this question myself, I will now defer to the discussion of exactly this point in a recent article by R.C. Atkinson and D. Tuzin, who are respectively the Chancellor of UC San Diego and Professor of Anthropology at the same institution (see [6] of the references at the end[32]). One of the main concerns of [6] is the sorry effect on undergraduates when programs designed solely to prepare students for graduate work *are imposed on all undergraduates regardless of whether they intend to go to graduate school or not.* Its conclusion is that when "courses designed for the few are applied to the many . . . ", "the quality of education available to the *average* student declined". It advocates the restoration of the values of a general education to the student as "a preparation for life as an informed, thinking adult." To do so, it recommends, among other possibilities, " a comprehensive **multi-track** (emphasis mine) system of departmental majors" (see the section on "Estrangement of Teaching and Research" in [6]). Here is a sensible approach to education that actively promotes "tracking" as a **positive** tool in education. If one replaces "undergraduate education" and "graduate school" in this article by respectively "high school mathematics education" and "four year college", then one would essentially arrive at the recommendation of separation into Groups 1 and 2 that I have just made. I think of this separation as a positive step in providing Group 1 with a mathematical training that would be enriching rather than demeaning. To make it work, what is needed is a good instructional program for Group 1. I believe that the IMP curriculum, *when properly modified*, could provide such an instructional program.

In the above, I have been exclusively concerned with the improvements one can make in the IMP curriculum in order to better serve both Groups 1 and 2. But I do not wish to imply that there are no other valid alternatives. On the contrary, even the traditional curriculum, with all its faults, is per-

[32] I wish to emphasize that the Atkinson- Tuzin article came to my attention *after* I had already completed the draft of this review. But this article is worth quoting because these authors obviously know a thing or two about politics in California.

fectly capable of being developed into an excellent instructional program for either group. As always, there is not one, but many paths to the truth.

At the moment, any curriculum that tries to do the impossible, i.e. teaching both Groups 1 and 2 adequately,[33] ends up wasting too much time paying lip service to both groups. Consequently, it fails, perhaps for lack of time, to address an obvious but important fact, that mathematics, far from being an esoteric and technical subject, has long played a basic rôle in our culture. How many high school students learn that a system created by a small number of Greeks more than twenty three centuries ago continues to be part of the living language of every scientist today? How many of them are ever told that one of the postulates set down by the same people was destined to play a central rôle in reshaping our perception of space through the works of Gauss, Bolyai, Lobachewsky, Riemann and Einstein? How many of them are taught that the lowly subject of quadratic equations, when fused with human imagination and the genius of Abel and Galois, was to lead to consequences which now control a good part of the theory of elementary particles, and hence also a good part of our understanding of the universe itself? And how many of them, after reading all these pedestrian accounts of mathematics weighed down by the burden of being all things to all students, are ever told that ultimately mathematics is worth learning because it is one of the supreme achievements of the human intellect, and because it has a beauty of its own much like poetry and music? Surely, if someone can make a fortune by writing about Gödel,[34] it behooves the educators to try to explain some of these central topics to the high school students. But this will not be possible if every curriculum designer has to *worry about an infinite laundry list of demands from all sides*. If we can realistically recognize the separation of the two groups, then we would have a much better chance. For example, for Group 1, we can devote a fixed amount of time and space to an exposition of the rôle of mathematics in culture without going into any technical details. On the other hand, for Group 2, such a discussion would give the right motivation to introduce the mathematics of axiomatic systems, the parallel postulate and non-Euclidean geometry, and the solvability of equations by radicals, before enlarging on these topics with the appropriate historical asides.

[33] And this includes both the IMP and the traditional curricula.

[34] The book *Gödel, Escher, Bach* by D.R. Hofstader (Basic Books, N.Y. 1979), was a national best seller for several months. One of its main concerns is the so-called Gödel incompleteness theorem in symbolic logic.

This is one experiment that deserves to be tried.

VI. Appendix 1: Why is the quadratic formula important?

Let us first recall the quadratic formula: given $ax^2 + bx + c = 0$, with $a \neq 0$, we get

$$x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right).$$

The first reason this formula is important is that its derivation uses the method of *completing the square*, which should be a basic part of every student's mathematical skill.[35] For example, later on she will be called upon to recognize that the equation $4x^2 + y^2 - 4x - 10 = 0$ defines an ellipse $4(x - \frac{1}{2})^2 + y^2 = 11$ centered at $(\frac{1}{2}, 0)$, and the only way she is going to know that is by completing the square.

Of course the most important reason for knowing the quadratic formula is that it is *the last word on the subject of solving quadratic equations*. So even if a, b, c are complex numbers, the formula will continue to yield both solutions. It is not often that a student gets to see one single formula that kills off a subject.

From another angle, this formula is important because it motivates the need for complex numbers: look at something less trivial like $x^2 + x + 1 = 0$. This then serves as the natural spring board to bring in polynomials of higher degree and the Fundamental Theorem of Algebra.

This formula is important for yet another reason: it is *algorithmic*. In this age of the computer, this fact is all the more significant. One can mention to the students the fact that there are corresponding formulas in degrees 3 and 4, but that they are far too complicated to be useful. For degrees 5 and up, of course there is no such general formula; so at this point one can try to explain to the students about Abel and Galois. The situation with degrees 5 and up in the context of the Fundamental Theorem of Algebra then affords an opportunity to explain the meaning of an *existence theorem* in mathematics. One can even discuss Newton and the latest work on the approximation of roots by computers.

Incidentally, the quadratic formula gives a concrete illustration of the abstract theorem that the roots of a real polynomial come in conjugate pairs.

[35] Professor Dan Fendel kindly informed me that in a unit of the IMP text that I have not read, one can find the technique of completing the square. This then makes the omission of the quadratic formula from the IMP curriculum all the more surprising.

Finally, the quadratic formula is important because it has very interesting applications, which would be inaccessible if we only know how to get an approximation of the roots by computers. To illustrate, I will consider two seemingly unrelated problems. The first one is: why do index cards have the approximate dimensions 3 by 5? The answer lies in the fact that the Greeks believed that a rectangle is most beautiful if the lengths of the two sides are in golden ratio. By definition, two numbers a and b (with $a > b > 0$) are in *golden ratio* if

$$\frac{a}{b} = \frac{a+b}{a}.$$

This is equivalent to:

$$\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) - 1 = 0.$$

Thus $\frac{a}{b}$ is a solution of $x^2 - x - 1 = 0$, so that

$$\frac{a}{b} = \frac{1}{2}\left(1 \pm \sqrt{5}\right). \tag{1}$$

Since $\frac{a}{b} > 0$, we can delete the root $\frac{1}{2}(1 - \sqrt{5})$. Hence two number a and b are in golden ratio exactly when

$$\frac{a}{b} = \frac{1}{2}\left(1 + \sqrt{5}\right).$$

This is approximately 1.618, which is close to $\frac{5}{3} \sim 1.666\ldots$. So we have at least explained the index cards.

Next, we considere the famous *Fibonacci numbers*: $1, 1, 3, 5, 8, 13, 21, 34, \ldots$. These numbers have a habit of showing up in the least expected places in all parts of applied mathematics. Our second problem is to get a formula for these numbers. Thus if a_n is the n-th Fibonacci number, then $a_1 = 1$, $a_2 = 1$, $a_3 = 3$, etc. and in general

$$a_{n+2} = a_{n+1} + a_n, \ \forall n \geq 1.$$

This last is a difference equation, so by the general theory (this is in every book on discrete mathematics, but [3] is nicer than most; see p.393):

$$a_n = \frac{1}{\sqrt{5}}\, r_1^{n+1} - \frac{1}{\sqrt{5}}\, r_2^{n+1},$$

where $r_1 > r_2$, and r_1 and r_2 are the roots of $x^2 - x - 1 = 0$. We have already solved this equation in (1). So we get:

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \tag{2}$$

This is an incredible formula: a_n is always a positive **integer**, whereas the right side looks like anything but an integer. In the classroom, checking through (2) for $n = 1, 2, 3, 4$ would work wonders for the students' faith in mathematics. But the important point here is that, even in the case of an equation with real roots, i.e., $x^2 - x - 1 = 0$, just knowing how to approximate the roots by computers would never yield anything like (2). *One must have the exact quadratic formula!*

Incidentally, by checking through (2) for small values of n, one can also convince the students why they need to know how to compute with square roots and fractions. This is a situation where no amount of "understanding" of the mathematics could help you if you do not know how to manipulate the symbols. Please note that the usual objection to technical drills is that they are boring and meaningless, but here is a concrete case where the manipulative skill that one acquires from drills is needed to do something very meaningful.

Beyond the obvious connection between the golden ratio and the Fibonacci numbers (established via $x^2 - x - 1 = 0$), there is a deeper one, namely,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}\left(1+\sqrt{5}\right). \tag{3}$$

Since this follows quite readily from (2), its proof will be omitted here. At the risk of belaboring the point, however, note once more that without the quadratic formula, (3) would be inaccessible. Furthermore, the proof of (3), when written out in detail, would show how a satisfying "conceptual" statement such as (3) needs to be backed up by a knowledge of manipulating the symbols.

VII. Appendix 2. Miscellany

(A) Some examples of imprecise exposition in the unit **Leave Room For Me!**.

Before we go into the details of some of the more problematic passages, a general comment is in order. The fact that calculus is presented in this unit in a very loose manner is taken to mean that this is only an early, informal excursion into calculus and therefore allowances must be made for the lack of precision. From the point of view of the uninformed student, however, everything in a text is supposed to represent solid knowledge unless it is explicitly stated to the contrary. Unfortunately, IMP does not state anything to this effect. More than that, the exposition in **Leave Room For Me!** contains some precise definitions and theorems as well as asks for precise answers and even proofs in the homework assignments. Clearly one cannot have it both ways. I will therefore treat this unit as a serious attempt at teaching calculus and judge it accordingly.

(1) p.22, lines 5 and 6: "learn more about the growth of functions; that is their rate of change". First of all, "growth of function" cannot by any stretch of the imagination or the meaning of the words be equated with "rate of change". Second, the phrase "rate of change" has no precise meaning from the point of view of everyday language, yet it is bandied about for the discussion of precise mathematics for five pages (= one day's work) without once being defined. Worse, a homework problem (which naturally asks for numerical answers) uses it. Is the student now expected to make guesses and then to act on them as if they were bona fide information?

(2) p.27, last line: "the "official" definition of slope". Since this is the 11th grade, if even at this level a mathematical definition cannot be given without an epithet in quotes, then something is wrong psychologically. It betrays a lack of conviction in mathematics itself, and it is this attitude that invites disapproval from the serious students and parents.

(3) p.38, middle of page: "average annual rate of population growth". Also p.55, next to last line: "average increase per hour". These phrases are again never defined. While their meaning is easier to guess in this case (but many will guess wrong), the very fact that the student has to guess is troublesome enough.

(4) p.63, middle of page: the *tangent line* to a graph at a point is **defined in terms of computer graphics**! This is a very serious abuse of the use of calculators in mathematics education. Since it could have been so easily avoided, the fact that it was not avoided must have been a deliberate decision. This decision then speaks for itself. See (2) above.

(5) p.67, passim: "rate of change of a function" is used as if everybody already knows its precise meaning. In fact, *nobody ever does* until she learns

what a derivative is; but on p.67, the derivative unfortunately has not yet been defined. (There is a subtle point here that is worth discussing in the context of mathematical *education*. The notion of average is a natural one, and hence so is the notion of rate of change (especially when it is constant, which is the main emphasis in the IMP text). But then the notion of instantneous rate of change must seem to be a contradiction in terms. This is why one must learn to be very precise in the teaching of calculus if cynicism on the part of the students is to be avoided.)

(6) p.67, line 10 from bottom: "the amount that the function changes per unit change in x". This is a phrase worthy of being included in the calculus texts of the early part of the 19th century, when the notion of the derivative was not yet understood. Alas, this is 1992.

(7) p.94 ff.: the way *inverse function* is defined, every function would admit an inverse function, which is absurd.

(B) Comments about two open-ended problems.

(1) THE BROKEN EGGS. As a problem for Group 1, it should not be enough for the students to "play with numbers until they come up with something like 301". Incentive should be given to the teacher to clearly explain that there is a general method to solve this problem *completely*, and also to give the complete solution. As a problem for Group 2, it would be better to simplify it (e.g., one left over when she put them into 2's, and two left over when she put them into 3's) and ask explicitly for the *complete* solution. This then becomes a problem entirely within the reach of every student (in Group 2), and it also gives them a clear idea of what kind of performance is expected of them.

(2) HOMEWORK #18 IN **SHADOWS**. Again, the way the problem is phrased does not encourage the student to look for *all* possible solutions and to justify her answer. Why not say instead: describe all the possible ways of drawing lines to cut off small triangles that are similar to the given triangle, and justify your answer.

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