The Critical Foundations of Algebra

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I will briefly discuss in this article what the National Mathematics Panel (NMP) report considers to be the critical foundations of algebra. The term *"critical foundations of algebra"* refers to the knowledge that students need in order to achieve algebra. The discussion of algebra itself is given in Chapter 3 of [NMP2] and in [Schmid-Wu]. Recall that the purpose of the NMP report is to foster greater knowledge and improved performance in school algebra by use of the best available scientific evidence.

The reason for the emphasis on school algebra is that it is in algebra that school students must face unflinchingly the concepts of abstraction and generality, the hallmarks of advance mathematics. Algebra is therefore their entry point into advance mathematics, science, and engineering.

A noteworthy feature of the NMP report is that all its considerations are based specifically on curricular content. I will give two examples. Whereas it is common to discuss the learning of fractions on the basis of the general psychology of learning (cf. the comments in [Wu2], pp. 31–33) as if fractions can only be learned through hands-on activities and diverse examples but not logical reasoning, the Learning Processes report ([NMP2], Chapter 4) bases the discussion of learning fraction on a clear definition of a fraction using the number line and on the use of mathematical reasoning. A second example is that, while discussions of assessment often center on psychometrics, and *only* psychometrics, the NMP Assessment report ([NMP2], Chapter 8) *begins* with a discussion of the need to achieve a good match of the mathematics assessment in grades K–8 with the critical foundations of algebra.

I will limit my discussion of the critical foundations of algebra to what is given in the Conceptual Knowledge and Skills report ([NMP2], Chapter 3), and will only touch lightly on the related issues in the Teachers and Teacher Education report ([NMP2], Chapter 5) and the Assessment report.

The Conceptual Knowledge and Skills report begins with a definition of school algebra (see The Major Topics of School Algebra, [NMP2], p. 3-4 to p. 3-15; see also [NMP1], p. 16, and especially [Schmid-Wu]). This definition is central to the whole NMP report, and in particular to the present discussion. Indeed, *without knowing precisely what school algebra is, it is meaningless to talk about how to prepare students for algebra.* The discussion below is, at each step, controlled by The Major Topics of School Algebra and what they demand of each learner.

In the most general terms, a serious problem in the current American mathematics education is that its K–7 curriculum is *flat*, in a sense to be explained. A main obstacle in the learning of school mathematics has always been how to cope with the increase in abstraction and cognitive complexity as one progresses through the grades. While the increase in cognitive complexity is, by design, true of all subjects, the problem of increasing abstraction may be special to mathematics and science. The whole number symbols 0, 1, 2, ... are already an abstraction, to be sure, so learning whole number arithmetic requires abstract thinking. But the leap from whole numbers to algebra, in terms of abstraction, is a different order of magnitude altogether. It may be said that a main concern of the NMP report is to help students negotiate this giant leap. We can explain this leap more fully, as follows. What students learn in whole number arithmetic, and thus in grades K–5 more or less, is how to compute with explicit numbers. For example, $27 \times 48 = ?$ Or, what is the division-with-remainder of 277 by 13? On the other hand, a typical concern in algebra is to learn the following identity and its applications:

$$(1-x)(1+x+x^2+\cdots+x^n) = 1-x^{n+1} \quad \text{for } \textit{all} \text{ numbers } x$$

Because x can assume *any* value, no explicit computation is possible in this case. Instead, one must forgo the explicitness and rely on *general* properties of numbers as a whole, such as the associative and commutative laws for $+$ and $\times$ and the distributive law, to verify that the numbers on both sides are indeed the same. In due course, one will also argue that the same reasoning shows that the identity is true when x is a square matrix, or a complex number, or a whole host of other possibilities. Another example from algebra is the concept of a function: instead of looking at a few numbers at a time and compute with them, students must now consider two infinite collections of numbers (the domain and its range) and the relationship between them, not necessarily through explicit computations but usually in abstract terms, e.g., the square root function or the logarithmic function. The difference between such considerations and whole number arithmetic cannot be more stark, and a beginner in algebra should rightfully feel disoriented.

In view of this difference, it would be a good curricular strategy to acclimatize students to the new algebraic realities by gradually introducing an appropriate amount of abstraction and generality all through grades grades 5–7. A natural way to do so would be to give fractions (positive or negative) their due by teaching a fraction as an abstract concept and use it as a ramp to lead students gently from arithmetic to algebra (see [Wu1] and the Conceptual Knowledge and Skills report in [NMP2], page 3-41). To give one simple example of how fractions can serve this purpose, the rule of adding fractions, when stated properly, would be the statement that *for all* fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

In this way, the symbolic notation and the concept of generality are introduced naturally, and some form of *proof* of this general fact will have to be given.[1] As another example, a sixth grader can be taught the cross-multiplication algorithm as the statement that

$$\textit{for } \textsf{all} \textit{ fractions } \frac{m}{n} \textit{ and } \frac{k}{\ell}, \quad \frac{m}{n} = \frac{k}{\ell} \quad \textit{if and only if} \quad m\ell = kn.$$

In the process, students learn about what it means for two fractions to be equal (they are the same point on the number line), learn the meaning of the standard mathematical statement "if and only if", and learn to use symbols to express generality. They are again led to a *proof*, which can be given in a way that is grade-level appropriate.

However, none of this is usually found in standard textbooks. What we find, instead, is that most American curricula of grades 5–7 downplay the inherent abstraction of the subject of fractions and teach it exclusively by the use of metaphors, analogies, and manipulatives. Precise reasoning therefore goes out the window. For example, if a (positive) fraction is a piece of pizza, students will have to fantasize how one may divide one piece of pizza by another, and how *that* may be somehow related to invert-and-multiply. If they cannot make such a fantastic leap of the imagination, then they are told that they lack conceptual understanding. If students ask for help, well then, give them more metaphors and more sophisticated manipulatives regardless of whether they help or not. And so on. The K–7 mathematics curriculum is therefore basically stripped of abstractions, and *general* statements about numbers are kept at bay. It is in

[1] Incidentally, notice that the lowest common denominator is not mentioned, but that is a different discussion.

this sense that we say the K–7 mathematics curriculum is flat. *Students are prevented by the curriculum from being ready for algebra.*

Beyond the issue of abstraction and generality, let us see in greater detail the critical role played by fractions in the learning of algebra. A large portion of school algebra is concerned with the solution of equations; this requires, at the very least, the ability to fluently compute with fractions. For example, to solve the equation

$$\frac{1}{5}x + 7 \;=\; \frac{1}{3}x - 4$$

is to assume that there is *some* number x that satisfies the equation, and then treat both sides of the equation as any two arithmetic expression and try to bring the x terms to the left and the constants to the right. So, adding $-\frac{1}{3}x$ to both sides gives

$$-\frac{1}{3}x + \frac{1}{5}x + 7 \;=\; -4.$$

Then adding -7 to both sides yields

$$-\frac{1}{3}x + \frac{1}{5}x \;=\; -4 - 7.$$

By the distributive law for positive and negative numbers,[2] we obtain for the left side

$$\left(-\frac{1}{3} + \frac{1}{5}\right) x \;=\; -\frac{2}{15}\,x\,,$$

so that

$$-\frac{2}{15}x \;=\; -11\,.$$

Multiplying both sides by $-\frac{15}{2}$ gives

$$x \;=\; (-\frac{15}{2})(-11) \;=\; \frac{165}{2}$$

We have therefore derived the *necessary condition* that IF the equation $\frac{1}{5}x + 7 = \frac{1}{3}x - 4$ has a solution, it must be $\frac{165}{2}$. (To verify that this *is* a solution, one must compute once more, a fact conveniently left out in textbooks.)

By describing the calculations in such excruciating details, we have succeeded in exposing clearly the need for students to be totally at ease with the arithmetic of positive and negative fractions in solving equations. Therefore, without the ability to compute

[2] In other words, rational numbers.

fluently with fractions, students have no hope of learning algebra.

At the moment, the reluctance in K–7 to engage in abstractions and allow the general laws of operations (associative, commutative, and distributive) to guide the discussion of negative numbers has led to an uncomfortable amount of learning-by-rote. The study of negative fractions has become one of memorizing a large number of rules with no reasoning, so that the question why it is true that

$$(\text{negative}) \times (\text{negative}) = \text{positive}$$

is, by anecdotal evidence, the Most Frequently Asked Question in the mathematics of grades 6–7.

Before one even gets to negative fractions, of course one must first study (positive) fractions. Unhappily, the non-learning of fractions is the bottleneck of the elementary school mathematics curriculum.

At the risk of harping on the obvious, let us briefly go over one of the main reasons that makes the learning of fractions such a huge challenge for children. Children do not have serious problems dealing with whole numbers because, given their intimate knowledge of their fingers, they get the illusions that whole numbers are things they "know". Even if they cannot do certain computations, they do not feel completely out of it. On the other hand, most fractions do not show up on command. For example, what is a natural manifestation of $\frac{2}{7}$ or $\frac{11}{9}$? A fraction is therefore inherently an abstract concept, something that is, at best, on the fringe of children's intuition. Mathematicians have learned through bitter lessons spread over almost two hundred years after Newton and Leibniz that, for abstract concepts, any understanding must be accompanied by *precise definitions* (to pin down what we are talking about) and *precise reasoning* (to insure that we are not talking nonsense). The essence of these basic principles (though not necessarily the attendant mathematical formalism) must therefore guide the teaching of fractions. But is there any scientific evidence for this claim beyond the historical one? A qualified yes, if we argue by elimination. We know from decades of experience that teaching fractions to students without definitions and without *precise mathematical reasoning* cannot get the job done. The pandemic of fraction-phobia among school children (see the comic strips of Peanuts and FoxTrot) should be eloquent testimony to the fact that the old way must go. Now that the use of the number line as the foundation for the presentation of fractions has been shown to make possible the formulation of definitions and the infusion of reasoning at every step (see [Wu2] and the Learning Processes report, [NMP2], page 4-40 ff.), such a presentation should be a legitimate first step in our re-thinking of this part of the mathematics curriculum. What is more relevant is the fact that, through the use of the number line, the subject of fractions achieves an internal coherence that connects all the concepts and skills via mathematical reasoning. It is this coherence that is so vividly emphasized in the NMP Final Report ([NMP1], pp. xvi–xvii, 20–22). We repeat, ***if we want students to achieve algebra, we cannot allow fractions to be presented, as it is commonly done, as a collection of factoids held together only by hands-on activities and manipulatives.***

There are two more curricular issues to discuss. The first concerns whole numbers. Fractions are an extension of whole numbers, and as such, a knowledge of whole numbers is a prerequisite to the learning of fractions. Before students take up fractions in earnest, usually in the fifth or sixth grade, they must be fluent in the four arithmetic operations with whole numbers. Such fluency, in particular, requires that the 10×10 multiplication table be committed to memory. As with fractions, whole numbers should also be taught in a coherent manner. All four algorithms should be be explained in terms of place value, and all four should be shown to be nothing but different manifestations of a single *leitmotif*: how to reduce all whole number computations to computations with single-digit numbers. One should emphasize not only the coherence within whole numbers but also the coherence in going from whole numbers to fractions: the four arithmetic operations for whole numbers and those for fractions are *conceptually identical*. Such coherence maximizes students' long-term retention of what they learn (see the Learning Processes report, [NMP2], page 4-7).

A second issue pertains to geometry. Some knowledge of geometry is necessary for the understanding of algebra. Basic facts about areas and volumes are obviously needed for doing some of the standard word problems, and these are usually found in most curricula of grades K–7. What is often missing is a working knowledge of similar triangles that is needed for an understanding of the relationship between linear equations in two variables, $ax + by = c$, and their graphs. For example, why are these graphs straight lines? Without this knowledge, the ability to write down the equation of a line satisfying certain geometric data (passing through two given points, passing through a given point with a given slope, etc.) becomes a matter of brute force memorization, hardly the right way to learn algebra or, for that matter, any kind of mathematics. Adequate preparation for algebra must therefore include a knowledge of what it means for two triangles to be similar as well as some basic criteria of similarity (e.g., AAA). At the moment, many state mathematics curricula do not provide students the opportunity to acquire this knowledge before they come to algebra.

It is pertinent to point out that learning about similar triangles in grades 5-7 does not require that every statement be proved. After all, one does not teach calculus by introducing epsilons and deltas before teaching differentiation or integration. It is enough that students feel comfortable working with similar triangles.

I would like to make a comment about why there has been no explicit discussion of *problem solving* in this article thus far. It would appear that in the average school mathematics classroom, mathematical reasoning does not make its appearance often. Because reasoning is synonymous with problem solving, one may surmise that "problem solving" then become the code words in mathematics education for reasoning, and the emphasis on problem-solving is nothing more than a plea for the inclusion of reasoning in the curriculum. However, it is a fact that essentially all of mathematics is an unending chain of problem solving. The preceding emphasis on coherence and logical reasoning all through the curriculum therefore automatically carries with it the emphasis on problem solving at each step. Far from neglecting problem solving, we are asking that the critical foundations of algebra be taught in a way that respects the integrity of mathematics, not *sometimes*, not *often*, but *all the time*. In other words, we are asking that problem solving be an integral part of every facet of mathematics instruction.

Let me conclude by making two brief references to the Teachers and Assessment reports. Student achievement is highly correlated with the quality of teaching. While we cannot yet, at this juncture, define precisely what accounts for effective teaching, it should be non-controversial to claim that effective teaching must *include* knowing the mathematics one teaches. The problem of how to empower our middle school teachers with the requisite knowledge about fractions, whole numbers, and similar triangle remains a major issue in mathematics education. As to assessment, NAEP carries a great deal of weight in promoting better mathematics education in many states, and this is all the more true in the absence of a national curriculum. With this in mind, the Assessment Task Group suggests that NAEP should try to focus on the critical foundations of algebra in the NMP report at least for grades 4 and 8, the two grades in which NAEP conducts assessment. Furthermore, the Task Group recommends that the NAEP assessment items be of the highest *mathematical* as well as psychometric quality. To this end, the recommendation is that mathematicians should be involved in greater numbers in the review and design of test items.

References

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