How good are the Common Core Mathematics Standards?*

Concord, CA

October 29, 2012

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*This is the text of a presentation at the SF East Bay CCSS Summit. I am indebted to R. A. Askey for a critical remark, and to Larry Francis for many corrections and improvements.

California has one of the best sets of math standards in the nation, maybe the best.

Why did it adopt the **CCSSM** (Common Core State Standards in Mathematics)?

Because a set of standards is a very complex object. No matter how good, there is room for improvement.

The CCSSM are not perfect by any means, but they are much better than the CA standards in the one area that truly matters: they prescribe how better mathematics can be taught in the classroom.

Our nation has had a **de facto national curriculum** in mathematics for several decades: it is the curriculum embedded in the textbooks.

Be it *reform* or *traditional*, the underlying mathematics of this curriculum has essentially remained the same, and it is invariably defective.

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We call the mathematics of this de facto national curriculum **TSM** (Textbook School Mathematics).[†]

[†]See H. Wu, Bringing the Common Core State Mathematics Standards to Life. *American Educator,* Fall 2011, Vol. 35, No. 3, pp. 3-13. http://www.aft.org/pdfs/americaneducator/fall2011/Wu.pdf.

No set of standards, state or national, challenged TSM until the CCSSM confronted TSM and called for systemic change.

This is where the CCSSM surpass the CA standards.

It would take several days to detail all the flaws of TSM.

In this short presentation, I can only give you two examples. One is the defective treatment of the slope of a line, and the other is the use of **LCD** (Least Common Denominator) in the definition of the sum of two fractions.

But first, a digression.

On October 9, Bill Evers of the Hoover Institute circulated an article by Lance T. Izumi of the Pacific Research Institute for Public Policy (PRI) entitled, *Romney bashes Obama's national ed standards that dumb down CA math.*

In this article, Izumi calls the CCSSM the "less rigorous national standards supported by President Obama" and "the lower national standards".

Izumi further opined that the CCSSM have “altered and distorted math education in California”.

He, like many others, is particularly concerned about the fact that the CA standards mandate Algebra in grade 8 whereas the CCSSM grade 8 standards are merely “pre-Algebra standards”.

So this is the crux of the matter: the CCSSM only have “pre-Algebra standards” in grade 8 whereas CA, like a few other states (MA, MN, etc.), claims to do full-fledged algebra.

According to the critics, the CCSSM are intellectually less challenging.

These critics know nothing about how TSM has crippled our nation's math education. They latch onto a slogan such as "Algebra in grade 8" and make it the benchmark of excellence.

But what is the reality behind the slogan?

Let us make a direct comparison of the grade 8 standards in CA and the CCSSM.

What they have in common:

Linear equations; simultaneous linear equations; functions and their graphs. *(These are topics in algebra.)*

Where they differ:

	CA	**CCSSM**
absolute value inequalities	$\surd$	HS alg(?)
polynomials	$\surd$	HS alg
rational expressions	$\surd$	HS alg
graphing linear inequalities	$\surd$	HS alg
quad. eq. and quad. formula	$\surd$	HS alg

	CA	**CCSSM**
Intuitive geometry: rotations, reflections, translations, and dilations	no	$\surd$
concepts of congruence and similarity	no	$\surd$
Correct def of slope of a line	no	$\surd$
reason why graph of $ax + by = c$ is a line	no	$\surd$
proof of Pythagorean Th. and converse using similarity	HS geom	$\surd$
volume formulas for cones, cylinders, and spheres	HS geom	$\surd$

Key observation: *The CCSSM in grade 8 do not teach less than the grade 8 CA standards.*

Rather, there is a trade-off: each does some things that the other doesn't.

The CCSSM give up:

polynomials,

quadratic equations,

in exchange for doing the following geometric topics:

rotations, reflections, translations,

congruence,

dilation and similarity.

Why this trade-off?

In order to learn about *linear equations and their graphs*, a major topic of Algebra I that is *taught by both CA and CCSSM in grade 8*, students must come to grips with the concept of the **slope** of a line.

The CCSSM try to provide students, perhaps for the first time, with the necessary geometric tools *to learn what slope really means.*

This explains why the CCSSM make the trade-off.

Am I saying that the CA standards—like all other standards in the nation—do not provide students with the necessary geometric tools to learn about slope?

Yes. This is TSM at work.

Am I saying that the CA standards—like all other standards in the nation—do not provide students with the necessary geometric tools to learn a major topic of Algebra I?

Yes. This is TSM at work.

Am I saying that the CA standards—like all other standards in the nation—make students learn linear equations and their graphs by rote?

Yes. This is TSM at work.

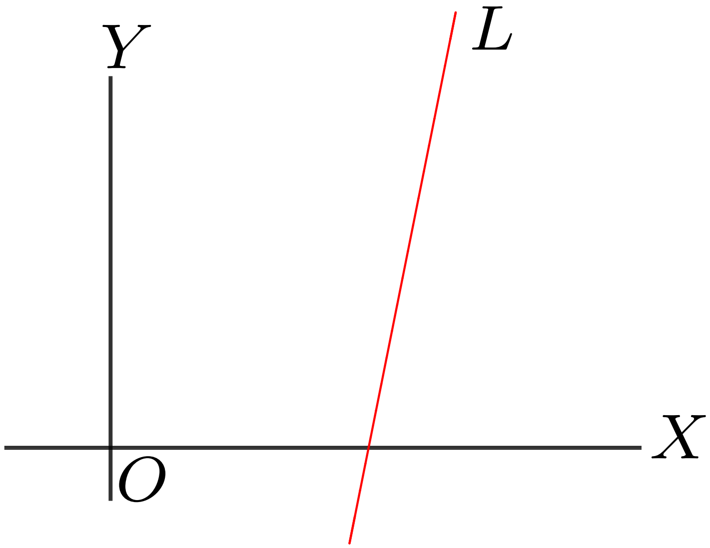
Am I saying that this glaring omission in the CA standards—like all other standards in the nation—is what allows the CA standards to prescribe Algebra I for grade 8?

Yes.

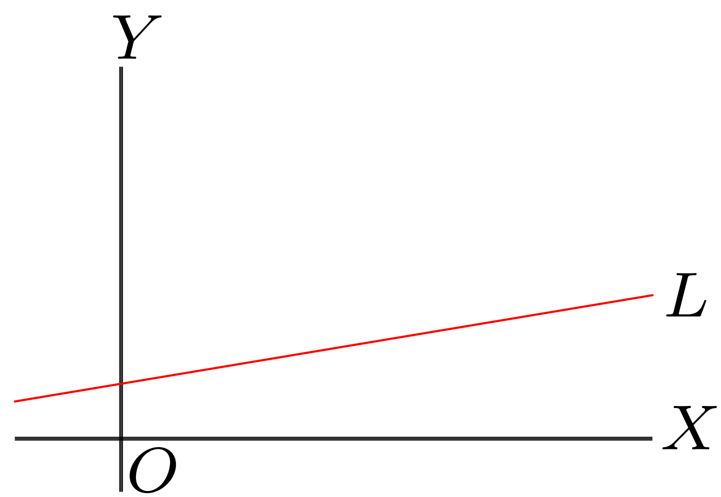
But the CCSSM try to fill the gap.

The intuitive meaning of slope:‡

Large slope (very steep)



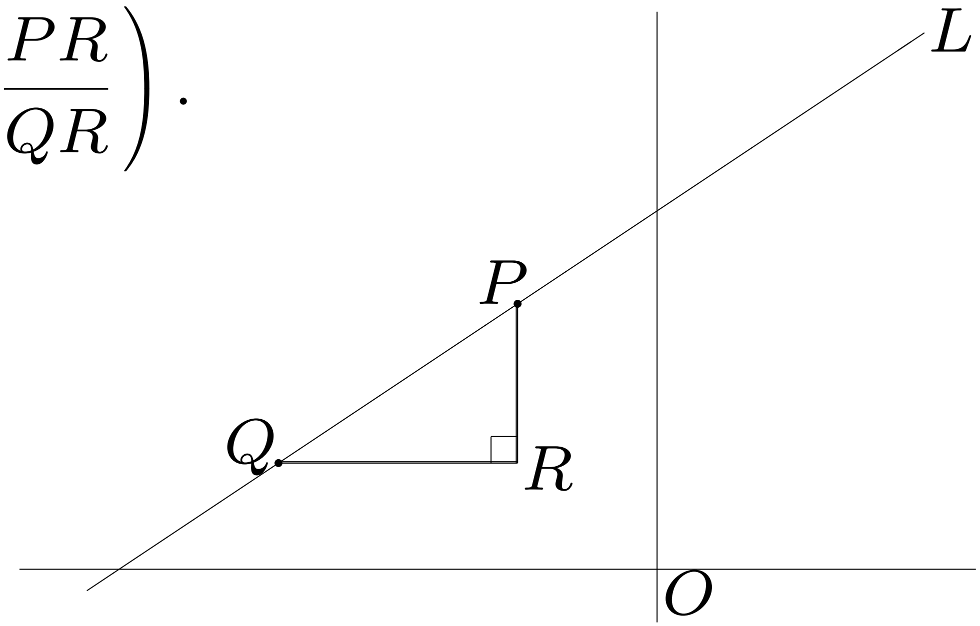
Small slope (almost horizontal)



‡For simplicity, I limit myself to lines leaning to the right in the whole discussion.

Let L be a nonvertical line in the coordinate plane and let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be distinct points on L. According to TSM, the definition of the **slope of L** is

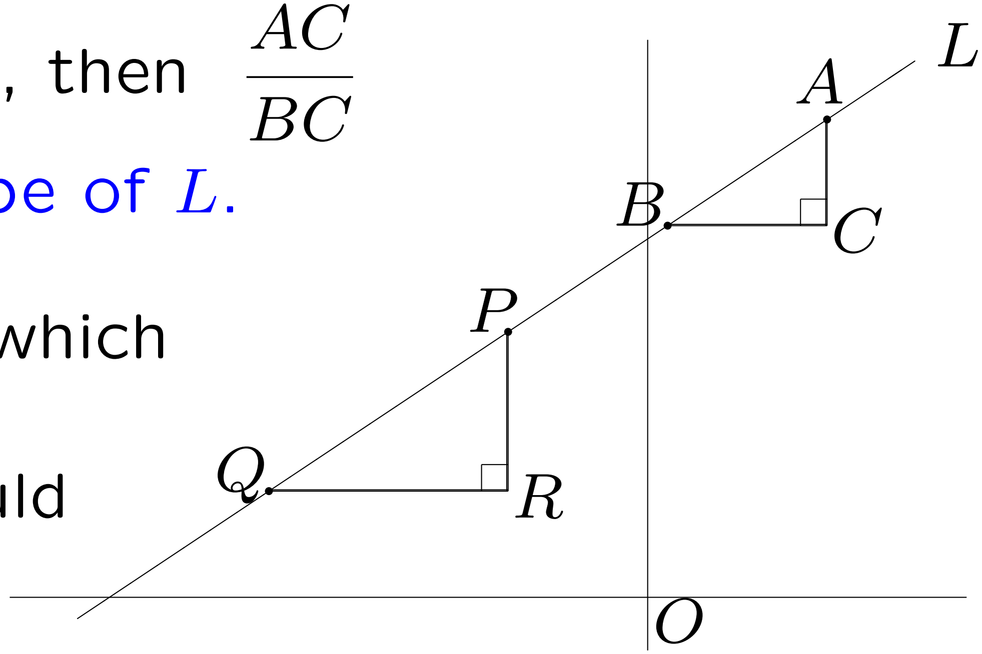
$$\frac{p_2 - q_2}{p_1 - q_1} \; \left(= \; \frac{PR}{QR} \right).$$



Lines have lots of points. What if two different points A and B on L are chosen instead?

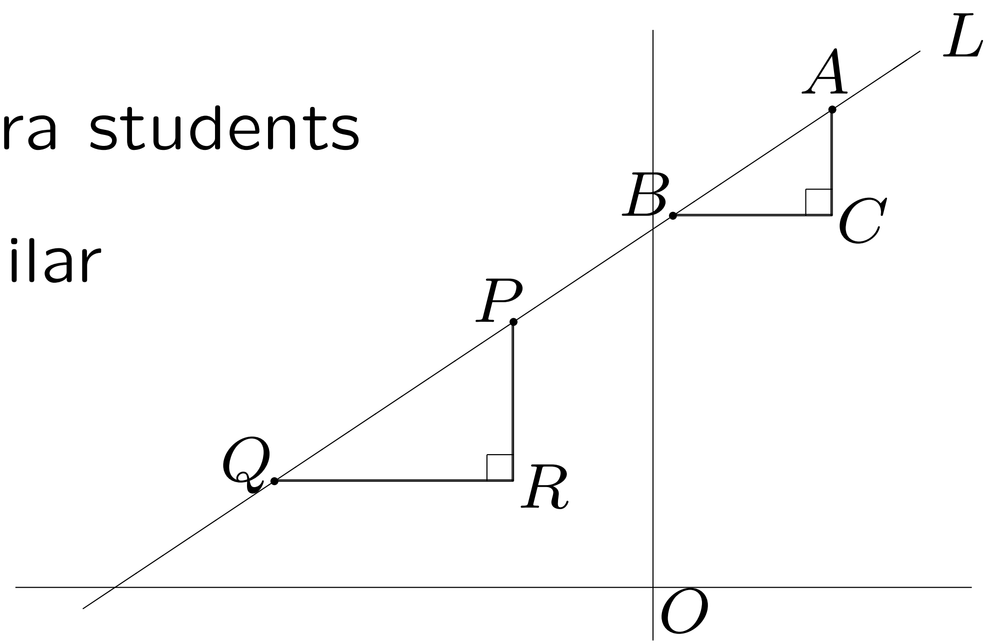
With A and B chosen, then $\dfrac{AC}{BC}$ would also be the slope of L.

But if $\dfrac{PR}{QR} \neq \dfrac{AC}{BC}$, which of the two ratios should be *the slope of L?*



Fortunately, $\dfrac{PR}{QR} = \dfrac{AC}{BC}$, and this is because $\triangle ABC \sim \triangle PQR$ (*similar triangles*).[§]

This is why algebra students need to know similar triangles.



[§]See, e.g., Section 4 of H. Wu, *Introduction to School Algebra*, http://math.berkeley.edu/~wu/Algebrasummary.pdf.

At the moment, students are *not* shown that

$$\frac{PR}{QR} = \frac{AC}{BC}$$

for *any* P, Q, A, B on the line.

They are simply told to memorize the definition of slope using two fixed points. They do not know that moving these points to another pair of points (on the line) does not change the slope.

We have been teaching a major topic of algebra across the nation *by rote*. Students also have had no choice but to learn *by rote.*

This is TSM at work.

This way of teaching slope by rote has serious consequences in mathematics learning.

According to a recent survey of *students' understanding of lines in algebra* by Valentina Postelnicu and Carole Greenes (Winter 2011-2012 issue of the *NCSM Newsletter*), the most difficult problems for students are those requiring the identification of the slope of a line from its graph.

No wonder.

If students do not realize they can use *any* two points on the line to compute its slope, they will naturally be confused about “how to measure rise and run.”

There is also anecdotal evidence to suggest that students cannot relate the slope of a line to the equation of the line because, once its slope has been computed using two fixed points (p_1, p_2) and (q_1, q_2), students do not know what to do with the "rise and run",

$$\frac{y - p_2}{x - p_1},$$

computed by using another pair (x, y) and (p_1, p_2).

As things stand right now, all the states which claim to be teaching Algebra I in grade 8 (including **California**) do so by making all their students learn a major topic of algebra by rote.

Isn't this too high a price to pay?

Let us revisit the choices posed by Izumi's article:

(I) Get all Algebra I topics done in grade 8 at all costs, including forcing students to learn a major topic by rote. (CA)

(II) Develop algebra naturally by respecting reasoning, and do only what is possible in grade 8. (CCSSM)

Is (II) a "dumb down" of (I)? Is (II) "“less rigorous" than (I)? **Of course NOT.**

Would it be correct to say that, by following **(II)**, the CCSSM are obstructing the teaching of algebra in grade 8?

On the contrary, the CCSSM are making the learning of algebra possible for all students for the first time.

Let us look at another example of how TSM distorts mathematics in the subject of fractions.

The twin pillars that support algebra-learning (according to the 2008 National Mathematics Advisory Panel Report) are similar triangles and rational numbers.

Now look at the teaching of rational numbers, in particular, *adding fractions.*

There are two CA standards on the addition of fractions:

Gr5 NS 2.0 Students perform calculations and solve problems involving addition, subtraction, and simple multiplication and division of fractions and decimals:

> 2.3 (*Paraphrase*) Solve simple problems, involving the addition and subtraction of fractions and mixed numbers.

Gr6 NS 2.0 Students calculate and solve problems involving addition, subtraction, multiplication, and division:

2.1 (*Paraphrase*) Solve problems involving addition, subtraction, multiplication, and division of positive fractions.

These standards are both mathematically accurate and grade-level appropriate, but they serve as an object lesson in how mathematically correct standards can be sabotaged by TSM.

Because they did not take TSM into account, these well-intentioned standards did not lead to good lessons on the addition of fractions.

How to add $\frac{7}{8}+\frac{5}{6}$:

TSM: Find the LCD of 8 and 6, which is 24. Note that $24 = 3 \times 8$ and $24 = 4 \times 6$. Therefore

$$\frac{7}{8}+\frac{5}{6} \;=\; \frac{(3 \times 7) + (4 \times 5)}{24} \;=\; \frac{41}{24}$$

Did that make any sense to you?

Adding is supposed to "put things together". But did you see any "putting together" in this description of $\frac{7}{8} + \frac{5}{6}$?

How is a student who has just mastered adding whole numbers supposed to learn this kind of "addition"?

The CCSSM approach to the addition of fractions is to take into account the likely distortion by TSM and carefully *prescribe* how addition should be taught.

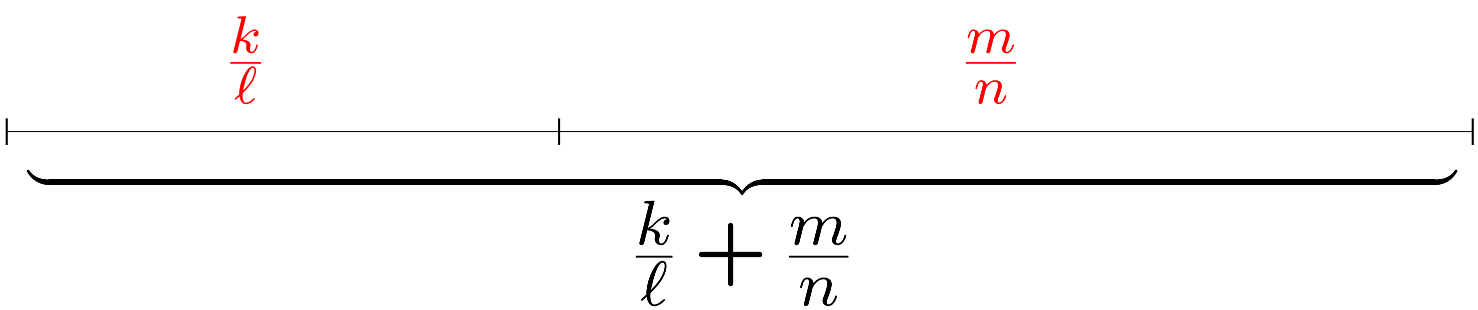
Grade 3 Understand a fraction as a number on the number line; represent fractions on a number line diagram. Explain equivalence of fractions in special cases.

Grade 4 (*Paraphrase*) Explain why a fraction $\frac{a}{b}$ is equivalent to a fraction $\frac{na}{nb}$ by using visual fraction models. Define addition of fractions as joining parts referring to the same whole. Then for two fractions with like denominators, $\frac{m}{n} + \frac{k}{n} = \frac{(m+k)}{n}$.

Grade 5 Add and subtract fractions with unlike denominators by replacing given fractions with equivalent fractions, so that we have fractions with like denominators. For example, $\frac{2}{3}+\frac{5}{4}=\frac{8}{12}+\frac{15}{12}=\frac{23}{12}$. (In general, $\frac{a}{b}+\frac{c}{d}=\frac{(ad+bc)}{bd}$.)

Brief illustration of these standards: fractions are identified with lengths of segments on number line.

First, the meaning of $\frac{k}{\ell} + \frac{m}{n}$: It is the length of the segment obtained by joining a segment of length $\frac{k}{\ell}$ and another of length $\frac{m}{n}$:



This definition of addition is *a direct extension* of the meaning of *adding whole numbers:* combining things.

We will use the language of **“5 copies of $\boldsymbol{\frac{m}{n}}$”** to refer to the joining of 5 segments each of length $\frac{m}{n}$.

In this language, a fraction such as $\frac{5}{7}$ is just the length of 5 copies of $\frac{1}{7}$.

Thus

$$\frac{4}{7}+\frac{8}{7} = \frac{4+8}{7} = \frac{12}{7},$$

because *the joining of 4 copies of* $\frac{1}{7}$ *and 8 copies of* $\frac{1}{7}$ *is clearly 12 copies of* $\frac{1}{7}$*.*

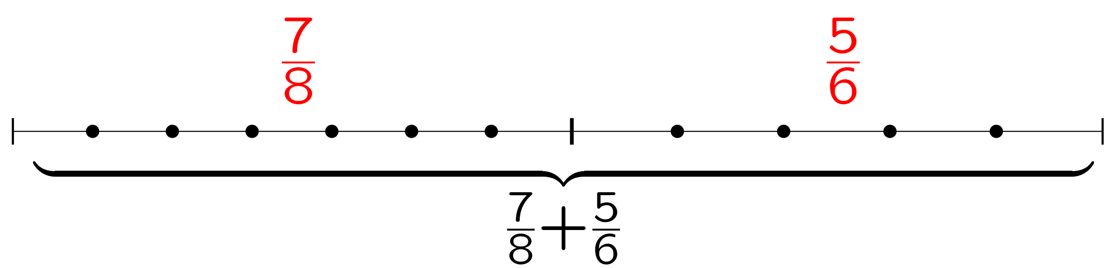
For the same reason, for any $\frac{k}{n}$ and $\frac{m}{n}$,

$$\frac{k}{n}+\frac{m}{n} = \frac{k+m}{n}$$

Now what is $\frac{7}{8} + \frac{5}{6}$?

We want the total length of the joining of 7 copies of $\frac{1}{8}$ and 5 copies of $\frac{1}{6}$.

Trouble is: we don't know how to relate $\frac{1}{8}$ to $\frac{1}{6}$.



No problem: by equivalent fractions, **we may regard $\frac{1}{8}$ and $\frac{1}{6}$ as two fractions with the same denominator:**

$$\frac{1}{8} = \frac{6 \times 1}{6 \times 8} = \frac{6 \times 1}{48}$$

$$\frac{1}{6} = \frac{8 \times 1}{8 \times 6} = \frac{8 \times 1}{48}$$

So $\frac{1}{8}$ is (the length of) 6 copies of $\frac{1}{48}$, and therefore,

$$\begin{aligned} \frac{7}{8} &= 7 \text{ copies of } \tfrac{1}{8} \\ &= 7 \times 6 \quad \text{copies of } \tfrac{1}{48} \\ &= \frac{42}{48} \end{aligned}$$

Similarly, $\frac{1}{6}$ is (the length of) 8 copies of $\frac{1}{48}$, and

$$\frac{5}{6} = \frac{8 \times 5}{48} = \frac{40}{48}$$

Now we can add:

$$\frac{7}{8}+\frac{5}{6} \;=\; \frac{42}{48}+\frac{40}{48} \;=\; \frac{82}{48}$$

In summary, the addition was done

(1) by *joining* segments, thereby reinforcing the concept of "adding", and

(2) without using LCD.

Of course, $\frac{82}{48} = \frac{41}{24}$ (by equivalent fractions), but the simplification is merely cosmetic, *not necessary*.

Altogether, these standards in the CCSSM guide students through three grades to get them to know the meaning of adding fractions: Addition *is* joining things together, even for fractions.[¶]

In particular, *NO LCD.*

[¶]For all this, see Chapter 14 of H. Wu, *Understanding Numbers in Elementary School Mathematics*, Amer. Math. Society, 2011.

The use of LCD is a special skill that *can be useful* in computations, but *it has no place in the* **definition** *of adding fractions.*

It is also bad pedagogy to introduce LCD into the *definition* of adding fractions. Students confuse LCD with GCF, and one should avoid discussing *two* difficult concepts (addition and LCD) at the same time.

There is also a valid reason in advanced mathematics (*about quotient fields of integral domains*) as to why it is *incorrect* to invoke LCD for the definition of adding fractions.

Let us backtrack to Izumi for a moment. Can we really believe that such improvements in classroom instruction would

alter and distort math education in California?

We have just seen two ways in which the CCSSM differ from other standards:

> They restore reasoning and reaffirm the logical structure of mathematics when both are missing in other standards.
>
> They anticipate the corrosive nature of TSM and provide incisive guidance on the proper course of correction.

The goal of the CCSSM is to see correct mathematics materialize in math classrooms.

But it will not materialize unless we have teachers who have the requisite content knowledge to make it happen: how to teach the addition of fractions properly (without LCD), how to define the slope of a line correctly (without reference to two chosen points), etc., etc.

At the moment, we have let our teachers down,

in college, by not helping them overcome their baggage of TSM,[‖]

in the field, by not providing them with content-based PD (professional development).

[‖]See H. Wu, The Mis-Education of Mathematics Teachers, *Notices Amer. Math. Soc.* 58 (2011), 372-384. http://math.berkeley.edu/~wu/NoticesAMS2011.pdf.

We need sustained **content-based** professional development for our math teachers (and a rigorous assessment system for students) in order to benefit from the CCSSM. No pain, no gain.

Sustained content-based professional development in mathematics requires immense funding and real expertise. At the moment we seem to have neither. So we must be willing to work our way up from scratch.

For CCSSM:

To adopt or not to adopt, that is not the question.

The question is whether California has the wisdom and the resolve to do the right thing for its children in K-12 mathematics: to commit to a long-term process of incremental implementation of the CCSSM in order to realize the vision of those standards.