Introduction to School Algebra [Draft]

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July 24, 2010

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Preface

This is a set of lecture notes on introductory school algebra written for middle school teachers. It assumes a knowledge of pre-algebra as given in the Pre-Algebra Institute of 2009:

http://math.berkeley.edu/~wu/Pre-Algebra.pdf
(hereafter referred to as the *Pre-Algebra notes*)

There is nothing fancy about the content of these lecture notes. With minor exceptions, it covers only the standard topics of a first year algebra course in school mathematics (grade 8 or grade 9). These notes may therefore be called **Introductory Algebra from a Somewhat Advanced Point of View.** If there is any merit to be claimed for them, it may be the sequencing of the topics and the logical coherence of the presentation. The exposition is formally self-contained in the sense that the readers are not assumed to know anything about algebra. In practice, though, the readers are likely to have taught, or will be teaching beginning algebra so that they are already familiar with the more routine aspect of the subject. For this reason, while I try to shed light on the standard computations whenever appropriate, I have on the whole slighted the drills that usually accompany any such presentation.

Many of our algebra teachers are experiencing real difficulty in carrying out their duties, mainly because they have been told to emphasize certain ideas that they (like most mathematicians) cannot relate to, such as that of a quantity that "changes" or "varies" or that the equal sign is a "method that expresses equivalence". At the same time, they are denied the explanations of key facts which form the backbone of introductory algebra, such as the proper way to use symbols, why the graph of a linear equation is a straight line, why fractional exponents are defined the way they are and what they are good for, or why the significance of the technique of completing the square goes beyond the proof of the quadratic formula. The fact that many teachers do not even recognize that these are key facts in algebra speaks volumes about the present state of pre-service professional development in mathematics. The main impetus behind the writing of these notes is to propose a remedy. It gives an exposition of algebra *ab initio*, assuming only a knowledge of the rational number system and some elementary facts about similar triangles. I am quite aware that the latter is not standard fare in either the school curriculum or texts for professional development, but the Pre-Algebra notes give an adequate exposition of this topic and we will make frequent references to it. One of the stated goals of these notes is to make a strong case that this aspect of the school mathematics curriculum must change. Otherwise, the exposition of these notes is entirely unexceptional, and all it does is meet the minimum requirements of any exposition in mathematics, so that the unfolding of ideas is achieved not by appealing to any abstruse philosophical discussions but by use of clear and precise definitions and logical reasoning.

An integral part of the learning of algebra is learning how to use symbols precisely and fluently. This point of view is eloquently exposed in Chapter 3 of the National Mathematics Advisory Panel Report, *Foundations for Success: Reports of the Task Groups and Subcommittees*, U.S. Department of Education, 2008, available at

http://www.ed.gov/about/bdscomm/list/mathpanel/report/conceptual-knowledge.pdf

I believe if there is any meaning at all to the phrase "algebraic thinking" in school mathematics, it would be "the ability to use symbols precisely and fluently". In this regard, there is a need to single out the first two sections of these notes for some special comments. A principal object of study in introductory algebra is polynomials or, in the language of advanced mathematics, elements of the polynomial ring $\mathbb{R}[x]$. Every exposition of school algebra must come to grips with the problem of how to properly introduce this abstract concept to beginners. The mathematical decision I made (which is of course not mentioned in the notes proper) is to exploit the theorem that $\mathbb{R}[x]$ is isomorphic to the ring of real-valued polynomial functions, so that *in the context of introductory school algebra*, the x in a polynomial $a_n x^n + \cdots + a_1 x + a_0$ may be simply taken to be a *number*. The purpose of Section 1 is to demonstrate how one can do algebra by taking x to be just a number, and school algebra then becomes generalized arithmetic, literally. Formal algebra in the sense of $\mathbb{R}[x]$ can be left to a later date, e.g., a second course in school algebra.

Section 1 is thus entirely elementary, and is nothing more than a direct extension of arithmetic. The exposition intentionally emphasizes its similarity to arithmetic. There is a danger, however, that precisely because of its elementary character, readers would consider it as something "they already know" and therefore not worthy of a second thought. I would like to explicitly ask algebra teachers to recognize that what is in Section 1 is *genuine algebra*, and that its simplicity is precisely the reason that algebra can be taught without any fanfare. At a time when *Algebra for All* is the clarion call of the day in mathematics education, such a simple approach to algebra merits serious consideration for implementation in the school classroom.

Section 2 addresses one of the main obstacles in the teaching of algebra: students' apparent inability to solve word problems. It seems to me sensible to separate this difficulty into two stages: first teach how to transcribe verbal information into symbols, and then teach the necessary symbolic manipulations to extract the solution from the symbolic expressions. The need for such a separation does not seem to be well recognized at present in mathematics education. Too many teachers have been conditioned in the routine of plunging headlong into the *solution* of a problem by using guess-and-check, and because guess-and-check is sometimes identified with conceptual understanding, their students follow suit. It come to pass that students also fail to recognize the critical need of the transcription process, and symbols are looked upon with suspicion. The anti-mathematical practice of relying solely on guess-and-check to achieve mathematical understanding gets recycled from generation to generation and algebra students fail to learn the most fundamental aspect of algebra, namely, the proper use of symbols. For the purpose of good education, I believe we should, and must, promote the importance of transcription. The main purpose of Section 2 is to do exactly that.

Another perennial problem in the learning of introductory algebra is the absence of reasoning even for the most basic facts of the subject, such as why the graph of a linear equation in two variables is a line, why the graph of a linear inequality is a half-plane, or why the maximum or minimum of a quadratic function $f(x) = ax^2 + bx + c$ is achieved at the point $x = -\frac{b}{2a}$. As a result, students scramble to *memorizing* all four forms of the equation of a line, though not always with success, and come out of the subject of quadratic functions and equations without learning the fundamental skill of completing the square except as a rote skill to *get* the quadratic formula. Concomitant with the absence of reasoning is the tendency to slight basic definitions. Indeed, if one is not fully aware of the precise definition of the graph of an equation, it would be difficult to *prove* what the graph of a linear equation must look like. Or, if one doesn't say what the definition of the graph of a linear inequality is or what the definition of a half-plane is, one cannot possibly prove that the graph of a linear inequality is a half-plane. This monograph tries to compensate for these common deficiencies by addressing each of them in detail.

There is at present an urgent need to emphasize precise definitions, coherence of presentation, and logical reasoning in the mathematics education of teachers.[1] This need may be greatest among algebra teachers because of the inherent abstraction of algebra. Without precise definitions and logical reasoning at each step of the mathematical processes, abstract mathematics is not learnable. The exposition in these notes fully reflects this sense of urgency.

[1]See the discussion in the article, *The mathematics K-12 teachers need to know,* http://math.berkeley.edu/~wu/

1 Symbolic Expressions

Basic Protocol in the use of symbols

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Rational expressions

Basic Protocol in the use of symbols

It can be argued that the most basic aspect of algebra is the use of symbols. Why symbols? When we try to assert that something is valid for a large collection of numbers instead of just for a few specific numbers (e.g., for all positive integers, or for all rational numbers), we have to resort to the use of symbols to express this assertion correctly and succinctly. For example, suppose we observe that $2 \times 3 = 3 \times 2$, $3 \times 4 = 4 \times 3$, $\frac{6}{17} \times \frac{4}{9} = \frac{4}{9} \times \frac{6}{17}$, $(-\frac{8}{3}) \times 82 = 82 \times (-\frac{8}{3})$, and so on, for any two numbers that we care to multiply, and we want to say summarily that

> for *any* two numbers, if we multiply them one way and, switching the order, we multiply them again, we get the same number.

Of course, what we wish to assert is what is known as the **commutative law of multiplication**. The question is how to say it completely, unambiguously, *and* succinctly. After many trials and errors through many centuries, starting with Diophantus around the third century[2] to René Descartes (1596–1650) [3], people finally settled on the use of symbols as we know it today. For the problem at hand, the accepted way of enunciating the commutative law of multiplication is to say:

$$ab = ba \qquad \text{for all numbers } a \text{ and } b$$

[2]Diophantus was a Greek mathematician who lived in Alexandria, Egypt (which was a Greek colony named after Alexander the Great). Unfortunately, his dates are unknown other than the fact that he *probably* lived in the third century A.D. His influence in the development of mathematics is quite profound, as evidenced by the fact that the terminology of *Diophantine equations* is standard in mathematics.

[3]A co-discoverer of analytic geometry with Pierre Fermat (1601-1665). He is also an important philosopher who is noted for the statement that, "I think, therefore I am".

Compared with the preceding indented verbal statement, the brevity resulting from the use of symbols should be obvious.

It would seem that the fruits of some thirteen centuries of development of the symbolic notation have not filtered down to our school curriculum, and the use of symbols in standard textbooks is reckless at best. Major misconceptions ensue. A main theme throughout these notes is to give careful guidance on the etiquette of using symbols in order to undo these misconceptions.

One of the misconceptions that accompanies the abuse of the symbolic notation is the concept of a *variable*. At present, *variable* occupies a prominent position in school mathematics, especially in algebra. In standard algebra texts as well as in the mathematics education literature, there may be no explicit definition of what a *variable* is, but students are asked to understand this concept nevertheless because it is considered the gateway to algebra. When students are asked to understand something which is left unexplained, learning difficulties inevitably arise. Sometimes, a *variable* is described as a quantity that changes or varies. The mathematical meaning of the last statement is vague and obscure. At other times it is asserted that students' understanding of this concept should be beyond recognizing that *letters can be used to stand for unknown numbers in equations*, but nothing is said about what lies "beyond" this recognition. For example, in the National Research Council volume, *Adding It Up* (The National Academy Press, 2001), there is a statement that students emerging from elementary school often carry the "perception of letters as representing unknowns but not variables" (p. 270). The difference between "unknowns" and "variables" is unfortunately not clarified. All this deepens the mystery of what a *variable* really is.

These notes will not make the understanding of a *variable* a prerequisite to the learning of algebra. There is no need for that in mathematics.[4] Instead, we will explain the correct way to use symbols, and once you understand that, you will feel

[4]In mathematics, a *variable* is an informal abbreviation for "an element in the domain of definition of a function", which is of course a perfectly well-defined concept (See Section 9). If, for example, a function is defined on a set of ordered pairs of numbers, it is referred to as "a function of two variables", and it must be said that, in that case, the emphasis is more on the word "'two" than on the word "variables". In the sciences and engineering, the word "variable" is bandied about with gusto. However, to the extent that mathematics is just a tool rather than the central object of study in such situations, scientists and engineers can afford to be cavalier about mathematical terminology. In these notes, we have to be more careful because we are trying to learn mathematics.

no compunction about pushing *variable* aside and go on with your study of algebra. However, the word *variable* has been in use for more than three centuries and, sooner or later, you will run across it in the mathematics literature. The point is not to pretend that this word doesn't exist but, rather, to understand enough about the use of symbols to put *variable* in the proper perspective. Think of the analogy with the concept of *alchemy* in chemistry; this word has been in use longer than *variable*. On the one hand, we do not want alchemy to be a basic building block of school chemistry, and on the other hand, we want every school student to acquire enough knowledge about the structure of molecules to know why alchemy is an absurd idea. We hope you will carry the analogous message about *variable* back to your classroom.

Let a letter x stand for a number, in the same way that the pronoun "he" stands for a man. All the knowledge accumulated about rational numbers[5] can now be brought to bear on this x. There should be no unease about the use of symbols any more than there should be unease about the use of pronouns. The analogy with a pronoun is apt, in the sense that, if one does not begin a sentence with a pronoun *without* saying what the pronoun stands for, one also never uses a symbol without saying what the symbol stands for. Here then is what might be called the **Basic Protocol in the use of symbols**:

> Each time one uses a symbol, one must specify precisely what the symbol stands for.

In a situation where we want to determine which number x satisfies an equality such as $2x^2 + x - 6 = 0$, the value of the number x would be unknown for the moment and x is then also called an **unknown**. In broad outline, this is all there is to it as far as the use of symbols is concerned.

A closer examination of this usage reveals some subtleties, however. Consider first the following three cases of the equality $xy = yx$:

(V1) $xy = yx$.

(V2) $xy = yx$ for all whole numbers x and y so that $0 \leq x, y \leq 10$.

(V3) $xy = yx$ for all real numbers x and y.

[5]Recall FASM (see §7 in Chapter 1 of the Pre-Algebra notes). Thus in terms of all arithmetic computations, we may regard x as a rational number.

The statement (V1) *has no meaning*, because we don't know what the symbols x and y stand for. To pursue the analogy with pronouns, suppose someone makes the statement, "He is 7 foot 6." Without indicating who "he" refers to, this statement is neither true nor false.[6] It is simply meaningless. If x and y in (V1) are real numbers, then (V1) is true, but there are other mathematical objects x and y for which (V1) is false.[7] On the other hand, (V2) is true, but it is a trivial statement because its truth can be checked by successively letting both x and y be the numbers 0, 1, 2, …, 9, 10, and then computing xy and yx for comparison. The statement (V3) is however both true and more profound. As mentioned implicitly above, this is the commutative law of multiplication among real numbers. It is either something you take on faith, or, in some context, a not-so-trivial theorem to prove. Thus, despite the fact that all three statements (V1)–(V3) contain the equality $xy = yx$, they are in fact radically different statements because the **quantifications** (i.e., the precise descriptions) of the symbols x and y are different. This reinforces the message of the above Basic Protocol that the quantification of a symbol is critically important.

Next, consider the question of which number(s) x satisfies $3x + 7 = 5$. Any such x is called a **solution** of $3x + 7 = 5$. The usual procedure (which we take for granted at this juncture but will take it up in §3 below) yields $3x = 5 - 7$, and therefore there is only one solution in this case, namely,

$$x = \frac{5-7}{3}$$

If we consider $3x + \frac{1}{2} = 13$ instead, we'd get

$$x = \frac{13 - \frac{1}{2}}{3}$$

Or consider $3x + 25 = 4.6$ and get

$$x = \frac{4.6 - 25}{3}$$

Or consider $5x + 25 = 4.6$ and get

$$x = \frac{4.6 - 25}{5}$$

[6] It is true if "he" refers to the basketball star Yao Ming, but is false for Woody Allen.

[7] For example, if x and y are certain 2×2 matrices.

And so on. There is an unmistakable pattern here: if we consider which number x satisfies $ax + b = c$ no matter what the numbers a, b, and c may be, with a, b, c ($a \neq 0$) understood to be *three fixed numbers* throughout this discussion, the solution x is

$$x = \frac{c-b}{a}$$

We have now witnessed the fact that some symbols stand for elements in an infinite[8] set of numbers, e.g., the statement that $xy = yx$ for *all real numbers x and y*, while in others, the symbols stand for fixed values throughout the discussion, e.g, the numbers a, b, and c in the preceding discussion of $ax + b = c$. In the former case, the symbols are called **variables**, and in the latter case, **constants**. We see that a variable so defined *does not vary or change.* It is simply an element in an infinite set of numbers, and each time we deal with x, we deal with a specific x and ***not*** an infinite collection of numbers all at one.

Expressions and identities

It is time to recall that in arithmetic, there are many occasions when the use of symbols is unavoidable. In addition to the commutative law of multiplication, the statements of the same law for addition, the **associative laws** for addition and multiplication, and also the **distributive law** require a similar use of symbols. In addition, the formulas for the addition, subtraction, multiplication, and division of fractions likewise cannot be stated without the use of symbols. We repeat these formulas here to emphasize this point: let $\frac{k}{\ell}$, $\frac{m}{n}$ be arbitrary rational numbers. In other words, k, ℓ, m, n are integers, and $\ell \neq 0$, $n \neq 0$. Then:

$$\begin{aligned} \frac{k}{\ell} \pm \frac{m}{n} &= \frac{kn \pm m\ell}{\ell n} \\ \frac{k}{\ell} \cdot \frac{m}{n} &= \frac{km}{\ell n} \end{aligned}$$

[8] As mentioned earlier, a variable is an element in the domain of a function, and the domain can be finite or infinite. But for school algebra, where functions are those defined on intervals of the number line, saying that a domain is "infinite" suffices for the purpose at hand.

$$\frac{\frac{k}{\ell}}{\frac{m}{n}} = \frac{kn}{\ell m}$$

We emphasize that in each of these formulas, *we don't need to know the exact value each of k, ℓ, m, n*, but so long as they are whole numbers, they will have to satisfy $\frac{k}{\ell} \pm \frac{m}{n} = \frac{kn \pm m\ell}{\ell n}$, etc. For example, with $k = 11$, $\ell = -7$, $m = 5$ and $n = 23$, then the above formulas imply that

$$\frac{11}{-7} \pm \frac{5}{23} = \frac{(11 \times 23) \pm (5 \times (-7))}{(-7) \times 23} = -\frac{218}{161} \quad \text{or} \quad -\frac{288}{161}$$

$$\frac{11}{-7} \cdot \frac{5}{23} = \frac{11 \times 5}{-7 \times 23} = -\frac{55}{161}$$

$$\frac{\frac{11}{-7}}{\frac{5}{23}} = \frac{11 \times 23}{5 \times (-7)} = -\frac{253}{35}$$

As a natural extension of these ideas, we now give some well-known algebraic **identities**. The term *identity* is used in mathematics to indicate, *informally*, that an equality is valid for a "large set" of numbers of interest. What "large" means will be clearly indicated in each situation and, in any case, is usually clear from context. The term "identity" *is definitely not a well-defined mathematical concept that requires a 100% precise definition*. However, since the meaning of this term seems at present to be endlessly (and one may say, unnecessarily) debated, we will now try to clarify its meaning as best we can. By a **number expression** or more simply an **expression** in a given collection of numbers x, y, $\ldots w$, we mean a number obtained from these x, y, $\ldots w$ by performing a specific combination of arithmetic operations (i.e., addition, subtraction, multiplication, and division). For example, if x, y, z are numbers, then

$$\frac{xy}{xyz+2} + x^3(16z - y^2) - z^{21}, \qquad \frac{x - y^3}{8 + \frac{5}{(yz)^2}}\ , \qquad x^4 + y^4 + z^4 - xyz$$

are examples of expressions in the numbers x, y, z (we have to assume $xyz \neq -2$ in the first expression and $y \neq 0$ and $z \neq 0$ in the second expression to avoid dividing by 0). *Later on in Section 8, we shall expand the meaning of* expression *after we have defined* taking the n-th root. You may have noticed that the above expressions would be ambiguous unless a **notational convention** concerning the arithmetic operations among the symbols is understood. With the help of parentheses, the correct order in carrying out the arithmetic operations in, for example,

$$\frac{xy}{xyz+2} + x^3(16z - y^2) - z^{21}$$

will always be understood to be

$$\{xy \cdot (xyz+2)^{-1}\} + \{(x^3)[(16z) - (y^2)]\} - \{z^{21}\}$$

The ungainly sight of the last expression should be reason enough for the adoption of this notational convention. Postponing the exact description of this notational convention to the latter part of this section so as not to disrupt the flow of the exposition, we may roughly describe this convention as follows: *do the multiplication indicated by the exponents first, then the multiplications, and finally the additions.* Recall in this connection that subtraction is nothing but addition in disguise: $a - b = a + (-b)$ by definition, for any two rational numbers a, b, and that multiplication includes the division in $\frac{xy}{xyz+2}$ above as this is nothing other than the multiplication $xy \cdot (xyz+2)^{-1}$.

Now we can give "an approximate definition" of an **algebraic identity**, or more simply an **identity**, as a statement that two given number expressions are equal for every number in a given collection under discussion (such as all whole numbers, all positive numbers, or all numbers[9]) *allowing for a small set of exceptions.* We emphasize again that an *identity* is not a precise concept within mathematics, but is no more than a terminology used loosely for convenience. In specific situations, there will be plenty of opportunities to discern what "the given collection under discussion" is and what the "small set of exceptions" means. A few examples will be given below.

The assertion that $ab = ba$ is true for all numbers a and b is an example of an identity, and so is $\frac{k}{\ell} \pm \frac{m}{n} = \frac{kn \pm m\ell}{\ell n}$ for all integers k, ℓ, m, n *provided* $\ell \neq 0$ *and* $n \neq 0$. Right here, we see that the latter identity is one which make allowance for the exceptions of $\ell = 0$ and $n = 0$. More is true. We have just stated this equality $\frac{k}{\ell} \pm \frac{m}{n} = \frac{kn \pm m\ell}{\ell n}$ for integers k, ℓ, m, n, but we know from considerations of *rational quotients*[10] that this equality remains true even if k, ℓ, m, n are arbitrary rational

[9] Recall: a number is just a point on the number line. Sometimes one says "all real numbers" in place of "all numbers" for emphasis.

[10] Recall that these are quotients $\frac{A}{B}$ where A and B are rational numbers. See Section 5 in Chapter 2 of the Pre-Algebra notes.

numbers. Therefore, in this form, this identity is valid for all rational numbers k, ℓ, m, n *provided* $\ell \neq 0$ *and* $n \neq 0$. The fact that the identity remains valid for all real numbers is then a consequence of FASM. But even here, there are a "small number of exceptions" to this general identity, namely, $\ell \neq 0$ and $n \neq 0$.

In case it helps to further illustrate the cavalier manner in which the terminology of *identity* is used, we give two advanced examples without attempting to define the relevant concepts. The equality $\log xy = \log x + \log y$ is an identity for all *positive* numbers x and y. The equality $1 + \cot^2 x = \csc^2 x$ is an identity for all numbers x *except* for all integral multiples of π.

We want to get more interesting identities. Consider the computation of the square, 104^2, for example. One can compute it directly, of course. But one can also proceed by appealing to the distributive law, as follows:

$$\begin{aligned} 104^2 = (100+4)^2 &= (100+4)(100+4) \\ &= \{(100+4)\times 100\} + \{(100+4)\times 4\} \qquad \text{(dist. law)} \\ &= \{100^2 + (4\times 100)\} + \{(100\times 4) + 4^2\} \qquad \text{(dist. law again)} \\ &= 100^2 + 2\times(100\times 4) + 4^2 \end{aligned}$$

At this point, it should be possible to mentally finish the computation as $10000 + 800 + 16 = 10816$. More than a trick, this idea of computing the square of a sum using the distributive law turns out to be almost omnipresent in algebraic manipulations of all kinds. It is a good idea to formalize it once and for all. We therefore have, in an identical fashion:

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{for all numbers } a \text{ and } b.$$

This is our first identity of note.

A similar consideration, but worth pointing out in any case, is the computation of the square of 497, for example. We recognize it as $(500-3)^2$, so that

$$\begin{aligned} 497^2 = (500-3)^2 &= (500-3)(500-3) \\ &= \{(500-3)\times 500 - (500-3)\times 3\} \qquad \text{(dist. law)} \\ &= \{500^2 - (3\times 500) - \{(500\times 3) - 3^2\} \qquad \text{(dist. law again)} \\ &= 500^2 - 2\times(500\times 3) + 3^2 \end{aligned}$$

(Note that the preceding computation furnishes a good review of the basic arithmetic of rational numbers: the distributive law for a difference, $a(b-c) = ab - ac$ for all numbers a, b, c, and the removal of parentheses by $-(a-b) = -a+b$ for all a, b.) Again, we stop the calculation at this point because it can now be finished in one's head: $250000 - 3000 + 9 = 247009$. The same computation also leads to:

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \text{for all numbers } a \text{ and } b.$$

It is a good illustration of the use of symbols, and the attendant generality the symbolic method brings, that the identity for $(a-b)^2$ can be obtained directly from the identity for $(a+b)^2$. Indeed, since the identity $(a+b)^2 = a^2 + 2ab + b^2$ is valid for *all* numbers, we may replace b by an arbitrary number $-c$ to get

$$(a+(-c))^2 = a^2 + 2a(-c) + (-c)^2 = a^2 - 2ac + c^2$$

Since $a + (-c) = a - c$ by definition, we get $(a-c)^2 = a^2 - 2ac + c^2$, and since c is arbitrary anyway, we may replace c by b to obtain $(a-b)^2 = a^2 - 2ab + b^2$ for any number b. Thus we retrieve the second identity by way of the first.

A third common identity can be introduce by a computation of another kind: $409 \times 391 = ?$ We recognize that $409 \times 391 = (400+9)(400-9)$, so that

$$\begin{aligned}
409 \times 391 &= \{(400+9) \times 400\} - \{(400+9) \times 9\} && \text{(the dist. law)} \\
&= 400^2 + (9 \times 400) - (400 \times 9) - 9^2 && \text{(the dist. law again)} \\
&= 400^2 - 9^2
\end{aligned}$$

It follows that $409 \times 391 = 160000 - 81 = 159919$. The same reasoning carries over to any two numbers a and b, so that

$$\begin{aligned}
(a+b)(a-b) &= (a+b)a - (a+b)b \\
&= a^2 + ba - ab - b^2 \\
&= a^2 - b^2
\end{aligned}$$

When the symbolic computation is given in such detail, we see that in the second line, the commutative law for multiplication was used. We have obtained our third identity:

$$(a+b)(a-b) = a^2 - b^2 \qquad \text{for all numbers } a \text{ and } b.$$

This particular product $(a+b)(a-b)$ for any two numbers a and b, turns out to be very common, and the usefulness of this identity far transcends the preceding computation of 409×391 and others like it. For example, if we read this identity backwards, i.e.,

$$a^2 - b^2 = (a+b)(a-b) \qquad \text{for all numbers } a \text{ and } b$$

then we obtain what is known as a **factorization** or **factoring** of $a^2 - b^2$, which merely means expressing $a^2 - b^2$ as a product, in the same sense that $24 = 3 \times 8$ is a factorization of 24. Knowing such a factorization for any two numbers a and b can be very useful. Thus, if $a + b \neq 0$, we can simplify the division $\frac{a^2-b^2}{a+b}$ to

$$\frac{a^2 - b^2}{a+b} = a - b$$

because $a^2 - b^2 = (a+b)(a-b)$, so that we can cancel the number $a+b$ in the numerator and the denominator. We explicitly point out that, insofar as a and b can be rational numbers (say, $\frac{17}{5}$ and $\frac{2}{7}$), we are using the cancellation law for rational quotients here.[11] One cannot over-emphasize the importance of the role played by complex fractions in school mathematics.

Mersenne primes and the finite geometric series

There is another identity that is equally elementary but has far-reaching applications in mathematics. This time, we start with a symbolic calculation: if a, b are any two numbers, then

$$\begin{aligned} (a^2+ab+b^2)(a-b) &= (a^2+ab+b^2)\,a - (a^2+ab+b^2)\,b \\ &= (a^3+a^2b+ab^2) - (a^2b+ab^2+b^3) \\ &= a^3 - b^3 \end{aligned}$$

Notice two features in the preceding calculation. First, if we call any of the products separated by two consecutive $+$'s a **term** of the number expression,[12] e.g., a^3, a^2b,

[11] See, for example, Section 5 of Chapter 2 of the Pre-Algebra notes.

[12] Recall that since a subtraction is an addition in disguise, this reference to $+$ includes automatically all the $-$'s.

ab^2, $\ldots b^3$, then the way to remember the expression $a^2 + ab + b^2$ is to observe that the power of a decreases by 1 and the power of b increases by 1 as we go through the terms from left to right. Second, the cancellation in the second line

$$(a^3 + a^2b + ab^2) - (a^2b + ab^2 + b^3)$$

is due to the matching of each term in the first pair of parentheses with a term in the second pair of parentheses, *except* for the first term a^3 and the last term b^3, and this is why the only survivors at the end are the two terms $a^3 - b^3$. The same pattern repeats itself if we multiply $(a^3 + a^2b + ab^2 + b^3)$ by $(a - b)$. Thus,

$$\begin{aligned}(a^3 + a^2b + ab^2 + b^3)(a - b) &= (a^3 + a^2 + ab^2 + b^3)a - (a^3 + a^2 + ab^2 + b^3)b \\ &= (a^4 + a^3 + a^2b^2 + ab^3) - (a^3b + a^2b^2 + ab^3 + b^4) \\ &= a^4 - b^4\end{aligned}$$

If we form the products

$$(a^4 + a^3b + a^2b^2 + ab^3 + b^4)(a - b),$$

$$(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)(a - b),$$

the results would be $a^5 - b^5$, $a^6 - b^6$. Let us write these down. For any two numbers a and b, we have

$$\begin{aligned}(a - b)\,(a^2 + ab + ab^2) &= a^3 - b^3 \\ (a - b)\,(a^3 + a^2b + ab^2 + b^3) &= a^4 - b^4 \\ (a - b)\,(a^4 + a^3b + a^2b^2 + ab^3 + b^4) &= a^5 - b^5 \\ (a - b)\,(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5) &= a^6 - b^6\end{aligned}$$

Activity Verify that $(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) = a^5 - b^5$.

At this point, it should not be difficult to discern a pattern. So let n be a positive integer and we form the product

$$(a^n + a^{n-1}b + a^{n-2}b^2 + a^{n-3}b^3 + \cdots + ab^{n-1} + b^n)(a - b).$$

Then we get the following sum:

$$\begin{aligned}&(a^n + a^{n-1}b + a^{n-2}b^2 + a^{n-3}b^3 + \cdots + b^n)\, a \\ &\qquad - (a^n \;\; + a^{n-1}b \;\; + a^{n-2}b^2 \;\; + \cdots + ab^{n-1} + b^n)\, b\end{aligned}$$

If we use the distributive law to expand both expressions in the numbers a and b, then *after the expansion*, the terms in each product that are vertically aligned cancel each other. What is left is then a^{n+1} and $-b^{n+1}$. Thus we have:

$$\begin{aligned} a^{n+1} - b^{n+1} &= (a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + a^{n-3}b^3 + \cdots + ab^{n-1} + b^n) \qquad (\sharp) \\ &\quad \textit{for any two numbers } a \textit{ and } b, \textit{ and any positive integer } n \end{aligned}$$

This gives a very useful factorization of the difference of two numbers raised to the same power $n+1$. If $n = 1$, it gives back $a^2 - b^2 = (a-b)(a+b)$. Therefore the general identity for an arbitrary positive integer n is nothing but a generalization of the earlier simple identity $a^2 - b^2 = (a-b)(a+b)$. The case $n = 2$ of this identity also comes up often and we call attention to it by stating it separately:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2) \qquad \text{for any numbers } a,\, b.$$

The special case of $b = 1$ in $(\sharp)$ is extremely interesting. It reads:

$$(a^{n+1} - 1) = (a-1)(a^n + a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1) \qquad \textit{for any number } a.$$

This factorization says that if you raise a whole number a which is bigger than 2 (e.g., $a = 215,608$) to any power (e.g., $n = 87$) and diminish it by 1, then the resulting number $a^{n+1} - 1$ is the product of two whole numbers both of which are bigger than 1 (this is where we need $a > 2$ as otherwise $a - 1$ would be either 0 or 1). Therefore $a^{n+1} - 1$ *is never a prime number*. We will discuss the case $a = 2$ presently. But with $a = 215,608$ and $n = 87$, we see that $a - 1 = 215,607$, so that

$$215,608^{87} - 1 = (215,607)(215,608^{86} + \cdots + 215608 + 1)$$

This shows that $215,608^{87} - 1$ is not a prime, because it is the product of $215,607$ with another very large number. Thus, even if we have no idea of what a number of this size means ($(215,608^{87} - 1)$ has 426 digits!), we know at least that it is not a prime. We can illustrate this factorization another way: if we know that $1,453,933,567 = 68^5 - 1$, then the number $1,453,933,567$ is not a prime because it is equal to $68^5 - 1$, which, in turn, is equal to $(68-1)(68^4+68^3+68^2+68+1)$, and is therefore divisible by 67.

If $a = 2$, then the preceding factorization only says

$$2^{n+1} - 1 = 2^n + 2^{n-1} + \cdots + 2^2 + 2 + 1$$

and there is no factorization. In fact, for some n, $2^{n+1} - 1$ *is* a prime. For example, $2^2 - 1 = 3$ ($n = 1$) and $2^3 - 1 = 7$ ($n = 2$), and both 3 and 7 are primes, although $2^4 - 1 = 15$ and 15 is certainly not a prime. Those numbers $2^p - 1$ which are primes for a positive integers p are called **Mersenne primes**.[13] We do not know if there are an infinite number of Mersenne primes, but the largest Mersenne prime that is known at present (July 2010) has 12978189 digits, corresponding to $p = 43112609$ (discovered on September 15, 2008). You can keep up-to-date by Googling "Mersenne primes".

But we are far from exhausting the charm of the identity for $(a^{n+1} - 1)$; we can read it backwards to obtain

$$(1 + a + a^2 + \cdots + a^{n-1} + a^n) \;=\; \frac{a^{n+1} - 1}{a - 1} \qquad \text{for any number } a \neq 1.$$

This now becomes a **summation formula** for the first $n + 1$ powers of a number a, starting with $a^0 = 1$ up to a^n. For example, if $a = 5$ and $n = 11$, then

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^{10} + 5^{11} = \frac{5^{12} - 1}{5 - 1} = \frac{5^{12} - 1}{4}.$$

Since $5^{12} - 1 = 244,140,624$, we have

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^{10} + 5^{11} \;=\; 61,035,156.$$

[13] Marin Mersenne (1588–1648) was a French monk, a scholar of science and mathematics, and the central clearinghouse of European science and mathematics of his time. There were no scholarly journals in those days and Mersenne, through his correspondence with the leading scientists and mathematicians of Europe, helped disseminate the latest discoveries to a wider audience. Among his regular visitors or correspondents were Descartes, Pascal, Fermat, and Huygens. He came upon the primes that are named after him in his (unsuccessful) search for a formula that would represent all primes.

If $a = -3$ and $n = 15$, then

$$1 - 3 + 3^2 - 3^3 + 3^4 - \cdots + 3^{14} - 3^{15} = \frac{3^{16} - 1}{-3 - 1} = -\frac{43,046,720}{4} = -10,761,480.$$

And finally, if $a = \frac{3}{4}$ and $n = 10$, we have

$$1 + \tfrac{3}{4} + (\tfrac{3}{4})^2 + (\tfrac{3}{4})^3 + \cdots + (\tfrac{3}{4})^{10} \;=\; \frac{\{(\frac{3}{4})^{11} - 1\}}{\frac{3}{4} - 1}$$

which is equal to

$$\frac{16,068,628}{4,194,304}.$$

It is roughly 3.8.

A sum of the form $1 + a + a^2 + \cdots + a^{n-1} + a^n$ is called a **finite geometric series** of n terms in a. We have just learned how to sum a finite geometric series. Geometric series appear everywhere in both science and mathematics.

Polynomials and order of operations

We next introduce polynomials. Underlying the whole discussion of polynomials will be a simple observation based on the distributive law, and we deal with this first. Suppose we have a sum

$$(18 \times 5^3) + (5^3 \times 23) + (69 \times 5^3)$$

One can compute this sum by multiplying out each term 18×5^3, $5^3 \times 23$, and 69×5^3, and then add the resulting numbers to get

$$(18 \times 5^3) + (5^3 \times 23) + (69 \times 5^3) = 2250 + 2875 + 8625 = 13750$$

Now if we reflect for a moment, we would realize that we wasted precious time doing three multiplications before adding. If we apply the distributive law, then the computation becomes easier:

$$\begin{aligned} (18 \times 5^3) + (5^3 \times 23) + (69 \times 5^3) &= (18 + 23 + 69) \times 5^3 \\ &= 110 \times 125 = 13750 \end{aligned}$$

(Notice that we have made use of the commutative law of multiplication to change $5^3 \times 23$ to 23×5^3 in the process.) You may think that with the advent of high speed computers, it doesn't make any difference whether we get the answer by multiplying three times and then add once, or (as in the second case) add three times and multiply once. This is true, but the difference in conceptual clarity between

$$(18 \times 5^3) + (5^3 \times 23) + (69 \times 5^3)$$

and

$$(18 + 23 + 69) \times 5^3$$

is enormous. This is because multiplication is a far more complicated concept than addition: $234 + 677$ merely means lumping 234 and 677 together, but 234×677 means adding 677 to itself *234 times*. It is therefore conceptually simpler to add three times and multiply once than to multiply three times and add once. Because conceptual clarity is very important in learning and doing mathematics, whenever we see terms involving the same numbers raised to a fixed power in the future (such as 5^3 in $(18 \times 5^3) + (5^3 \times 23) + (69 \times 5^3)$), we shall always collect them together by the use of the distributive law. For example, we shall always rewrite

$$(181 \times 2^5) + (67 \times 2^5) + (2^5 \times 96) - (257 \times 2^5)$$

as

$$87 \times 2^5 \quad (= (181 + 67 + 96 - 257) \times 2^5),$$

Similarly, we write

$$\{24 \times 59^{14}\} - \{(\tfrac{3}{5})^8 \times 89\} + \{59^{14} \times 73\} + \{59^{14} \times 66\} + \{25 \times (\tfrac{3}{5})^8\} + \{(\tfrac{3}{5})^8 \times 11\}$$

as

$$\{163 \times 59^{14}\} - \{53 \times (\tfrac{3}{5})^8\},$$

where $163 = 24 + 73 + 66$ and $-53 = -89 + 25 + 11$. Recall yet again that we talk about both of the above expressions as a "sum" even in the presence of the terms $-(257 \times 2^5)$ and $-\{(\frac{3}{5})^8 \times 89\}$ because $-(257 \times 2^5) = +\{-(257 \times 2^5)\}$ and $-\{(\frac{3}{5})^8 \times 89\} = +\{-((\frac{3}{5})^8 \times 89)\}$.

In an entirely similar manner, suppose we are given a sum of multiples of *nonnegative integer* powers of a fixed number x, where **multiple** here means simply multiplication by any number and not necessarily by a whole number, and "nonnegative integers" refers to whole numbers 0, 1, 2, ... Then we would automatically collect together the terms involving the same power of x as before. For example, we would rewrite

$$\frac{1}{2}x^3 + 16 - 8x^2 + \frac{1}{3}x^3 - x^5 - 6x^2 + 75x + 2x^3$$

as

$$-x^5 + \frac{17}{6}x^3 - 14x^2 + 75x + 16.$$

Observe that we have followed three **conventions** in writing the latter sum involving the powers of a fixed number x:

(i) *Parentheses are suppressed with the understanding that exponents be computed first, multiplications second, and additions third.* (This was already used earlier.)

(ii) *Powers of x are placed last in each term* (so that instead of $-x^2 14$, we write $-14x^2$).

(iii) *The terms are written in* decreasing *powers of the number x in question.* (We make the *ad hoc* definition in this situation that $\boldsymbol{x^0 = 1}$ regardless of whether x is 0 or not.[14] The term 16 is then the term $16x^0$; incidentally, this is where we need the zeroth power of x.)

The number in front of a power of x is called the **coefficient** of that particular power of x, and a sum of multiples of nonegative integer powers of x is called a **polynomial** in x. We emphasize that, in this terminology, the definition in (iii) says that, *for a polynomial in x,* we define $x^0 = 1$ regardless of whether x is 0 or not.

A multiple of a single nonnegative power of x, such as $58x^{12}$ is called a **monomial**. Thus, a monomial is a polynomial with only one term. The highest power of x with a nonzero coefficient in a polynomial is called the **degree** of the polynomial. The terminology about "nonzero coefficient" refers to the fact that the preceding polynomial $-x^5 + \frac{17}{6}x^3 - 14x^2 + 75x + 16$ could be written as $0 \cdot x^{37} - x^5 + \frac{17}{6}x^3 - 14x^2 + 75x + 16$, but the 37-th power of x clearly doesn't count. This polynomial has

[14] For a fuller discussion of the 0-th power of a number, see Section 8 below.

degree 5, and not 37 (and not any whole number different from 5, for that matter.) Moreover, -1 is the coefficient of x^5, 0 is the coefficient of x^4, and -14 is the coefficient of x^2, because, strictly as a *sum* of the powers of x, this polynomial is in reality

$$(-1)x^5 + 0x^4 + \frac{17}{6}x^3 + (-14)x^2 + 75x + 16x^0.$$

Similarly, 16 is the coefficient of x^0.

As is well-known, a polynomial of degree 1 is called a **linear polynomial**, and one of degree 2 is called a **quadratic polynomial**. Because a general quadratic polynomial has only three terms, $ax^2 + bx + c$, it is sometimes called a **trinomial** in school mathematics; the terminology of "trinomial" is not used in advanced mathematics, so you should use it as sparingly as possible. We will discuss quadratic polynomials in some detail in the last two sections. A polynomial of degree 3 is called a **cubic polynomial**.

There is no reason why we must restrict ourselves to polynomials in one variable. If x, y, z, etc., are numbers, then sums of multiples of the products of nonnegative powers of x, y, z, etc., are called **polynomials in $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$**, etc. For example, $19x^3y^{21} - 8y^9z^5 - xyz + 31$ is such a polynomial.

Here we should recall the earlier reference to carrying out the arithmetic operations on a polynomial in a particular order. This is the so-called **order of operations**, a topic as wrongly over-emphasized in school mathematics as the insistence on having all fractions in lowest terms.[15] This is just a *convention* and, like all other conventions in mathematics, it has no mathematical substance. I suggest that you explain to your students, as clearly as you can, what this convention is all about, why we adopt it, and then go on to more important topics, such as those on which we will be spending a lot of time in the days ahead. While we are discussing conventions, we may mention a few others.

(iv) *In symbolic expressions, we use a dot $\cdot$ in place of $\times$ for the multiplication between specific numbers, e.g.,* $24 \cdot 95$ *instead of* 24×95*.*

[15] For a fuller discussion of the issues involved, see *"Order of operations" and other oddities in school mathematics*, http://math.berkeley.edu/∼wu/order3.pdf.

(v) *We usually omit even the dot* $\cdot$ *between a letter and a number, e.g., write* $42x^2$ *instead of* $42 \cdot x^2$ *unless we wish to achieve an extra degree of clarity.*

(vi) *We also write* $1x$ *simply as* x*, and we agree to omit all terms of the form* $0x^m$ *where* m *is any whole number.*

The reason for (iv) is to avoid confusing the letter x with the multiplication symbol $\times$.

You have seen polynomials before. The so-called *expanded form* of the five-digit whole number 75018, for example, is

$$(7 \times 10^4) + (5 \times 10^3) + (0 \times 10^2) + (1 \times 10^1) + (8 \times 10^0)$$

which is a fourth-degree polynomial in the number 10. Of course the expanded form of any k-digit whole number is a polynomial of degree $(k - 1)$ in 10. On the other hand, the so-called complete expanded form of a decimal such as 32.58,

$$(3 \times 10^1) + (2 \times 10^0) + (5 \times 10^{-1}) + (8 \times 10^{-2}),$$

is *not* a polynomial in 10, for the reason that it contains *negative* powers of 10.

It should be pointed out that, *as a polynomial in 10, the expanded form of a whole number is very special: each coefficient is a single digit whole number.* Thus none of the following polynomials in 10 is the expanded form of a whole number:

$$\begin{aligned} (35 \times 10^2) + (2 \times 10^1) + (7 \times 10^0) \\ (3 \times 10^3) - (1 \times 10^2) + (7 \times 10^1) + (4 \times 10^0) \\ (5 \times 10^2) + (\tfrac{2}{3} \times 10^1) + (7 \times 10^0) \end{aligned}$$

The first is not the expanded form of a whole number because 35 is not a single digit, the second because the coefficient of 10^2 is -1, which is not a whole number, and the third because $\frac{2}{3}$ is not a whole number. However if we choose to rewrite the first of these three polynomials in 10 as

$$(3 \times 10^3) + (5 \times 10^2) + (2 \times 10^1) + (7 \times 10^0),$$

then it is the expanded form of 3527.

Because polynomials are just numbers, we can add, subtract, multiply, and divide them as usual. With the exception of division, the other three arithmetic operations produce another polynomial in a routine manner. (Division of polynomials does not generally produce a polynomial and will be looked at separately.) Consider the product of two first degree polynomials, for example. If a, b, c, d are the coefficients of two **linear polynomials** (i.e., polynomials of degree 1) in x, then

$$\begin{aligned}
(ax+b)(cx+d) &= (ax+b)(cx) + (ax+b)d \qquad \text{(dist. law)} \\
&= acx^2 + bcx + adx + bd \qquad \text{(dist. law)} \\
&= acx^2 + (ad+bc)x + bd \qquad \text{(dist. law)}
\end{aligned}$$

Of course we had to collect terms of the same degree using the distributive law and rearranging the terms so that they are in descending powers of x at the end. By the definition of a polynomial, this has to be done automatically anyway. The main point is to emphasize the role played by the distributive law and to showcase the fact that multiplying polynomials is no different from the usual operations with numbers. If the arithmetic of numbers (whole numbers and fractions) is taught correctly, such operations with polynomials are just more of the same and would not be a problem. *In particular, the uncivilized mnemonic device called FOIL is to be studiously avoided.*

We have mentioned the need to sometimes look at an equality backwards, and now we have to repeat this message. What we obtained above,

$$(ax+b)(cx+d) \;=\; acx^2 + (ad+bc)x + bd$$

is nothing but routine applications of the distributive law. However, when this equality is read backwards, it becomes

$$acx^2 + (ad+bc)x + bd \;=\; (ax+b)(cx+d)$$

In general, if the polynomials $p(x)$, $q(x)$, $r(x)$ in x satisfy $p(x) = q(x)r(x)$, then we say $q(x)r(x)$ is a **factorization** of $p(x)$ if the degrees of both $q(x)$ and $r(x)$ are positive. (Thus $\frac{5}{3}x^3 - 2x^2 + \frac{2}{3} = (\frac{1}{3})(5x^3 - 6x^2 + 2)$ is not a factorization of $\frac{5}{3}x^3 - 2x^2 + \frac{2}{3}$.) In this terminology, what we have obtained is a factorization of $acx^2 + (ad+bc)x + bd$ as a product $(ax+b)(cx+d)$, where it is understood that

$a \neq 0$ and $c \neq 0$. For example, we get

$$\frac{1}{2}x^2 + \frac{5}{4}x - 3 \;=\; (2x-3)(\frac{1}{4}x+1)$$

by letting $a = 2$, $b = -3$, $c = \frac{1}{4}$, and $d = 1$. With some practice, the factorization of $\frac{1}{2}x^2 + \frac{5}{4}x - 3$ can be done directly. One way is the following. Since it is much easier to deal with integers rather than rational numbers, we rewrite the polynomial by using the distributive law to take out the denominators of all the coefficients, as follows.

$$\frac{1}{2}x^2 + \frac{5}{4}x - 3 \;=\; \frac{1}{4}(2x^2 + 5x - 12)$$

Then we recognize that

$$(2x^2 + 5x - 12) \;=\; (2x-3)(x+4)$$

because the zero-degree term (i.e., 12) of $2x^2 + 5x - 12$ has to be the product of the zero-degree terms -3 and 4 of $2x - 3$ and $x + 4$, and the coefficient 2 of $2x^2 + 5x - 12$ has to be the product of the coefficients 2 and 1 of $2x - 3$ and $x + 4$, respectively. So a few trials and errors would get it done. Hence, we obtain as before,

$$\frac{1}{2}x^2 + \frac{5}{4}x - 3 = \frac{1}{4}(2x^2 + 5x - 12) = \frac{1}{4}(2x-3)(x+4)$$

At present, the teaching of factoring quadratic polynomials with integer coefficients figures prominently, not to say obsessively, in an algebra course. For this reason, some perspective on this subject is called for. One should keep in mind that all it is is learning to decompose two whole number A and C into products of integers $A = ac$ and $C = bd$ so that $B = ad + bc$ and a given quadratic polynomial $Ax^2 + Bx + C$ can be written as $acx^2 + (ad + bc)x + bd$ (which equals $(ax+b)(cx+d)$). There is no denying that beginning students ought to acquire some facility with decomposing numbers into products. It is also important that they can effortlessly factor a simple quadratic polynomial $x^2 + 2x - 35$ as $(x+7)(x-5)$. But it sometimes happens that *if a little bit of something is good, a lot of it can actually be bad for you.* This seems to be the case here, when the teaching of a small skill gets blown up to be a major topic, with the consequence that other topics that are more central and more substantial (such as learning about the graphs of linear equations or solving rate problems correctly) get slighted. The teaching of algebra should avoid this pitfall. Please also keep in mind the fact that once the quadratic formula becomes available (see Section 12), there will be a two-step algorithm to accomplish this factorization no matter what the coefficients of the quadratic polynomial may be.

We give one more illustration of the multiplication of polynomials; each step except the last makes use of the distributive law.

$$\begin{aligned}(5x^3 - \frac{1}{2}\,x)(x^2 + 2x - 4) &= (5x^3 - \frac{1}{2}\,x)x^2 + (5x^3 - \frac{1}{2}\,x)2x - (5x^3 - \frac{1}{2}\,x)\,4 \\ &= (5x^5 - \frac{1}{2}\,x^3) + (10x^4 - x^2) - (20x^3 - 2x) \\ &= 5x^5 + 10x^4 - \frac{41}{2}\,x^3 - x^2 + 2x\end{aligned}$$

By the way, reading this equality backwards gives a factorization that is (for a change) not so trivial:

$$5x^5 + 10x^4 - \frac{41}{2}\,x^3 - x^2 + 2x \;=\; (5x^3 - \frac{1}{2}\,x)(x^2 + 2x - 4)$$

Note the fact that if $p(x)$ and $q(x)$ are polynomials of degree m and n, respectively, then the degree of the product $p(x)q(x)$ is $(m + n)$. In other words, *the degree of a product is the sum of the degrees of the individual polynomials.* For example, the preceding calculation which multiplies a degree 3 polynomial with a degree 2 polynomial yields a polynomial of degree 5 ($= 3 + 2$).

Question: Is the sum of two n-th degree polynomials always an n-th degree polynomial?

Rational expressions

A quotient (i.e., division) of two polynomials in a number x is called a **rational expression in** x. Here is an example:

$$\frac{3x^5 + 16x^4 - 25x^2 - 7}{x^2 - 1}$$

We note that in the case of rational expressions, we need to exercise some care in not allowing division by 0 to take place. For example, in the preceding rational expression, x can be any number *except* ± 1 because if $x = \pm 1$, then $x^2 - 1 = 0$ and the denominator would be 0.

In writing rational expressions, it is usually understood that only those values for which the denominator is nonzero are considered.

In middle school, we are mainly interested in rational numbers only and, as a consequence, *all computations with numbers tacitly assume that the numbers involved are rational numbers.* With this mind, since x is a (rational) number, a rational expression is just a complex fraction and can therefore be added, subtracted, multiplied, and divided like any other complex fraction. For example, in case $x = \frac{1}{2}$ in the foregoing rational expression, we would be looking at the complex fraction

$$\frac{3(\frac{1}{32}) + 16(\frac{1}{16}) - 25(\frac{1}{4}) - 7}{(\frac{1}{4}) - 1},$$

which is equal to $16\frac{5}{24}$. In general, no matter what x is, we can compute with rational expressions in x in the usual way:

$$\frac{5x^3+1}{x^8+x-2} + \frac{2x^7}{x^3+4} = \frac{(5x^3+1)(x^3+4) + (2x^7)(x^8+x-2)}{(x^8+x-2)(x^3+4)}$$

and

$$\frac{x^2+1}{x^2+4x-7} \cdot \frac{6}{3x^4-5} = \frac{(x^2+1)(6)}{(x^2+4x-7)(3x^4-5)}$$

and

$$\frac{\frac{2x+1}{x^2-3}}{\frac{4x^3-x+11}{2x}} = \frac{(2x+1)(2x)}{(x^2-3)(x^3-x+11)}$$

These are the same as any computation with complex fractions. It is important to realize that these computations are exactly the same as those in complex fractions and not just "analogous to" them. There is so much in algebra that is just a revisit of arithmetic.

Because the cancellation law is valid for complex fractions (i.e., $\frac{AB}{AC} = \frac{B}{C}$ for all rational numbers A, B, C, with $A \neq 0$, $C \neq 0$),[16] some rational expressions can be simplified. Sometimes the cancellation presents itself, as in

$$\frac{(5x^4-x^3+2)(2x-15)}{(14x^2+3x-28)(5x^4-x^3+2)}$$

[16] Again, compare Section 5 in Chapter 2 of the Pre-Algebra notes.

Here, the number $(5x^4 - x^3 + 2)$ in both the numerator and denominator can be cancelled, resulting in

$$\frac{(5x^4 - x^3 + 2)(2x - 15)}{(14x^2 + 3x - 28)(5x^4 - x^3 + 2)} = \frac{2x - 15}{14x^2 + 3x - 28}$$

Sometimes, the cancellation can be less obvious. For example, the rational expression

$$\frac{x^3 - 8}{x^2 + 2x + 4}$$

can be simplified to $x - 2$ because, by an earlier identity,

$$x^3 - 8 = x^3 - 2^3 = (x - 2)(x^2 + 2x + 4)$$

and we can cancel the number $(x^2 + 2x + 4)$ from the numerator and denominator. (As you will learn when we come to quadratic equations, it turns out that in the case of this particular rational expression in x, x can be any number because $x^2 + 2x + 4$ is never equal to 0. Therefore, we actually have an identity $\frac{x^3-8}{x^2+2x+4} = x - 2$ for all x.)

In beginning algebra, often there is too much emphasis on simplifying rational expressions. This is a left-over from the questionable practice of teaching fractions by insisting on the reduction of all fractions to lowest terms at all costs.

It remains to round off this discussion by mentioning that, just as one can easily define polynomials in x, y, z, etc., one can likewise define rational expressions in x, y, z, etc.

EXERCISES

1. If a, b are two numbers, what are $(a + b)^3$ and $(a - b)^3$? (These are useful identities to bear in mind.)

2. Is the following true or false for any numbers s and t ?

$$(s^2 - t^2)^2 + (2st)^2 = (s^2 + t^2)^2$$

Do you see why such an identity could be of interest?

3. Sum $5^6 + 5^7 + 5^8 + \cdots + 5^{27}$.

4. If y is a nonzero number, what is $1 + \frac{1}{y} + \frac{1}{y^2} + \frac{1}{y^3} + \cdots + \frac{1}{y^{19}}$?

5. If y is a nonzero number and n is a positive integer, what is $\frac{1}{y^3} + \frac{1}{y^4} + \cdots + \frac{1}{y^n}$?

6. Sum $\frac{1}{4^3} - \frac{1}{4^5} + \frac{1}{4^7} - \frac{1}{4^9} + \cdots - \frac{1}{4^{33}}$.

7. If x is a nonzero number and n is a positive integer, what is

$$-1 + \frac{1}{x^3} - \frac{1}{x^6} + \frac{1}{x^9} - \cdots - \frac{1}{x^{2n \cdot 3}} \ ?$$

8. If a is a number, what is $(\frac{2}{3}a^2 - \frac{3}{4}a - 1)(\frac{1}{2}a^2 + \frac{1}{3}a - \frac{4}{9})$? What is $(a^2 - \frac{5}{3}a - \frac{2}{3})(a^2 + \frac{5}{3}a - \frac{2}{3})$?

9. If x and y are numbers and $x \neq y$, $\dfrac{1}{x^2 - y^2} - \dfrac{1}{x^2 + y^2} = ?$

10. If x is a number different from 2, -3, and -1, what is

$$\frac{2}{x-2} + \frac{3}{x+3} - \frac{1}{x+1} = ?$$

11. If x is a number that makes all the denominators nonzero in the following, simplify:

$$\frac{\frac{2x^3 - 9x^2 - 5x}{(x-2)^2}}{\frac{x^2 - 3x - 10}{x^4 - 16}}$$

12. Factor $4x^2 - 12x + 9$, and $25x^2 + 10x + 1$ for any x.

13. Factor $a^3 + b^3$ for any numbers a and b. Factor $a^{2n+1} + b^{2n+1}$ for any positive integer n. Show that $5^{89} + 6^{89}$ is never a prime.

14. Factor $s^4 + s^2t^2 + t^4$ for any numbers s and t. Factor $s^{4k} + s^{2k}t^{2k} + t^{4k}$ for any positive integer k.

15. Simplify (assuming x and y are not both 0):

$$(i)\ \frac{4x^4 - 9y^4}{4x^4 + 12x^2y^2 + 9y^4}, \quad (ii)\frac{15x^3y^4}{-60x^2y^7}$$

16. If x, y, z are numbers, compute $(x+y+z)(x-y-z)$. (Obviously, you can compute it by brute force. Equally obvious, such is not the expectation of this problem, that all you can do is to compute by brute force.)

2 Transcription of Verbal Information into Symbolic Language

Equations and inequalities

Word problems are the bugbears of students (and teachers too). Many experience difficulty transcribing verbal information into equations or inequalities. Part of this difficulty stems from students' habit, probably acquired in their primary grades, of not trying to understand what the problem says but looking instead for so-called "key words" as part of the rote skill to replace words by symbols. Thus, "increase by" is $+$, "less than" is $-$, "of" is $\times$, etc. We will not pretend to be psychologists and offer advice on how to convince students to actually read the words in front of them. What we propose to do is to *assume* that they are willing to read and try to offer a bridge that leads them from words to symbols.

We first pause to formally define an **equation** in a symbol x, which is always assumed to be a number. Many readers would feel incredulous at this point that they would be called upon to do something this ridiculous: *define* an equation? Have they not worked with equations all their lives? Perhaps. But let us see what an equation is in the way mathematics is taught in schools. First x is a "variable", which means it is some symbol and all you know is that it varies. Then something like

$$3x - 5 \;=\; 7x - 1$$

is given. You immediately set about "solving" it by going through the motions. What is $3x - 5$, and what is $7x - 1$? Both are combinations of a "variable" and some numbers, and therefore both "vary" so that you don't know what they are. Yet you are supposed to accept that they are equal and you go about computing with them as if they were plain, ordinary numbers. Are you making any sense? Is mathematics so incomprehensible that it is reduced to a collection of symbolic manipulations devoid of any meaning, and you go through them because "you are supposed to"?

Such thoughts should give you pause, and make you aware that you have to take a fresh look at what an equation in x is all about.

An **equation in x** is an invitation to determine the totality of all the *numbers* x so that two given expressions in x are equal. Any number x that makes the equality of the expressions valid is called a **solution**. In this terminology, to **solve an equation** is to obtain *all* the solutions of the equation. For example, suppose one expression is $3x - 5$ and the other is $7x - 1$. The equation $3x - 5 = 7x - 1$ asks for the collection of (all the) numbers x_0 so that $3x_0 - 5 = 7x_0 - 1$. As is well known, the only solution in this case is -1, and we will discuss this in the next section. In textbooks and education materials, the whole question is usually presented as the following symbolic statement with no preamble:

$$\textit{Solve} \quad 3x - 5 \;=\; 7x - 1.$$

Notice that such a statement violates the Basic Protocol in the use of symbols because the symbol x has not been quantified and we have no idea what it is.

We hasten to add that, after a while, we too will get tired of repeating the cumbersome statement about "for which collection of numbers x is it true that $3x - 5 = 7x - 1$?" We will sooner or later lapse into the same cryptic statement "Solve $3x - 5 \;=\; 7x - 1$." At that point, though, we hope that it will be clear what we mean by this cryptic statement and there will be no misunderstanding about this intentional abbreviation. Before reaching that point, however, it is a good idea to remind ourselves of the real meaning of an equation, because if we don't, the process of "solving an equation" would degenerate into a sequence of meaningless moves that you see in textbooks. If you cannot make sense to yourself, how do you make sense to your students, and if you don't do that, how can you be a teacher?

Given an equation in x, it could happen that *every* x is a solution, there are some (finite number of) solutions, or there is no solution. An equation for which every number is a solution is what we called in Section 1 an identity. For example, $(x+1)^2 = x^2+2x+1$ is an identity in x. On the other hand, the equation $x^3-1 = -\frac{5}{2}x^2-\frac{1}{2}x$ for a number x is an equation with exactly three solutions: $\frac{1}{2}$, -1, -2.[17] The equation $x^2+1 = x^2$ has no solution, however, as is easily seen. Because an equation in x involves only one number x, it is usually called **an equation of one variable** out of respect for tradition. This terminology is retained in the mathematics literature because, as we said, it was used in the past and, like "identity", it is convenient to have around. However, you should also see that, strictly speaking, *we don't need this terminology for solving equation*, so please don't lose sleep over what "variable" means.

Equations in a collection of (yet-to-be-determined) numbers are similarly defined. We will deal with equations in two variables in §4.

In a similar vein, an **inequality** in a number x is an invitation to determine the totality of all the numbers x that makes one number expression in x bigger than (or, bigger than or equal to) the other. Again, the inequality may be valid for all x, for some x, or for no x. The inequality $x^2+1 < 0$, for example, is satisfied by no x. As in the case of equations, *the explicit statement that an inequality in x is an invitation to determine the totality of all the numbers x that makes one number expression in x bigger than (or, bigger than or equal to) the other will sometimes be omitted in the future.*

Some examples of transcription

There is no hope of getting the correct solution to a word problem if we work with

[17] Although we will not prove this fact here, but it is not at all difficult to prove that there can be no more than 3 solutions, and since $\frac{1}{2}$, -1, -2 are easily seen to be solutions, they are the only ones.

the wrong equations or inequalities. Therefore the focus of this section is on this critical link in the solution of word problems, namely, the process of transcribing verbal information into equations or inequalities. The importance of a correct transcription of the verbal information into symbolic expressions in the context of problem-solving seems not to be fully recognized by teachers and educators alike. Many teacher jump directly from reading the verbal information in a problem to attempting to get a solution, often by guess-and-check. The fact that there is an intermediate step between the "reading" and the "solving" is one that deserves emphasis. Our purpose here is to isolate this intermediate step and call attention to the need of addressing this critical issue. Without worrying about how to solve equations or inequalities (i.e., the exact determination of all the numbers x that make the equality or inequality valid), our task is to make sure that students learn to set up problems correctly. To this end, you as a teacher must take the lead. You will be asked to do plenty of such transcriptions in this section, with the hope that the extra practice will enable you to gain the needed facility and confidence to help your students.

We will give a few illustrative examples. In these examples, notice that *the starting point is always a systematic, sentence by sentence transcription of the verbal data into symbolic language.* Then all the information is pulled together at the end to arrive at the correct equation(s). Let us begin with a simple one.

> *Let $\frac{a}{b}$ be a fraction. If $\frac{a}{b}$ of 57 is taken away from 57, what remains exceeds $\frac{2}{3}$ of 57 by 4. Express this information as an equation in $\frac{a}{b}$.*

Solution We know from the meaning of the multiplication of fractions that "$\frac{a}{b}$ of 57" is just $\frac{a}{b} \times 57$. (We put in the $\times$ symbol here for clarity, as $\frac{a}{b} \cdot 57$ would look somewhat odd, while writing it as $57 \cdot \frac{a}{b}$ might confuse it with a mixed number. The main purpose of the symbolic language is to add clarity and brevity to the verbal expression. Consequently, whatever convention there is concerning the use of symbols should be ignored when clarity or brevity is jeopardized.) Thus the statement, "If a fraction $\frac{a}{b}$ of 57 is taken away from 57", becomes

$$57 - (\frac{a}{b} \times 57)$$

because of the exact definition of subtraction (see Section 3 in Chapter 1 of the Pre-Algebra notes. According to the given information, this number is 4 bigger than $\frac{2}{3}$ of

57, i.e., 4 bigger than $\frac{2}{3} \times 57$. So we transcribe this information directly:

$$\{57 - (\frac{a}{b} \times 57)\} - (\frac{2}{3} \times 57) = 4$$

This is then the equation in the fraction $\frac{a}{b}$ that we must solve.

The following example is a bit more complicated.

> *Johnny has three siblings, two brothers and a sister. His sister is half the age of his older brother, and three-fourths the age of his younger brother. Johnny's older brother is four years older than Johnny, and his younger brother is two years younger than Johnny. Let J be the age of Johnny, A the age of Johnny's older brother, and B the age of his younger brother. Express the above information in terms of J, A, and B.*

Solution The first thing to take note of is that the given data of the problem involves Johnny's sister, but we are asked to "express the above information in terms of J, A, and B", i.e., the sister is left out. There are many ways to deal with this situation, and one of them is to transcribe all the information by bringing in the sister, and then try at the end to omit any reference to her and still faithfully transcribe the given verbal data.

So let S be the age of the sister. "His sister is half the age of his older brother" then becomes $S = \frac{1}{2}A$, and "His sister is ...three-fourths the age of his younger brother" becomes $S = \frac{3}{4}B$. "Johnny's older brother is four years older than Johnny" becomes $A = J + 4$, while "his younger brother is two years younger than Johnny" translates to $B = J - 2$.

At this point, the two equations, $A = J + 4$ and $B = J - 2$, would appear to be the answer because they are the only equations directly involving A, B, and J. But these two equations fail to capture the part of the given information about how the brothers are related to the sister, which gives information on how the brothers are related to each other. So we go back to look at $S = \frac{1}{2}A$ and $S = \frac{3}{4}B$. They show that both $\frac{1}{2}A$ and $\frac{3}{4}B$ are equal to S, and therefore equal to each other. Thus we also have $\frac{1}{2}A = \frac{3}{4}B$. Now we have collected all the given information concerning J, A and B:

$$A = J + 4, \qquad B = J - 2, \qquad \frac{1}{2}A = \frac{3}{4}B.$$

We next give an example requiring the use of inequalities:

> *Erin has 10 dollars and she wants to buy as many of her two favorite pastries as possible. She finds that she can buy either 10 of one and 9 of the other, or 13 of one and 6 of the other, and in both cases she will not have enough money left over to buy more of either pastry. If the prices of the pastries are x dollars and y dollars, respectively, write down the inequalities satisfied by x and y.*

Solution With x and y understood, Erin spends a total of $\$(10x + 9y)$ in the first option, and then $\$(13x+6y)$ in the second option. The key point is that in either case, "she will not have enough money left over to buy more of either pastry". Consider then the first option: the total number of dollars left over is $10 - (10x + 9y)$. If this amount exceeds or equals x, then Erin would be able to purchase one more of this pastry. Such not being the case, we have $10 - (10x + 9y) < x$. Moreover, she cannot spend more than \$10, and so $10 - (10x + 9y) \geq 0$. We combine these two inequalities into the following *double inequality*:[18]

$$0 \leq 10 - (10x + 9y) < x$$

Note that it is strict inequality and not $10 - (10x + 9y) \leq x$, as the possibility of equality (implied by $\leq$) would mean that Erin could buy one more of the x-pastry. Similarly,

$$0 \leq 10 - (10x + 9y) < y$$

The same consideration applies to the second case. The answer is then the collection of four double inequalities:

$$\begin{array}{ll} 0 \leq 10 - (10x + 9y) < x, & 0 \leq 10 - (10x + 9y) < y, \\ 0 \leq 10 - (13x + 6y) < x, & 0 \leq 10 - (13x + 6y) < y \end{array}$$

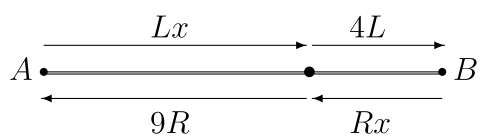
As a final illustration, we do a problem that is a trifle more sophisticated than the previous three. It is very instructive.

[18] See Section 6 in Chapter 2 of the Pre-Algebra notes.

Two women started at sunrise and each walked at constant speed. One went straight from City A to City B while the other went straight from B to A. They met at noon and, continuing with no stop, arrived respectively at B at 4 pm and at A at 9 pm.

If the sunrise was x hours before noon, and if L is the speed of the woman going from A to B and R is the speed of the woman going from B to A, transcribe the information above into equations using the symbols L, R and x.

Solution For ease of discussion, we will refer to the woman going from City A to City B as the First Woman, and the other as the Second Woman. Before looking at a correct solution, we first look at one that may be what most people would write down, and explain why it is not good enough. The reasoning goes as follows.



The distance between City A and City B is fixed, and both the First Woman and Second Woman walked this distance in the time given: the First Woman walked x hours before noon, and then another 4 hours (from noon till 4 pm), while the Second Woman also walked x hours before noon but continued for another 9 hours (from noon till 9 pm). So the First Woman walked a total of $x + 4$ hours while the Second Woman walked a total of $x + 9$ hours. Given that the former walked with speed L (let us say, miles per hour), the total distance she walked in $x + 4$ hours is of course $L(x+4)$ miles. Similarly the total distance the Second Woman walked in $x+9$ hours is $R(x+9)$ miles. By a previous remark, both distances are the same as both women walked between cities A and B. Therefore we get

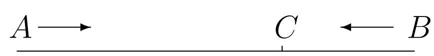
$$L(x + 4) = R(x + 9)$$

This is supposed to be the answer to the problem. But is it?

What this equation fails to capture is the information that the two women met at noon after walking x hours in opposite directions from City A and City B. This means that the total distance they covered after walking the first x hours (which is $Lx + Rx$ miles) is the distance between the cities, which is $L(x+4)$ or $R(x+9)$ miles, as we have seen. Therefore the additional piece of information that must be incorporated into the symbolic transcription is $Lx + Rx = L(x+4)$ or $Lx + Rx = R(x+9)$. Either one would do. Therefore, a solution to the problem is the following set of equations:

$$L(x+4) = R(x+9), \qquad R(x+9) = Lx + Rx$$

Now one may reason slightly differently. Let the meeting point of the two women be C:



Consider the distance between A and C. The First Woman covered it in x hours (before noon) while the Second Woman covered it in 9 hours (after noon). But in x hours the First Woman would walk Lx miles, while in 9 hours the Second Woman would walk $9R$ miles. Therefore $Lx = 9R$. Similarly, if we consider the distance between C and B, we get in exactly the same fashion that $4L = Rx$. A correct solution is then:

$$Lx = 9R, \qquad 4L = Rx$$

We will leave as an exercise that these two solution are "the same", in a precise sense.

EXERCISES

Do **not** attempt to solve any of the following problems. Do only what the problems tells you to do, which is always about writing down the needed equations or inequalities rather than getting the answers.

1. A train travels s km in t hours. At the same constant speed, how far does it travel in 5 hours? In T hours? How long does it take the train to travel 278 km? x km? If the speed of the train is tripled (3 times as fast), how long will it take to travel s km? And if the speed is n times faster?

2. The sum of the squares of three consecutive integers exceeds three times the square of the middle integer by 2. If the middle integer is x, express this fact in terms

of x. If the smallest of the three integers is y, express the same fact in terms of y.

3. Paulo read a number of pages of a book with N pages, then he read 43 pages more and finished three-fifths of the book. If p is the number of pages Paulo read the first time, write an equation using p and N to express the above information.

4. A whole number has the property that when the square of half this number is subtracted from 5 times this number, we get back the number itself. If y is this number, write down an equation for y.

5. Helena bought two books. The total cost is 49 dollars, and the difference of the squares of the prices is 735. If the prices are x and y dollars, express the above information in terms of x and y.

6. I have two numbers x and y. Take 20% of x from x, then what remains would be 4 less than y. If however I enlarge y by 20%, then it would exceed x by 5. Express this information in equations in terms of x and y.

7. I have \$4.60 worth of nickels, dimes and quarters. There are 40 coins in all, and the number of nickels and dimes is three times the number of quarters. If N, D, and Q denote the number of nickels, dimes, and quarters, respectively, write equations in terms of these symbols to capture the given information.

8. There are two whole numbers. When the large number is divided by the smaller number, the quotient is 9 and the remainder is 15. Also, the larger number is 97.5% of ten times the smaller number. If x is the larger number and y is the smaller number, express the given information in equations in terms of x and y.

9. We look for two whole numbers so that the larger exceeds the the smaller by at least 10, but that the cube of the smaller exceeds the square of the larger number by at least 500. If the larger number is x and the smaller number is y, transcribe the above information in terms of x and y.

10. If the digits of a three-digit number are reversed, the sum of the new number and the original number is 1615. If 99 is added to the original number, the digits are reversed. Let the hundreds, tens, and ones digits of the original numbers be a, b, and c, respectively. Write equations in a, b and c to express the given information. (*Caution:* Be very careful with the writing of your symbolic expressions.)

11. Two marathon runners run at constant speeds. The first runner runs D kilometers in A hours, and the second runner in B hours. If they start running at the same time from separate cities, D kilometers apart, towards each other, they are 11 kilometers apart after 1 hour. If the first runner runs twice as fast and, again, they do the same, then they would be 5 kilometers apart after one hour. Express this information in symbolic language in terms of A, B, and D.

12. A sum of money is to be divided equally among x people, each receiving y dollars. If there are 3 more people, each person would receive 1 dollar less, and if there are 6 fewer people, each would receive 5 dollars more. Write equations in x and y to express this information.

13. A man walked from one place to another in $5\frac{1}{2}$ hours. If he had walked $\frac{1}{4}$ of a mile an hour faster, the walk would have taken $36\frac{2}{3}$ fewer minutes. If the distance he walked is x miles and his speed was v miles an hour, express the above information in terms of x and v.

14. The denominator of a fraction exceeds twice the numerator by 2, and the difference between the fraction and its reciprocal is $\frac{55}{24}$. If the numerator is x and the denominator y, write equations in terms of x and y to express the above information.

15. A video game manufacturer sells out every game he brings to a game show. He has two games, an A Game and a B Game. He can bring 50 of A Games and B Games in total to the show. Each A Game costs \$75 to manufacture and will bring in a net profit of \$ 125. Each B Game costs \$165 to manufacture and will bring in a net profit of \$185. However, he only has \$6,000 to spend on manufacturing. If he brings x A Games and y B Games, describe in terms of x and y how he can maximize his profit.

16. Here are the instructions of a "magic trick":

(1) Grab a calculator.
(2) Key in the first three digits of your phone number (NOT the area code).
(3) Multiply by 80.
(4) Add 1.
(5) Multiply by 250.
(6) Add the last 4 digits of your phone number.
(7) Add the last 4 digits of your phone number again.
(8) Subtract 250.
(9) Divide number by 2.

Let the 3-digit number which is the first three digits of your phone number be denoted by x, and let the 4-digit number which is the last four digits of your phone number be denoted by y. Writing down the equation that shows that you *always* get back your phone number at the end.

17. Here are the instructions of another "magic trick":

(1) Pick any 3-digit number between 000 and 999.
(2) Reverse the order of digits.
(3) Subtract the larger from the smaller, getting another three digit number.
(4) Reverse those digits.
(5) Add this number to the last one.

Let the hundreds digit, the tens digit, and the ones digit of the 3-digit number be a, b, c, respectively. Write down the equation that show that you *always* get 1089 or 0 at the end.

3 Linear Equations in One Variable

In the preceding section, we have seen how equations involving a number x arise naturally. In this section, we make a first attempt at solving the simplest kind of equations which will turn out to be basic, namely, equations which ask for numbers x so that two given polynomials in x of degree at most 1 are equal. These are called **linear equations of one variable.** Examples are: $12x - 7 = 5x + 13$, $-\frac{5}{6}x + 1 = 23x - 4$, and $9 = 27x - 4$.

Now these are equations that you can solve with one eye closed. We will perhaps make you feel some discomfort by carefully analyzing the usual procedure of such solutions and ask if it makes any sense. Let us first look at how a simple equation such as $2x - 3 = 4x$ is solved in the manner it is taught in schools.

Step 1: $2x - 3 \;=\; 4x$

Step 2: $(2x - 3) - 4x \;=\; 4x - 4x.$

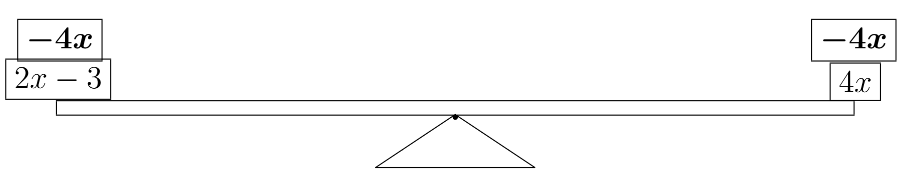
Step 3: $-2x - 3 \;=\; 0$

Step 4: $(-2x - 3) + 3 \;=\; 0 + 3$

Step 5: $-2x \;=\; 3$

Step 6: $x \;=\; -\frac{3}{2}$

How is Step 2 justified? Don't forget we are doing things the *usual way*, and the usual way is that x is a *variable*. Therefore we concede that we really don't know what $2x - 3$ or $4x$ is other than the fact they are supposed to be equal. Equal as what? We don't know, but surely an equality, whatever that is, should mean something. If $2x - 3 = 4x$, then it seems reasonable that adding the same object, $-4x$, to both $2x - 3$ and $4x$ would maintain the equality. Of course we have to offer *some* explanation, so we set up an analogy. Imagine that we have a balance and on the two sides of the balance are $2x - 3$ and $4x$, which balance each other out.



Thus putting $-4x$ on both sides will not "tip the balance", and this explains Step 2.

We have other concerns with this way of solving the equation, e.g., there is the same objection to Step 4. However, let us first analyze Step 2. Using the balance as an ***analogy*** is not good, because how can we be certain that the analogy is valid beyond a reasonable doubt? In addition, we cannot reason with a quantity, a *variable*, which cannot be defined precisely. For the case at hand, there is no way we can explain how the two quantities involving the *variable* x, $2x - 3$ and $4x$, are equal.

The right way of solving an equation is actually far simpler. It goes as follows. If we want a solution to $2x - 3 = 4x$, the way to look for it is to use the time-honored method of *pretending that we already know what it is*, let us say x_0, and then make use of this information to find out what it could be. Once we have this information, we do a simple check to verify that what we believe to be true is true.

So we assume that a number x_0 is a solution of $2x - 3 = 4x$. Thus

$$2x_0 - 3 \;=\; 4x_0$$

We emphasize that this is an equality between two *numbers* and, as such, we can bring to bear all we know about numbers[19] on the equality. Therefore, the following computation is entirely routine:

Step 1′: $2x_0 - 3 \;=\; 4x_0$

Step 2′: $(2x_0 - 3) - 4x_0 \;=\; 4x_0 - 4x_0$.

Step 3′: $-2x_0 - 3 \;=\; 0$

Step 4′: $(-2x_0 - 3) + 3 \;=\; 0 + 3$

Step 5′: $-2x_0 \;=\; 3$

Step 6′: $x_0 \;=\; -\frac{3}{2}$

In view of the fact that these are assertions about numbers, every one of the steps from 1′ to 6′ is transparent and there are no more dark clouds about a *variable* hanging over us. Some additional comments, however, are useful. Take the transition

[19] We use what we know about rational numbers, and remember FASM.

from Step $2'$ to Step $3'$. The right side is clear: $4x_0 - 4x_0 = 0$. If the details of the the left side are made explicit, they would go something like this:

$$\begin{aligned}
(2x_0 - 3) - 4x_0 &= (2x_0 + (-3)) + (-4x_0) && \text{(by definition of subtraction)} \\
&= (2x_0 + (-4x_0)) + (-3) && \text{(comm. and assoc. laws of +)} \\
&= -2x_0 + (-3) && \\
&= -2x_0 - 3 && \text{(definition of subtraction)}
\end{aligned}$$

What is notable here is the invocation of the commutative and associative laws of addition in the second line.[20] When we do computations with specific numbers, the use of the commutative or associative law is probably looked on with a trace of suspicion. For example, one hardly needs the associative and commutative laws to claim that $(15 - 7) - 16 = (15 - 16) - 7$, because the left side is $8 - 16 = -8$ whereas the right side is $-1 - 7 = -8$. There is no need for these laws. But now look at

$$(2x_0 + (-3)) + (-4x_0) = (2x_0 + (-4x_0)) + (-3)$$

Here *we don't know what x_0 is and no explicit computation is possible.* How can we claim this is valid except by invoking Theorem 1 in the Appendix of Chapter 1 in the Pre-Algebra notes?

It is only when we do algebra and have to compute with an unknown number that the import of the general laws (associative and commutative laws of + and ×, and the distributive law) begins to be apparent. Each time we solve an equation, we depend crucially on these laws.

A second comment is that what has been accomplished in Steps $1'$ to $6'$ is that ***if*** there is a solution x_0 to $2x - 3 = 4x$, then $x_0 = -\frac{3}{2}$. This has nothing to say whether $-\frac{3}{2}$ is a solution of $2x - 3 = 4x$ or not. Of course, having come so far, it is simple to check that such is the case:

$$2(-\frac{3}{2}) - 3 = -3 - 3 = -6 \quad \text{and} \quad 4(-\frac{3}{2}) = 2(-3) = -6$$

Our solution of $2x - 3 = 4x$, is now complete.

A third comment is that perhaps you begin to appreciate the earlier remark that there is no need for the concept of a *variable.* We have solved the equation by dealing

[20]See Theorem 1 in Appendix of Chapter 1 in Pre-Algebra notes.

strictly with *numbers*, and by observing the Basic Protocol in the use of symbols. This is a lesson you should bring back to your classroom.

The parallel between Steps $1'$–$6'$ and Steps 1–6 earlier (if we replace the x_0 in the former by x) is the reason that, despite the flawed reasoning in Steps 1–6, the solution obtained there is correct. However, this raises the specter that, in order to do mathematics correctly, even the solution of a simple problem such as $2x - 3 = 4x$, requires such a cumbersome explanation as given in the preceding paragraphs. Our final comment on Steps $1'$–$6'$ is intended to lay such fears to rest. On the one hand, you have to thoroughly understand what it means to solve an equation, and everything we have said is therefore an integral part of a teacher's repertoire. On the other hand, school students need to see such an explanation of a solution only once (with occasional reminders perhaps later on) and, because of the parallel between Steps 1–6 and Steps $1'$–$6'$, they can formally get the solution by the usual procedure of Steps 1–6.

Now that we have a fresh understanding of Steps 1–6, we will use this language to describe the general structure of solving a linear equation. There are two components.

(I) Solve equations in x of the form $ax = b$, where a, b are constants with $a \neq 0$. Clearly the solution is $\frac{b}{a}$, as one can check:

$$a\,(\frac{b}{a}) \;=\; b$$

Notice that the fact $a \neq 0$ guarantees that the fraction $\frac{b}{a}$ is well-defined. For example, the solution to $3x = -7$ is $-\frac{7}{3}$ $(= \frac{-7}{3})$.

(II) *Any linear equation* $Ax + B = Cx + D$ (*where* A, B, C, D *are constants and* $A \neq C$) *can be brought to the form* $ax = b$. In greater detail, this means: any number x that satisfies the former equation also satisfies the latter equation for some appropriate constants a and b.

The reason is that if x is a number so that $Ax + B = Cx + D$, then

$$(Ax + B) + (-Cx - B) \;= (Cx + D) + (-Cx - B)$$

Therefore, by Theorem 1 in the Appendix of Chapter 1 in the Pre-Algebra notes, we have $Ax - Cx = D - B$, i.e., $(A - C)x = D - B$. In other words, the original equation $Ax+B = Cx+D$ is now in the form of $ax = b$ with $a = A-C$ and $b = D-B$.

By part (I), the solution to $(A-C)x = D-B$ is

$$x \;=\; \frac{D-B}{A-C}$$

We now check that $(D-B)/(A-C)$ is indeed a solution of the original equation $Ax + B = Cx + D$, as follows. On the one hand,

$$\begin{aligned} A\left(\frac{D-B}{A-C}\right) + B &= \frac{AD-AB}{A-C} + B \\ &= \frac{AD-AB}{A-C} + \frac{BA-BC}{A-C} \\ &= \frac{AD-AB+AB-BC}{A-C} \\ &= \frac{AD-BC}{A-C} \end{aligned}$$

On the other hand,

$$\begin{aligned} C\left(\frac{D-B}{A-C}\right) + D &= \frac{CD-CB}{A-C} + D \\ &= \frac{CD-CB}{A-C} + \frac{DA-DC}{A-C} \\ &= \frac{CD-BC+AD-CD}{A-C} \\ &= \frac{AD-BC}{A-C} \end{aligned}$$

It follows that

$$A\left(\frac{D-B}{A-C}\right) + B \;=\; C\left(\frac{D-B}{A-C}\right) + D$$

or, what is the same thing, $(D-B)/(A-C)$ is a solution of $Ax + B = Cx + D$.

Notice that we need the assumption of $A \neq C$ to make sense of the fraction $\frac{D-B}{A-C}$.

We should now make contact with the terminology of school mathematics. What we did in part (II) is sometimes referred to as **isolating the variable**.[21] The process

[21] At the risk of sounding like a broken record, note that we freely use the word *variable* without saying what it means, because it doesn't matter.

of going from $Ax + B = Cx + D$ to $Ax - Cx = D - B$ is usually referred to as **transposing the terms $\boldsymbol{Cx}$ and $\boldsymbol{B}$ to the other side**.

For example, the solution to $3x - 1 = 8x + 7$ is

$$\frac{7+1}{3-8} = -\frac{8}{5}$$

It remains to examine the case of $Ax + B = Cx + D$ where A, B, C, D are constants and $A = C$. In this case, *it is time to remember what an equation means.* We are trying to determine the collection of all numbers x so that $Ax + B = Cx + D$. Suppose there is such a number x, then the same procedure as above leads to $(A - C)x = D - B$, which is $0 = D - B$. If $D \neq B$, then we have 0 being equal to a nonzero number, so that the assumption that such a number x exists is absurd. We conclude that there is no such number x. On the other hand, if $D = B$, then we have $0 = 0$, which is fine. In fact, let us go back to square one: we already assumed $A = C$ to begin with, and we have now further assumed $B = D$. Then of course the two sides of the equation $Ax + B = Cx + D$ are the same number for *every* x. Thus in this case we have an identity $Ax + B = Cx + D$ and all numbers x are solutions.

We summarize the whole discussion in the following theorem:

Theorem *Given a linear equation $Ax + B = Cx + D$, where A, B, C, D are constants. Then:*

(*i*) *The equation has a unique solution $(D - B)/(A - C)$ if $A \neq C$.*

(*ii*) *The equation has no solution if $A = C$ but $B \neq D$.*

(*iii*) *Every number is a solution if $A = C$ and $B = D$.*

The theorem should not be memorized. Rather, one should be totally fluent in repeating the steps in its proof in each case. Here is another example. To solve $-\frac{2}{3}x + 4 = -\frac{1}{5}x + 5\frac{1}{3}$, we transpose $-\frac{1}{5}x$ to the left: $(-\frac{2}{3}x + 4) - (-\frac{1}{5}x) = 5\frac{1}{3}$. On the left side we have $-\frac{2}{3}x + 4 + \frac{1}{5}x = -\frac{2}{3}x + \frac{1}{5}x + 4 = \frac{-7}{15}x + 4$. Thus the equation becomes $\frac{-7}{15}x + 4 = 5\frac{1}{3}$. We next transpose 4 to the right side: $\frac{-7}{15}x = 5\frac{1}{3} - 4$, and so $\frac{-7}{15}x = \frac{4}{3}$. Thus the solution is $\frac{4}{3}/\frac{-7}{15} = -\frac{20}{7}$.

To summarize, solving a linear equation in a number x depends on two simple ideas: by transposing terms, we isolate x on one side of the equation, and then we solve an equation of the type $ax = b$. From this point of view, the common practice of classifying linear equations into one-step equations, two-step equations, three-step equations and fourth-step equations, and then *teach* the solving of linear equations according to this classification simply does not make sense. You should avoid using this classification when you teach this topic.

It remains to point out that sometimes a linear equation is disguised as one involving rational expressions. For example, consider a number x that satisfies

$$\frac{2}{3x-1} \;=\; \frac{4}{x+\frac{1}{3}}$$

by the cross-multiplication algorithm (which is valid also for complex fractions), this equation is equivalent to

$$2(x+\frac{1}{3}) \;=\; 4(3x-1)$$

We can either use the distributive law on both sides directly, or we can avoid computations with fractions at this early stage and simply multiply both sides by 3 to get

$$2(3x+1) \;=\; 4(9x-3)$$

Then $6x+2 = 36x-12$ and $14 = 30x$. In other words, $x = \frac{7}{15}$.

EXERCISES

1. Solve: (i) $2x-8 = 15+\frac{4}{3}x$. (ii) $\frac{7}{3}x+2 = \frac{3}{2}-\frac{2}{5}x$. (iii) $\frac{11}{9}-\frac{5}{3}x = -6x+\frac{1}{18}$. (iv) $ax+6 = 8-7ax$, where a is a nonzero number. (v) $4bx+13 = 2x+26b$, where b is a number not equal to $\frac{1}{2}$. (vi) $\frac{1}{2}-\frac{8}{3}x = \frac{5}{6}x+\frac{2}{3}$. (vii) $\frac{2}{5}ax-17 = \frac{1}{3}ax-\frac{15}{2}$.

2. Given an equation $3x-8 = ax+7$, where a is number. For what values of a does the equation have a unique solution? have no solution? have an infinite number of solutions?

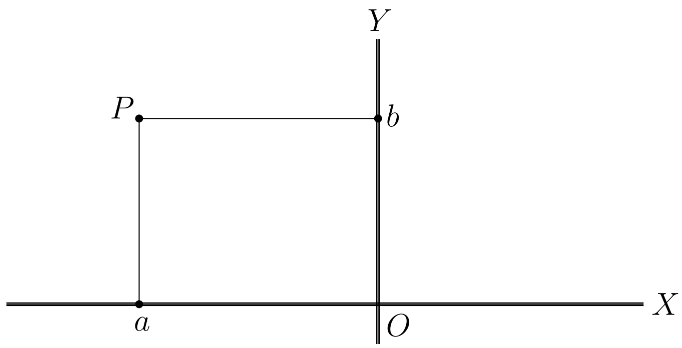
3. Solve: (a) $\dfrac{3-x}{x-1} \;=-\; \dfrac{3}{2}$. (b) $\dfrac{5}{2x-3} \;=\; \dfrac{4}{2-x}$.

4 Linear Equations in Two Variables and Their Graphs

Coordinate system in the plane

Before discussing the graphs of linear equations, we have to **set up a coordinate system in the plane**, in the sense that we will uniquely associate to each point of the plane an ordered pair of numbers, and vice versa. Because this is a standard process, we will merely outline the main points of how you can do this in the school classroom. In the process, we get to see why we took the trouble to prove that opposite sides of a parallelogram are equal (Theorem G4 in Chapter 5 of the Pre-Algebra notes).

Choose two perpendicular lines in the plane which intersect at a point to be called O. The horizontal line is traditionally designated as the ***x*-axis**, and the vertical one the ***y*-axis**. Recall that these may be regarded as number lines (cf. the discussion of distance in §1 of Chapter 5 in the Pre-Algebra notes). We may henceforth identify every point on these **coordinate axes** (as the x and y axes have come to be called) with a number. As expected, we choose the positive numbers on the x-axis to be on the right of O so that O is the zero of the x-axis, and we choose the positive on the y-axis to be above O on the y-axis so that O is also the zero of the y-axis. Then to each point P in the plane we associate an *ordered* pair of numbers in the following way. Let us agree to call any line parallel to the x-axis a **horizontal line**, and also any line parallel to the y-axis a **vertical line**. Then through P draw two lines, one vertical and one horizontal, so that they intersect the x-axis at a number a and the y-axis at a number b, respectively.Then the ordered pair of numbers $\boldsymbol{(a, b)}$ are said to be the **coordinates of *P*** (relative to the chosen coordinate axes); ***a*** is called the ***x*-coordinate** and ***b*** the ***y*-coordinate of *P*** (relative to the chosen coordinate axes).



Now, by construction, $PaOb$ is a parallelogram. Because opposite sides of a parallelogram are equal, the length of the segment from P to b, $|Pb|$, is just $|a|$. Likewise, the length of the segment from P to a, $|Pa|$, is just $|b|$. Since the line L_{Pa} is parallel to the y-axis and the y-axis is perpendicular to the x-axis, we see that L_{Pa} is perpendicular to the x-axis (Theorem G3 of Chapter 5 in the Pre-Algebra notes). For the same reason, L_{Pb} is perpendicular to the y-axis. Thus, $|a|$ is in fact the distance from P to the y-axis, and $|b|$ is the distance from P to the x-axis. We have therefore obtained a different interpretation of the coordinates of P:

> *The x-coordinate of P is the distance from P to the y-axis if P is in the right half-plane of the y-axis, and is minus this distance from P to the y-axis if P is in the left half-plane of the y-axis. The y-coordinate of P is likewise the distance from P to the x-axis if P is in the upper half-plane of the x-axis, and is minus this distance from P to the x-axis if P is in the lower half-plane of the x-axis.*

Conversely, with a chosen pair of coordinate axes understood, then given an ordered pair of numbers (a, b), there is one and only one point in the plane with coordinates (a, b). Precisely, this is the point of intersection of the vertical line passing through $(a, 0)$ and the horizontal line passing through $(0, b)$. These two lines being unique, by virtue of the Parallel Postulate, the point of intersection is also unique. There is thus a *bijection between all the points in the plane and all the ordered pairs of numbers.*

With this bijection understood, we proceed to adopt the usual abuse of notation by ***identifying a point with its corresponding ordered pair of numbers.*** In the plane, we define $\boldsymbol{(a,b) = (c,d)}$ to mean that the points represented by (a,b) and (c,d) are the same point. Since we have just shown that every point corresponds to one and only one ordered pair of numbers, we see that

$$(a,b) = (c,d) \quad \text{is equivalent to} \quad a = c, \quad b = d$$

Again, note that there is no ambiguity as to what the *equality* between two ordered pairs of numbers means.

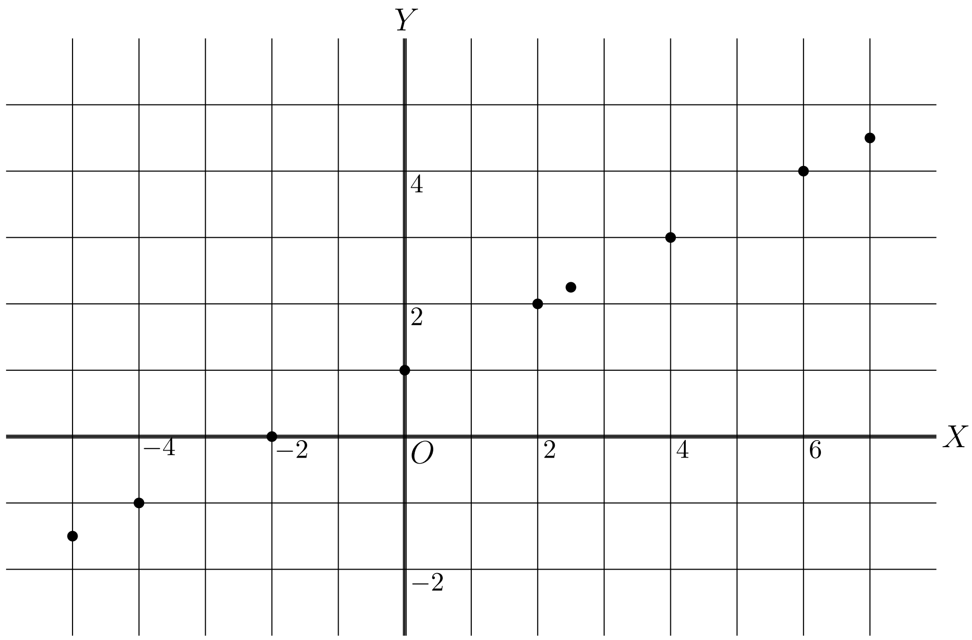
Linear equations in two variables

An equation in the numbers x an y such as $x - 2y = -2$ is an example of **a linear equation in two variable**, namely, x and y. A **solution** of this equation is an *ordered*[22] pair of numbers (x_0, y_0) so that x_0 and y_0 **satisfy** the equation $x - 2y = -2$, in the sense that $x_0 - 2y_0 = -2$. We observe that in this situation, it is easy to find all the solutions with a prescribed first number x_0 or a prescribed second number y_0. For example, with the first number prescribed as 3, then we solve the linear equation in y, $3 - 2y = -2$, to get $y = \frac{5}{2}$. Therefore $(3, \frac{5}{2})$ is the sought-for solution. Or, if the second number is prescribed to be -1, then we solve the linear equation in x, $x - 2(-1) = -2$, to get $x = -4$. The solution is now $(-4, -1)$. Relative to a pair of coordinate axes in the plane, the collection of *all* the points (x_0, y_0) in the coordinate plane so that each pair (x_0, y_0) is a solution of the equation $x - 2y = -2$ is called the **graph** of $x - 2y = -2$ in the plane. Using the above method of getting all the solutions of the equation $x - 2y = -2$, we can plot as many points of the graph as we please to get a good idea of the graph. For example, the following picture contains the following points (given by the dots) on the graph, going from left to right:

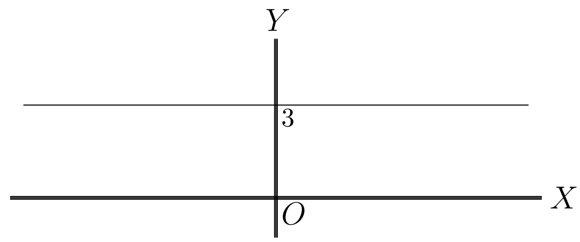
$$\begin{array}{ccccc} (-5,-1.5), & (-4,-1), & (-2,0), & (0,1), & (2,2), \\ (2.5,2.25), & (4,3), & (6,4), & (7,4.5). & \end{array}$$

[22] We emphasize that (x_0, y_0) being an ordered pair means $(x_0, y_0) \neq (y_0, x_0)$, unless of course $x_0 = y_0$. Thus $(3,5)$ is not the same as $(5,3)$, and this is most obvious when we think in terms of the graph: $(3,5)$ and $(5,3)$ are two distinct points which are symmetric with respect to the "diagonal" line $x - y = 0$.

These points strongly suggest that the graph of $x - 2y = -2$ is a (straight) line, and we will presently *prove* that such is the case.



Consider next the graph of the linear equation in *two* variables, $y = 3$, which, as an equation in two variables, is in reality the abbreviated form of the equation $0 \cdot x + 1 \cdot y = 3$. The collection of all solutions of $y = 3$ is then *exactly* all the pairs $(s, 3)$, where s is an arbitrary number, for the following reason. Every one of these $(s, 3)$'s is clearly a solution because $0 \cdot s + 1 \times 3 = 3$. Are there perhaps other pairs of numbers which are also solutions? For example, $(s, 3.1)$? But $0 \cdot s + 1 \times 3.1 = 3.1 \neq 3$, so $(s, 3.1)$ is not a solution of $y = 3$. Similarly, if a number t is not equal to 3, then (s, t) is not a solution of $y = 3$ no matter what t may be. This shows that the preceding assertion about the pairs $(s, 3)$ is true. In terms of the graph, the points with coordinates $(s, 3)$ always lie on the **horizontal line** (i.e., parallel to the x-axis) passing through the point $(0, 3)$ on the y-axis, and since s is arbitrary, these points $(s, 3)$ then comprise the complete horizontal line passing through $(0, 3)$. In short, *the graph of the equation* $y = 3$ *in the plane is exactly the horizontal line passing through the point* $(0, 3)$ *on the* y*-axis.*



Similarly, the graph of the equation $x = -2$ (as an equation in two variables) is the **vertical line** (i.e., parallel to the y-axis) passing through the point $(-2, 0)$ on the x-axis. In a similar manner,

> the graph (in the plane) of $x = c$ for any number c is the vertical line passing through the point $(c, 0)$ on the x-axis, and the graph (in the plane) of $y = b$ for any number b is the horizontal line passing through the point $(0, b)$ on the y-axis.

Since there is only one horizontal (respectively, vertical) line passing through a given point of the plane (*do you know why?*), it follows that

> every vertical line is the graph of the equation $x = c$, where $(c, 0)$ is the point of intersection of the line and the x-axis, and every horizontal line is the graph of an equation $y = b$, where $(0, b)$ is the point of intersection of the line with the y-axis.

Both of these simple facts are probably known to you, but the precise reasoning may have been missing. We have supplied so much detail to explain them because your students should understand that these facts are not facts to be memorized by brute force, but are consequences of careful reasoning and the precise definition of a graph.

We next treat the general case. A **linear equation in two variables** x and y is an equation in the numbers x and y which is either of the form $ax + by = c$ where a, b, c are given numbers, commonly referred to as **constants** and *at least one of a and b is nonzero*, or can be rewritten in this form after transposing and using the four arithmetic operations. Thus $-2x = \frac{2}{5}y + 7$ and $6 + \frac{3}{8}y = 179 - 5x$ are examples of linear equations of two variables. A **solution** of this equation is an ordered pair of numbers x_0 and y_0, written in the expected fashion as (x_0, y_0), so that they **satisfy** the equation $ax + by = c$, in the sense that $ax_0 + by_0 = c$. The **graph of $\boldsymbol{ax + by = c}$** (in the plane) is the collection of *all* the points in the plane with coordinates (x_0, y_0) (relative to a given pair of coordinate axes), so that each ordered pair of numbers x_0 and y_0 is a solution of $ax + by = c$. As we have seen, and will continue to bear witness, the study of linear equations of two variables is grounded in the study of linear equation of one variable.

Graphs of linear equations in two variables: background

Armed with these precise definitions, we are now in a position to state the following theorem.

Theorem *The graph of a linear equation in two variables is a straight line. Conversely, every straight line in the plane is the graph of a linear equation in two variables.*

This theorem establishes a correspondence between lines in the plane and the graphs of linear equations in two variables: the graph of a linear equation $ax+by = c$ is a line L, and every line L is the graph of some linear equation of two variables. It is customary to call $ax + by = c$ **the equation of the line** L if L is the graph of $ax + by = c$, and say that L **is defined by $\boldsymbol{ax + by = c}$**.[23] Incidentally, this theorem explains why equations of the form $ax + by = c$ are called *linear* equations, because their graphs are *lines*.

The reasoning in the proof of this theorem, given in this and the following two subsections, provides the key to the understanding of almost everything about linear equations in introductory algebra.

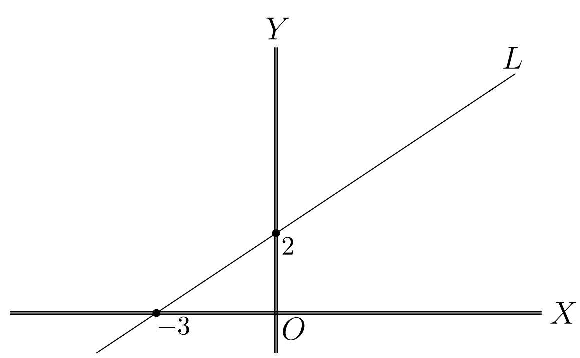
We want to make a minor, but significant simplification in the subsequent discussion of this Theorem. Suppose we start with a linear equation $ax + by = c$. If $b = 0$, then by the definition of a linear equation in two variables, $a \neq 0$. The equation may

[23] Assertion (*iii*) later in this section shows that the equation of a given line is unique up to a constant multiple, so the terminology of *the* equation of a line is justified.

therefore be rewritten as $x = c'$, where c' is the constant $c' = \frac{c}{a}$. In this case, we have seen that the graph is a vertical line. The first part of the Theorem is therefore true in this case, and we may assume from now on that $b \neq 0$ in a given equation $ax + by = c$. Such being the case, we may rewrite the equation as $by = -ax + c$, and therefore $y = mx + k$, where $m = -\frac{a}{b}$ and $k = \frac{c}{b}$. On the other hand, we have seen that a vertical line ℓ is the graph of $x = c'$ (i.e., $x + 0 \cdot y = c'$), where $(c', 0)$ is the point at which ℓ intersects the x-axis. In other words, the second part of the Theorem is true for vertical lines. *We may therefore assume from now on that a given line is not a vertical line.* Equivalently,

> we may assume in the subsequent discussion of this Theorem that a given linear equation is of the form $y = mx + k$, where m, k are constants, and that a given straight line is non-vertical.

Let us first begin the consideration of the Theorem by looking at a concrete case such as $y = \frac{2}{3}x + 2$. Why is the graph of this equation a straight line? The reasoning in this case will shed light on the general case. So let G be the graph of $y = \frac{2}{3}x + 2$. Notice that the point $(0, 2)$ on the y-axis and the point $(-3, 0)$ on the x-axis are on G, because $2 = \frac{2}{3} \times 0 + 2$ and $0 = \frac{2}{3} \times (-3) + 2$. Let L be the (straight) line joining $(0, 2)$ and $(-3, 0)$. We are going to prove that G is the line L.



How to show that the two sets G and L are the same? Obviously, we first have to show that every point on the graph G lies on the line L. But this is not enough because G could just be *part* of L and not all of L. So we must also show that every point of L is a point of G. Therefore, we must show two things:

(α) Every point on the graph G is a point on the line L.

(β) Every point on the line L is a point on the graph G.

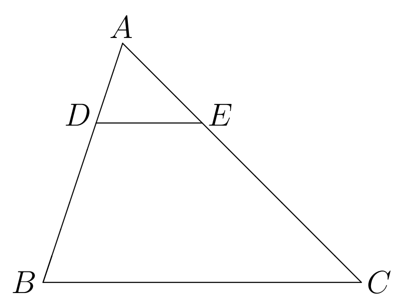
The proofs of these assertions require some preparation. We first review some facts concerning similar triangles[24] and prove a basic property about lines (concerning slope) which is almost universally mishandled in introductory algebra textbooks. First recall the definition of two geometric figures being **similar**: *one is mapped onto the other by a dilation followed by a congruence.* One then proves that two triangles are similar if and only if the corresponding angles are equal and the ratios of (the lengths of) the corresponding pairs of sides are equal. The proof of this fact follows quite easily from the definition of a dilation and the **Fundamental Theorem of Similarity**, which states:

Let $\triangle ABC$ be given, and let D, E be points on AB and AC respectively. If

$$\frac{|AB|}{|AD|} = \frac{|AC|}{|AE|} = \frac{m}{n}$$

for some positive integers m and n, then $DE \parallel BC$ and

$$\frac{|BC|}{|DE|} = \frac{m}{n}$$



However the following is less obvious (**AA** stands for angle-angle).

The AA criterion of similarity *If two triangles have two pairs of equal angles, they are similar.*

[24]Se Chapter 6 of the Pre-Algebra notes.

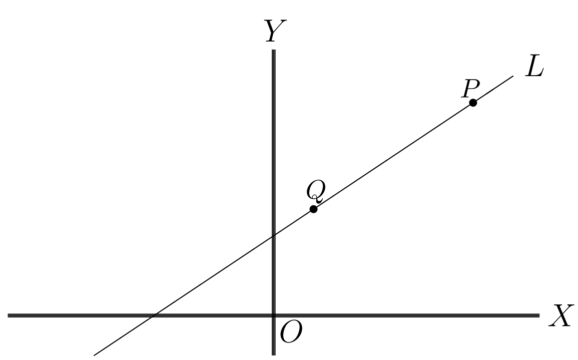
In order to use this criterion effectively, one needs to know when two angles are equal. In this context, we recall also the following fact (see Theorems G18 and G19 in Chapter 6 of the Pre-Algebra notes):

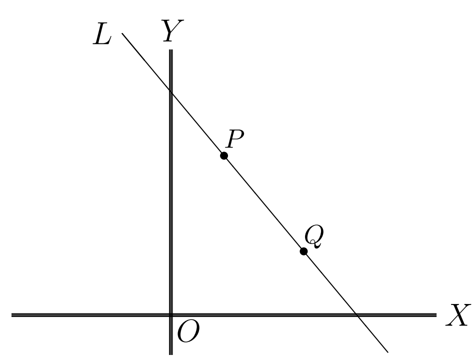
> *If a line L meets two other lines ℓ_1 and ℓ_2 (we call L a* **transversal** *of the lines ℓ_1 and ℓ_2), then the following three conditions are equivalent:*
>
> (1) $\ell_1 \parallel \ell_2$.
>
> (2) *A pair of alternate interior angles of the line L are equal.*
>
> (3) *A pair of corresponding angles of the line L are equal.*

Recall what it means to say (1), (2) and (3) are *equivalent*. It means if we know any one of these three assertions to be true, then the other two must be true too. For example, if we know two lines are parallel (i.e., if (1) holds), then a pair of alternate interior angles and corresponding angles they make relative to any transversal are equal. (It would then follow trivially that *all* pairs of alternate interior angles and corresponding angles are equal.) Or, if a transversal of two lines has a pair of equal corresponding angles (i.e., if (2) holds), then the two lines are parallel.

Graphs of linear equations in two variables: concept of slope

We apply the preceding facts to the study of straight lines by proving that, once a pair of coordinate axes have been set up, associated *to each non-vertical straight line L is a number called* slope which measures the amount of "slant" of the line. Thus coordinate axes as given, let a pair of *distinct* points P, Q on L be chosen.

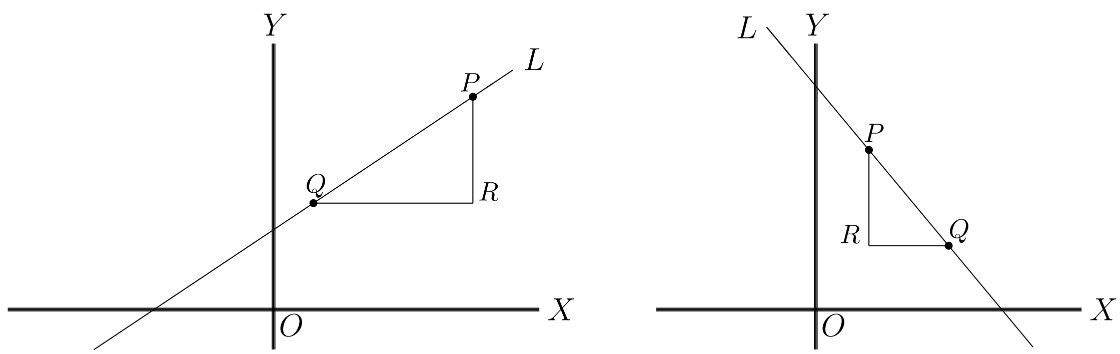




Let the coordinates of P, Q be (p_1, p_2) and (q_1, q_2), respectively. Consider the quotient

$$\frac{p_2 - q_2}{p_1 - q_1} \quad \left(= \frac{q_2 - p_2}{q_1 - p_1} \right)$$

Observe that the denominator is never 0 because if $q_1 - p_1 = 0$, then $p_1 = q_1$ and the fact that P, Q are distinct would mean that the line L is a vertical line. Because we have excluded vertical lines, this is impossible. Thus the denominator of this quotient is never zero, and the quotient makes sense. Also observe that this quotient is positive for the line on the left, which is slanted this way $/$, and negative for the line on the right, which is slanted this way $\backslash$, because, for the line on the left, $p_2 > q_2$ and $p_1 > q_1$ while for the line on the right, $p_2 > q_2$ but $p_1 < q_1$.[25] This observation concerning the sign of the slope of a line can be seen perhaps more transparently in terms of the geometry. Form a right triangle $\triangle PQR$ so that the lines PR and QR are parallel to the coordinate axes, as shown:



If we denote the length of the segment $\overline{PR}$ as usual by $|\overline{PR}|$, etc., then

$$\frac{p_2 - q_2}{p_1 - q_1} = \frac{|\overline{PR}|}{|\overline{QR}|}$$

for the line on the left, and

$$\frac{p_2 - q_2}{p_1 - q_1} = -\frac{|\overline{PR}|}{|\overline{QR}|}$$

for the line on the right. Thus the quotient $\frac{p_2 - q_2}{p_1 - q_1}$ has been directly expressed as a

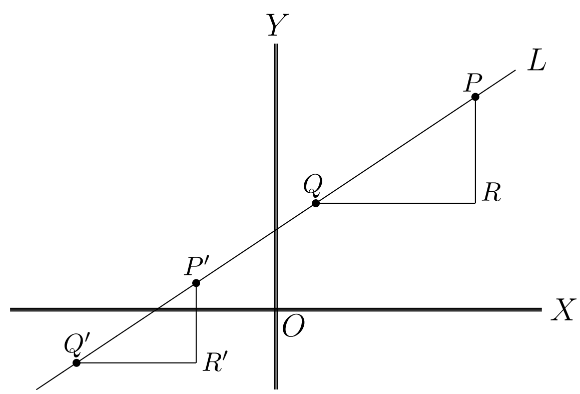
[25]For those who feel uneasy about the imprecision of describing a line as "slanting this way $/$ or that way $\backslash$", see the side remark at the end of the Appendix in §6.

positive or negative quotient of the lengths of the legs of a right triangle, positive for lines slanting like / and negative for lines slanting like \.

We claim that *this quotient* $\frac{p_2-q_2}{p_1-q_1}$ *in terms of the coordinates of* P *and* Q *does not depend on the particular choices of the points* P *and* Q *on the line* L.

Before proving this claim, let it be noted that the truth of this claim is something that is consistently overlooked in standard algebra textbooks (as of 2010). As we shall see, this omission has serious implications on the learning of linear equations. We will prove this claim for the line on the left; the proof for the line on the right is entirely similar and will be left as an exercise.

Thus let points P' and Q' be chosen on the line L as shown on the left, and let their coordinates be (p_1', p_2') and (q_1', q_2'), and let right triangle $\triangle P'Q'R'$ be formed so that the lines $P'R'$ and $Q'R'$ are parallel to the coordinate axes.



Our goal is to show that

$$\frac{p_2' - q_2'}{p_1' - q_1'} = \frac{p_2 - q_2}{p_1 - q_1}$$

By rewriting these quotients in terms of the lengths of the legs of $\triangle PQR$ and $\triangle P'Q'R'$, this equality becomes

$$\frac{|\overline{P'R'}|}{|\overline{Q'R'}|} = \frac{|\overline{PR}|}{|\overline{QR}|}$$

This is the same (using the cross-multiplication algorithm) as

$$\frac{|\overline{P'R'}|}{|\overline{PR}|} = \frac{|\overline{Q'R'}|}{|\overline{QR}|}$$

Now if we can show that the triangles $\triangle PQR$ and $\triangle P'R'Q'$ are similar, then this equality would follow because it merely states that the ratios of corresponding sides of these similar triangles are equal. So why are these triangles similar? By the AA criterion for similarity, we only have to show that two pairs of corresponding angles are equal. $\triangle PQR$ and $\triangle P'R'Q'$ being right triangles, there is already one pair of equal angles. We get a second pair by observing that the line L meets both lines PR and $P'R'$ and $\angle QPR$ and $\angle Q'P'R'$ are corresponding angles relative to the transversal L. But $PR \parallel P'R'$ because they are both parallel to the y-axis, $\angle QPR$ and $\angle Q'P'R'$ are equal . Thus we have two pairs of equal corresponding angles in $\triangle PQR$ and $\triangle P'R'Q'$, and the AA criterion implies their similarity. The desired equality of the quotients is now completely proved.

It should be remarked that in the picture, we have made P higher than Q (relative to the x-axis), and also P' higher than Q' to make the argument more intuitive. But the validity of the argument by no means depends on having P' higher than Q'. Why this is so is left as a classroom activity.

Activity Convince your neighbor that if Q is higher than P, the preceding proof is still valid.

The important conclusion to draw from the preceding discussion is this: given any straight line which is not vertical, choose *any* two points P and Q on the line with coordinates (p_1, p_2) and (q_1, q_2), respectively, and form the quotient

$$\frac{p_2 - q_2}{p_1 - q_1} ,$$

then we can be confident that the resulting number will always be the same regardless of where P and Q may be. In other words, for a given line L, this number does not depend on the choice of the P and Q on L and is therefore associated with the line L itself. We call this number the **slope of the line L**. By a previous observation, the slope of a line that slants this way / is positive, and the slope of a line that slants this way \ is negative. A little experimentation would reveal that if the slope of a line is close to 0 (be it positive or negative), then the line is close to being horizontal, and if it is large be it positive or negative (in the sense of being far away from 0 in either direction on the number line), then the line would be close to being vertical.

Activity Suppose a line L passes through $(2, -3)$ and $(-4, 1)$. If P is a point on L with x-coordinate $\frac{2}{3}$, what is the y coordinate of P?

Textbooks usually define the slope of a line by picking two points on the line and then declaring the quotient *formed from the coordinates of these two points* — in the manner shown above — to be *the slope* of the line. A priori, the quotient resulting from a different choice of points on the line could be a different number so that a line *could* have many slopes. This would render any discussion of "*the* slope" of the line nonsensical. For example, suppose instead of a straight line we have the circle of radius 5 around the origin $(0, 0)$, denoted by $\mathcal{C}$. If we take the two points $(-5, 0)$ and $(3, 4)$ on $\mathcal{C}$, and form the usual quotient, we get $\frac{4-0}{3-(-5)} = -\frac{1}{2}$. On the other hand, taking another pair of points $(5, 0)$ and $(0, 5)$ on $\mathcal{C}$ leads to the quotient of $\frac{5-0}{0-5} = -1$. For the curve $\mathcal{C}$, the quotients formed from different pairs of points on it are therefore not always the same. Yet for a line, these quotients will always be the same. The question is *why?*. We have answered this question by the use of similar triangles. Be sure your students know the answer too, because

> *the exploitation of the fact that one can compute the slope of a line by using* any *two points to one's liking is a powerful tool in dealing with all kinds of questions related to linear equations.*

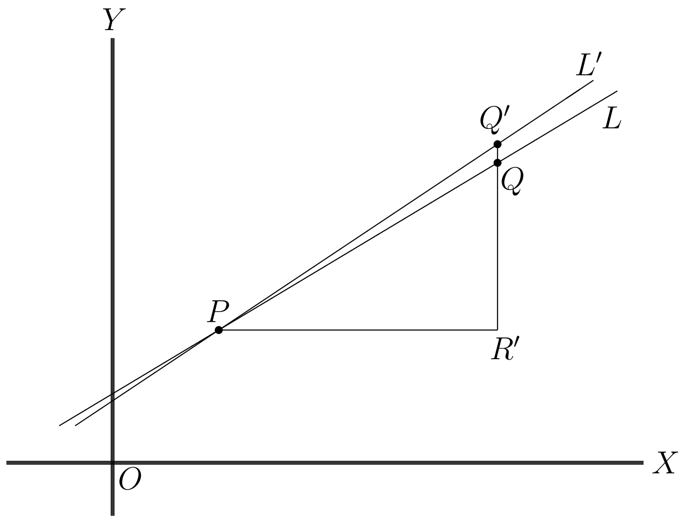
The discussion in the next subsection will amply bear out this assertion.

Proof that graphs of linear equations are lines

We are now in a position to take on the task of showing that the first part of the Theorem is true, i.e., the graph of a linear equation $y = mx + k$ is a straight line. Consider first the seemingly obvious question: if two lines have the same slope and pass through the same point, are they identical? The answer is affirmative.

Key Lemma *If two straight lines have the same slope and pass through the same point, then they are the same line.*

Proof Let the lines be L and L', let their slopes be equal, and let them both pass through the point P. A priori, they are distinct lines and will therefore be schematically represented as such. Our task is to show that they are in fact the same. In the following, we only take up the case of positive slope. The case of negative slope can be handled the same way.



Take an arbitrary point Q' on L', and form right triangle $\triangle PQ'R'$ so that the line PR' is parallel to the x-axis, and the line $Q'R'$ is parallel to the y-axis. Let the line $Q'R'$ intersect L at a point Q. Recall that we are trying to show the lines L and L' coincide, and this would be true as soon as we can show that Q and Q' coincide because, if so, then there is only one line passing through the two points P and Q. We shall make use of the fact that the slope of a line can be computed using *any* two points on the line. The equality of the slopes of L' and L is now expressed as:

$$\frac{|\overline{Q'R'}|}{|\overline{PR'}|} = \frac{|\overline{QR'}|}{|\overline{PR'}|}$$

Since the denominators of these two quotients are equal, the numerators must be equal as well, i.e., $|\overline{Q'R'}| = |\overline{QR'}|$. Therefore $Q = Q'$ and L and L' coincide. The proof of the Key Lemma is complete.

Activity We have just made the claim that if

$$\frac{|\overline{Q'R'}|}{|\overline{PR'}|} = \frac{|\overline{QR'}|}{|\overline{PR'}|}$$

then $|\overline{Q'R'}| = |\overline{QR'}|$. Let us make sure you know why this is so by asking you to prove the following. (a) First assume that the three numbers $|\overline{Q'R'}|$, $|\overline{PR'}|$ and $|\overline{QR'}|$ are fractions. Then prove this claim. (b) In general, apply FASM to finish the proof.

Armed with the Key Lemma, let us first take a look at the previous special case of the Theorem when $m = \frac{2}{3}$ and $k = 2$, i.e., the equation $y = \frac{2}{3}x + 2$. As before, let

L be the line joining $(-3, 0)$ and $(0, 2)$,

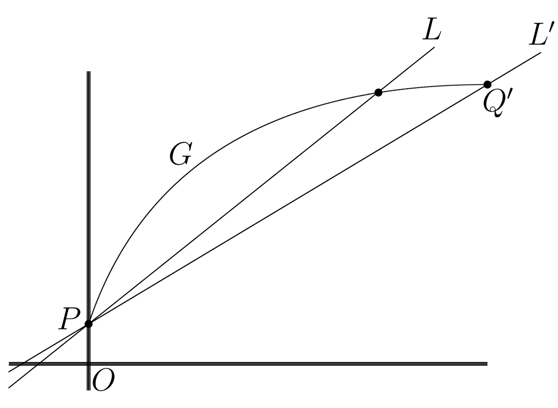
G be the graph of $y = \frac{2}{3}x + 2$.

Recall that our strategy is to prove that G coincides with L by proving:

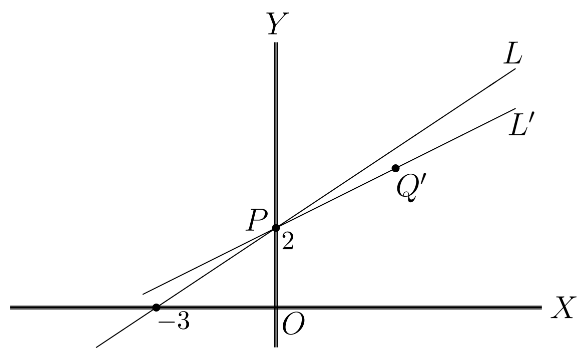
(α) Every point on the graph G is a point on the line L.

(β) Every point on the line L is a point on the graph G.

We first prove (α). Let the point $(0, 2)$ on the y-axis be denoted by P. Take an arbitrary point Q' on the graph G distinct from P, and we must prove that Q' lies on L. We do so by showing that the line L' joining P to Q' coincides with L so that, in particular, Q' lies on L. So why do L and L' coincide? There is no a priori reason that they do, because if the graph G is really "curved", as shown,



then L and L' would be distinct. What we are going to show is that, because G is the graph of a *linear* equation $y = \frac{2}{3}x + 2$, L and L' must have the same slope so that this picture is impossible. In particular, because the lines L and L' both pass through P, they will have to coincide because of the Key Lemma.

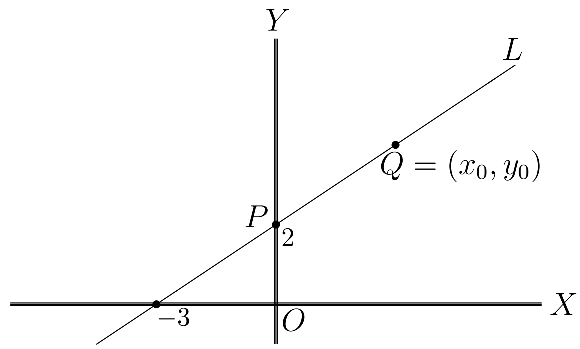


The slope of L can be computed from any pair of points on L, in particular, from $(0, 2)$ (i.e., P) and $(-3, 0)$: it is $\frac{0-2}{-3-0} = \frac{2}{3}$. What about the slope of L'? We compute it using the points $(0, 2)$ and Q', where Q' is the previously chosen point on the graph G distinct from $(0, 2)$. Let the coordinates of Q' be (x_0, y_0). Then the slope of L' is $\frac{y_0-2}{x_0-0}$. But we know more: since Q' is on the graph G of $y = \frac{2}{3}x + 2$, x_0 and y_0 satisfy $y_0 = \frac{2}{3}x_0 + 2$, by definition. It follows that the slope of L' is

$$\frac{y_0 - 2}{x_0 - 0} = \frac{(\frac{2}{3}x_0 + 2) - 2}{x_0} = \frac{\frac{2}{3}x_0}{x_0} = \frac{2}{3}$$

(We can cancel x_0 because Q' being distinct from $(0, 2)$ $(= P)$ implies that $x_0 \neq 0$.) So both lines have the same slope $\frac{2}{3}$ and pass through the same point P. They must coincide and step (α) is proved.

To prove step (β) for the equation $y = \frac{2}{3}x + 2$, we must show that if a point Q lies on the line L joining P $(= (0, 2))$ and $(-3, 0)$, then Q also lies on G. This means, if the coordinates of Q are (x_0, y_0), then we must show $y_0 = \frac{2}{3}x_0 + 2$. We prove this by computing the slope of L in two different ways, first using the two points $(-3, 0)$ and $(0, 2)$, and then using $(0, 2)$ and $Q = (x_0, y_0)$.



As we have seen, the former leads to the fact that the slope of L is $\frac{2}{3}$. Now the latter gives

$$\frac{y_0 - 2}{x_0 - 0} = \frac{y_0 - 2}{x_0}$$

Thus $\frac{y_0-2}{x_0} = \frac{2}{3}$, which implies $y_0 - 2 = \frac{2}{3}x_0$, so that $y_0 = \frac{2}{3}x_0 + 2$, as desired. This completes the proof of step (β), and therewith also the proof that the graph of $y = \frac{2}{3}x + 2$ is the line L joining $(0, 2)$ and $(-3, 0)$.

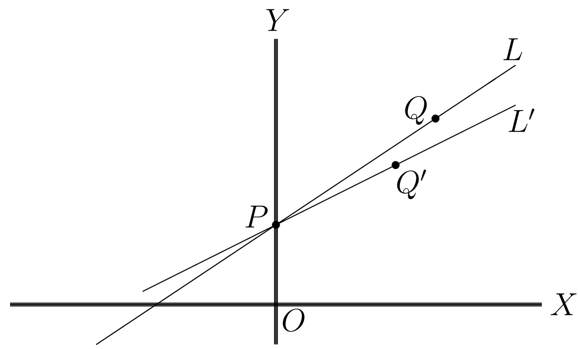
Observe how the preceding proof depends critically on the fact that we can compute the slope of a line by using two points of our choosing.

We now give the general ***proof that the graph of the general equation $y = mx + k$ is a line.*** Let any two points P and Q be chosen on the graph G of $y = mx + k$, and let L be the line joining P and Q. For simplicity, we simply take P to be the point $(0, k)$ on the y-axis (check that $(0, k)$ is on the graph of $y = mx + k$!). We will use the same method to show that L and G are the same, i.e., we go through the same two steps:

(α) Every point on the graph G is a point on the line L.

(β) Every point on the line L is a point on the graph G.

We begin with step (α). Take a random point Q' on the graph G distinct from P, and we will prove that Q' lies on L. We do so by proving that the line L' joining P to Q' coincides with L, so that Q' will be a point on L.



As in the case of $y = \frac{2}{3}x + 2$, we will prove the coincidence of L and L' by showing that they have the same slope. It then follows from the Key Lemma that $L = L'$ because they also pass through the same point P.

To compute the slopes of L and L', we could do it as in the special case of $y = \frac{2}{3}x + 2$, but having worked through a special case gives us more confidence in the general case and we can now afford to think more abstractly. We claim:

> (††) For any two distinct points on the graph of a linear equation $y = mx+k$, the slope of the line joining them is always equal to m.

(*Caution*: it is tempting to assert instead that "the slope of the graph of $y = mx + k$ is m", but at this particular juncture, we do not as yet know that the graph of $y = mx + k$ is a line, so we cannot talk about the slope of the graph of $y = mx + k$.) Indeed let the two points on the graph of $y = mx + k$ be (p_1, p_2) and (q_1, q_2). The slope of the line joining them is then $(q_2 - p_2)/(q_1 - p_1)$.[26] Being on the graph, the coordinates of these points satisfy, by definition, the equations

$$p_2 = mp_1 + k \quad \text{and} \quad q_2 = mq_1 + k$$

Therefore, the slope is

$$\frac{q_2 - p_2}{q_1 - p_1} = \frac{(mq_1 + k) - (mp_1 + k)}{q_1 - p_1)} = \frac{m(q_1 - p_1)}{q_1 - p_1} = m$$

This proves our claim.

[26] Note that this quotient always makes sense because $q_1 - p_1$ is never 0. The reason for the latter is that, if it were, we would have $p_1 = q_1$. But (p_1, p_2) and (q_1, q_2) being points on the graph of $y = mx + k$, we have $p_2 = mp_1 + k$ and $q_2 = mq_1 + k$. Thus also $p_2 = q_2$ and the two points (p_1, p_2) and (q_1, q_2) would not be distinct. Hence $q_1 - p_1$ is never 0.

Because L is the line joining the points P and Q on the graph G of $y = mx + k$, and L' is the line joining P and Q' on G, it immediately follows from this claim that both of their slopes are m. So $L = L'$, as desired. The proof of step (α) is complete.

Now, step (β): why every point of L lies on the graph G. The reason is very simple now. Take any point R on L distinct from P and let the coordinates of R be (r_1, r_2). Then we must prove $r_2 = mr_1 + k$. We have just seen that the slope of L is m. Let the coordinates of P be (p_1, p_2). Then the slope of L computed using P and R is still equal to m, i.e.,

$$\frac{r_2 - p_2}{r_1 - p_1} = m$$

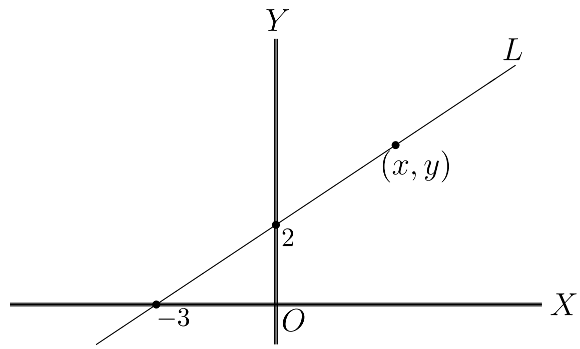
This implies $r_2 - p_2 = m(r_1 - p_1)$, or equivalently, $r_2 = mr_1 + (p_2 - mp_1)$. But P being on G means $p_2 = mp_1 + k$, so that $p_2 - mp_1 = k$. Altogether, we obtain $r_2 = mr_1 + k$. Thus R lies on G, and we have completely proved the first part of the Theorem.

Let us make a useful observation about an intermediate step in this proof for future references. According to the statement (††), the line joining any two points of the graph of a linear equation in two variables, $y = mx + k$, must be m. But now we know that the graph of $y = mx + k$ *is* a line. Hence we can conclude:

(†) The slope of the line which is the graph of $y = mx + k$ is m.

Every line is the graph of a linear equation

We proceed to finish the proof of the Theorem by showing that every straight line is the graph of a linear equation. Let us begin as usual with a special case: we look at our standby, the line L that joins the points $(-3, 0)$ and $(0, 2)$. What equation is it the graph of? Recall that the slope of L (obtain by computing with $(-3, 0)$ and $(0, 2)$) is $\frac{2}{3}$.

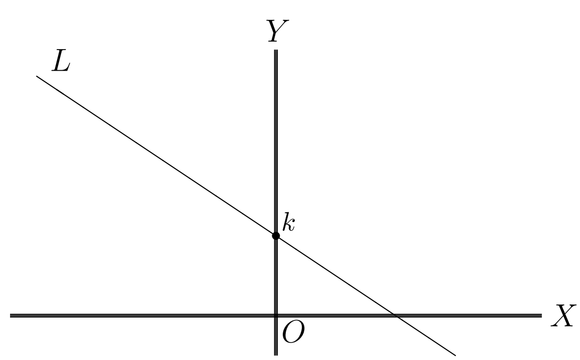


Therefore, for any point (x', y') on L, but not equal to $(-3, 0)$, the slope of L computed by using (x', y') and $(-3, 0)$ is still $\frac{2}{3}$, i.e., $\frac{y'-0}{x'-(-3)} = \frac{2}{3}$. Simplifying, we get $y' = \frac{2}{3}x' + 2$. This leads us to consider the linear equation in two variables

$$y = \frac{2}{3}\,x + 2$$

Let ℓ be the graph of this equation. By the first part of the Theorem, ℓ is a line. Since both $(-3, 0)$ and $(0, 2)$ are easily seen to be solutions of $y = \frac{2}{3}\,(x - (-3))$, the line ℓ passes through both of these points $(-3, 0)$ and $(0, 2)$. Thus both L and ℓ being lines which pass through the same two points $(-3, 0)$ and $(0, 2)$ must be the same line. It follows that L is the graph of $y = \frac{2}{3}x + 2$.

It remains to tackle the general case. The idea of the proof is essentially the same as in the special case. Given a (non-vertical) straight line L, we must find a linear equation whose graph is exactly L. Since L and the y-axis are not parallel, they must meet at some point. Let L intersect the y-axis at $(0, k)$. Let the slope of L be m. We are going to show that L is the graph G of the equation $y = mx + k$. For the sake of variety, let us assume m is negative so that we have the following picture:



By the first part of the Theorem, the graph G of $y = mx + k$ is a line. By the observation (†) at the end of the last subsection, the slope of G is m. Since $(0, k)$ is obviously a solution of $y = mx + k$, G also passes through the point $(0, k)$. Therefore G and L are two lines which have the same slope m and pass through the same point $(0, k)$. By the Key Lemma, G and L are the same line. It follows that the given line L is the graph of $y = mx+k$. This completes the proof of the Theorem.

Useful facts and examples

The preceding proof of the Theorem may seem long, but almost every piece of reasoning in the proof will show up in subsequent discussions of linear equations and straight lines. We now give a demonstration of this fact by extracting four useful consequences from the proof.

We first introduce two standard concepts. If a line ℓ intersects the y-axis at $(0, k)$, then the number k or the point $(0, k)$ is called the ***y*-intercept** of the line ℓ. Similarly, if ℓ intersects the x-axis at $(c, 0)$, then c or the point $(c, 0)$ is called its ***x*-intercept**. Observe that *if a non-vertical line has 0 y-intercept, then it also has 0 x-intercept.* Recall that we are only dealing with equations of the form $y = mx + k$ and lines which are not vertical.

> (i) *The graph of* $y = mx + k$ *is a line with slope* m *and* y*-intercept* k. *Conversely, the nonvertical line with slope* m *and* y*-intercept* k *is defined by* $y = mx + k$.

The first part of (i) follows from the Theorem that the graph of $y = mx + k$ is a line, and the slope of this line is m by (†). The fact that it has y-intercept follows immediately from the fact that $(0, k)$ is a solution of $y = mx + k$. The converse follows from the first part that the line defined by $y = mx + k$ has slope m and y-intercept k, so that by the Key Lemma, it must coincide with the given line.

A second useful fact is a restatement of the fact that the slope of a line can be computed using any two points on the line.

> (ii) *The equation of the line passing through two given points* (p_1, p_2) *and* (q_1, q_2), *where* $p_1 \neq q_1$, *is* $(y - p_2) = m(x - p_1)$, *where* $m = \frac{q_2 - p_2}{q_1 - p_1}$.

This is because if L is the given line and ℓ is the graph of $(y - p_2) = m(x - p_1)$, then because (p_1, p_2) is obviously a solution of $(y - p_2) = m(x - p_1)$, ℓ passes through (p_1, p_2). Moreover, the equation $(y - p_2) = m(x - p_1)$ can be rewritten as $y = mx + (p_2 - mp_1)$, so that ℓ has slope m (see (†)). By the Key Lemma, $\ell = L$.

In practice, one can also approach the equation $(y - p_2) = m(x - p_1)$ differently. The line L obviously has slope $m = \frac{q_2 - p_2}{q_1 - p_1}$, so it must be defined by an equation of the form $y = mx + k$ for some constant k. It suffices to determine what k is, and that can be done by recalling that (p_1, p_2) lies on the graph of $y = mx + k$, i.e., (p_1, p_2) is a solution of the equation so that $p_2 = mp_1 + k$. Thus $k = p_2 - mp_1$. One sees in Example 2 below that the actual computation of k is even simpler than the abstract description.

The third fact is a direct consequence of (i), and is usually glossed over in standard texts of all levels.

> (iii) *The lines defined by the two equations* $ax + by = c$ *and* $a'x + b'y = c'$*, are the same if and only if there is a nonzero number* λ *so that* $a' = \lambda a$*,* $b' = \lambda b$*, and* $c' = \lambda c$*.*

We prove (iii) as follows. First of all, if there is a nonzero number λ so that $a' = \lambda a$, $b' = \lambda b$, and $c' = \lambda c$, then the solutions of $ax + by = c$ and $a'x + b'y = c'$ are clearly identical because $a'x + b'y = c'$ may be rewritten as $\lambda(ax + by) = \lambda c$. Since the line defined by a linear equation is just the set of all its solutions, the lines defined by $ax + by = c$ and $a'x + b'y = c'$ have to be the same. Conversely, suppose the two lines defined by $ax + by = c$ and $a'x + b'y = c'$ coincide. Let us call it ℓ. Let the y-intercept of ℓ be μ. Rewrite $ax + by = c$ as $y = -\frac{a}{b}x + \frac{c}{b}$. Then since $(0, \mu)$ is on ℓ, $\mu = \frac{c}{b}$. Furthermore, by (ii) above, the slope of ℓ is $-\frac{a}{b}$. Reasoning likewise with the equation $a'x + b'y = c'$, we get that $\mu = \frac{c'}{b'}$ and the slope of ℓ is $-\frac{a'}{b'}$. Hence we get $\frac{c}{b} = \frac{c'}{b'}$ and $\frac{a}{b} = \frac{a'}{b'}$. This is the same as saying

$$\frac{b'}{b} = \frac{c'}{c} \quad \text{and} \quad \frac{b'}{b} = \frac{a'}{a}$$

A straightforward computation then shows that letting $\lambda = \frac{a'}{a}$ would get the job done.

In the situation of (iii), we can retrieve the equation $ax + by = c$ of ℓ from the equation $a'x + b'y = c'$ by multiplying both sides by $\frac{1}{\lambda}$. For this reason, one normally regards any two equations defining a line as "the same", and speaks of **the defining equation of a line**.

The final and fourth fact is a consequence of (i) and (iii):

> *(iv) A line ℓ defined by $ax + by = c$ with $a \neq 0$ and $b \neq 0$ has slope $-\frac{a}{b}$ and x-intercept $\frac{c}{a}$.*

By (iii), ℓ is also defined by $y = (-\frac{a}{b})x + \frac{c}{b}$, as this equation can be obtained from $ax + by = c$ by multiplying both sides by $\frac{1}{b}$. By (i), ℓ has slope $-\frac{a}{b}$. It is also obvious that $(\frac{c}{a}, 0)$ is a solution of $y = (-\frac{a}{b})x + \frac{c}{b}$, so ℓ has x-intercept $\frac{c}{a}$.

Next, we give some examples of how to write down the equation of a line.

Example 1 What is the equation of the line passing through the point $(2, -1)$ with slope $\frac{2}{3}$?

According to (i), the equation has the form $y = \frac{2}{3}x + k$, and we only need to find out what k is. It is not necessary to directly compute the y-intercept. Since the line contains $(2, -1)$ (and since the line is the graph of $y = \frac{2}{3}x + k$), we know $-1 = (\frac{2}{3})2 + k$, from which $k = -\frac{7}{3}$. Thus the equation is $y = \frac{2}{3}x - \frac{7}{3}$.

This is the proper place to comment on a common misconception. Sometimes it is taught in classrooms, and written up in textbooks, that the equation of this line is

$$\frac{y - (-1)}{x - 2} = \frac{2}{3}$$

This is not correct because the point $(2, -1)$ would not satisfy this equation as we would have $(2 - 2)$ in the denominator on the left side. Therefore the graph of $\frac{y-(-1)}{x-2} = \frac{2}{3}$ contains every point of the line passing through the point $(2, -1)$ with slope $\frac{2}{3}$ except the point $(2, -1)$ itself. However, if we re-express this equation as $y - (-1) = \frac{2}{3}(x - 2)$, then certainly this is an equation whose graph is the desired line. That said, we should add that the equation $\frac{y-(-1)}{x-2} = \frac{2}{3}$ contains the correct geometric conception of the desired line, because what it says is that this line consists of all the points (x, y), distinct from $(2, -1)$, so that the slope of the line containing (x, y) and $(2, -1)$ is $\frac{2}{3}$. The point of this comment is therefore that even correct thinking needs to be complemented by correct technical execution.

EXAMPLE 2 What is the equation of the line passing through $(-1, 3)$ and $(\frac{1}{2}, 4)$?

Call this line ℓ. The slope of ℓ is

$$\frac{4-3}{\frac{1}{2}-(-1)} = \frac{2}{3}$$

so the equation of ℓ has the form $y = \frac{2}{3}x + k$, where the constant k is determined by the fact that $(-1, 3)$ is a solution of the equation since $(-1, 3)$ lies on ℓ. Thus $3 = \frac{2}{3}(-1) + k$, and $k = \frac{11}{3}$. The equation of ℓ is $y = \frac{2}{3}x + \frac{11}{3}$.

There is another way to approach this problem. Let (x, y) be an arbitrary point on ℓ. We can compute the slope of ℓ by using (x, y) and $(-1, 3)$, getting

$$\frac{y-3}{x+1} = \frac{2}{3}$$

This is equivalent to $y - 3 = \frac{2}{3}(x + 1)$, which is again $y = \frac{2}{3} + \frac{11}{3}$.

The preceding solutions need to be complemented by an observation. We have used the point $(-1, 3)$ instead of $(\frac{1}{2}, 4)$ as the reference point in both solutions, but of course the outcome would have been the same had $(\frac{1}{2}, 4)$ been used. For example, suppose in the first solution we make use of the fact that $(\frac{1}{2}, 4)$ lies on the graph of $y = \frac{2}{3}x + k$; then we have $4 = \frac{2}{3} \cdot \frac{1}{2} + k$, so that

$$k = 4 - \frac{1}{3} = \frac{11}{3}$$

as before. In the second solution, if we use the point $(\frac{1}{2}, 4)$ as the point of reference, then we would have

$$\frac{y-4}{x-\frac{1}{2}} = \frac{2}{3}$$

It is simple to see that this equation is again $y = \frac{2}{3}x + \frac{11}{3}$.

It is worth emphasizing that none of these methods should be memorized by brute force beyond the fact that the equation of a non-vertical line is of the form $y = mx+k$ for some constants m and k, where m is the slope of the line. Instead, one should get to know the reasoning underlying these procedures and do a simple computation each time to get at the equation.

EXAMPLE 3 What is the x-intercept of the line joining the points $(-4, 6)$ and $(2, 1)$?

The slope of the line is $\frac{6-1}{-4-2} = -\frac{5}{6}$, so the equation of the line is $y = -\frac{5}{6}x + k$, for some constant k. Since it contains the point $(2, 1)$, we have $1 = -\frac{5}{6} \cdot 2 + k$, and so $k = 1 + \frac{5}{3} = \frac{8}{3}$. The equation of the line is therefore $y = -\frac{5}{6}x + \frac{8}{3}$. The point where this line intersects the x-axis has y-coordinate equal to 0; let it be $(c, 0)$. Since $(c, 0)$ lies on the line, we also get $0 = -\frac{5}{6}c + \frac{8}{3}$, which is the same as $\frac{5}{6}c = \frac{8}{3}$. Multiplying through by $\frac{6}{5}$, we get $c = \frac{16}{5}$. So the x-intercept is $\frac{16}{5}$.

We conclude this section with a comment on the concept of *dilation* in the plane. We claim:

If D is a dilation of the coordinate plane with center at the origin O and with scale factor r $(r > 0)$, then for any point (a, b),

$$D(a, b) \;=\; (ra, rb)$$

If we define the multiplication of a point (x, y) by a number c as

$$\boldsymbol{c(x, y) \overset{\text{def}}{=} (cx, cy),}$$

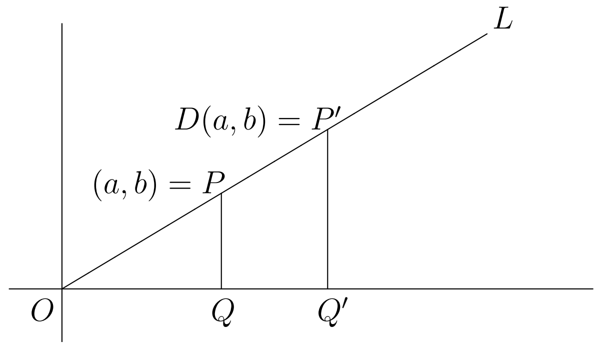
then we can rewrite the preceding claim as

$$D(a, b) = r\,(a, b)$$

The reason for the claim is as follows. Denote the points (a, b) and $D(a, b)$ by P and P', respectively. First recall the definition of P': on the ray L from O to P, P' is the point so that the distance $|OP'|$ from O to P' is r times the distance $|OP|$ from O to P, i.e.,

$$|OP'| = r\,|OP|$$

For simplicity, first assume both a and b are positive. Then we drop perpendiculars PQ and $P'Q'$ from P and P' to the x-axis, as shown.



Notice that the x-coordinates of P and P' are $|OQ|$ and $|OQ'|$, respectively. Because $PQ \parallel P'Q'$, the theorem on corresponding angles of parallel lines implies that their corresponding angles relative to the transversal L are equal, so that triangles OPQ and $OP'Q'$ are similar (AA criterion). Therefore

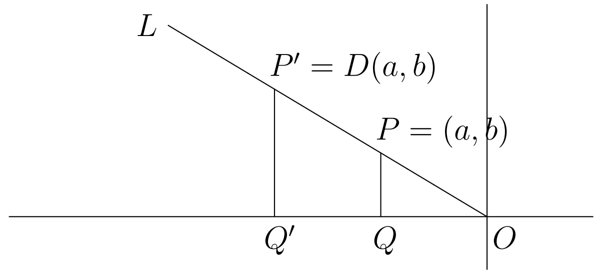
$$\frac{|OQ|}{|OQ'|} = \frac{|OP|}{|OP'|}$$

But we have seen that $\frac{|OP|}{|OP'|} = r$, so we get

$$\frac{|OQ|}{|OQ'|} = r,$$

or what is the same things, $|OQ| = r\,|OQ'|$. Thus the x-coordinate of $D(a, b)$ is r times the x-coordinate a of (a, b). By dropping perpendiculars from P and P' to the y-axis and repeating the same argument, we get in a similar manner that the y-coordinate of $D(a, b)$ is also r times the y-coordinate b of $D(a, b)$. This shows $D(a, b) = (ra, rb) = r\,(a, b)$, as claimed.

In case one or both of a and b is negative, consider the case $a < 0$ but $b > 0$ for definiteness. Then the preceding picture becomes



In this case, the only difference is that $|OQ| = -a$ and $|OQ'|$ is equal to the negative of the x-coordinate of $D(a, b)$, so that from $|OQ| = r\,|OQ'|$, we conclude the x-coordinate of $D(a, b)$ is again r times the x-coordinate a of (a, b) as before. Similarly, the y-coordinate of $D(a, b)$ is again r times the y-coordinate b of (a, b) . The proof is complete.

EXERCISES

1. (a) Write down all the linear equations of two variables whose graphs pass through $(-2, 1)$. (b) Write down three linear equations of two variables whose graphs all have slope $\frac{3}{2}$, and compute the y-intercept in each case.

2. Write down a proof of the independence of the definition of slope from the particular pair of points chosen on a given line which is slanted this way: $\backslash$.

3. (i) Find the equation of the line joining $(2, -1)$ and $(-3, -11)$. What is its x-intercept? (ii) Find the equation of the line joining $(-\frac{1}{4}, \frac{2}{3})$ and $(5, \frac{3}{2})$. What is its y-intercept?

4. (i) What is the equation of the line with x-intercept equal to -2 and slope $-\frac{1}{3}$? What is its y-intercept? (ii) What is the equation of the line with x-intercept equal to $\frac{2}{5}$ and y-intercept equal to $-\frac{4}{3}$?

5.(i) Find the equation of the line passing through $(1, -1)$ with slope -1. What is its y-intercept? (ii) Find the equation of the line passing through $(-\frac{2}{5}, 3)$ with slope $\frac{1}{7}$. What is its x-intercept?

6. (i) What is the y-intercept of the graph of $x = -5y + 7$? What is its slope? Does the point $(352\frac{2}{5}, -70\frac{1}{5})$ lie on the graph? (ii) What is the x-intercept of the line passing through $(5, -\frac{2}{3})$ and $(-\frac{4}{3}, \frac{1}{2})$? What is its slope?

7. Do the graphs of $6x - 2y = 7$ and $\frac{1}{5}x - \frac{1}{15}y = 329$ intersect? Explain why or why not *using what we have done so far.*

8. (i) Let L be the line passing through $(1, -2)$ with slope m. For which value of m would L pass through $(20, 72)$? (ii) Let ℓ be the line with slope m passing through $(\frac{1}{2}, \frac{3}{4})$. For which value of m would ℓ pass through $(\frac{5}{3}, \frac{1}{3})$?

9. (i) Let L be the line joining $(1, 2)$ to $(p, -4)$, where p is some number. For what value of p would L pass through $(10, 25)$? (ii) Let ℓ be the line joining $(-\frac{3}{2}, 4)$ and $(\frac{4}{5}, q)$, where q is some number. For what value of q would ℓ pass through $(2, -3)$?

10. Does the line joining $(3, -2)$ and $(6, 2)$ contain the point $(9, 6)$? Explain it two different ways.

11. Practice explaining to an eighth grader why the line joining the origin and the point $(-1, -1)$ is the graph of a linear equation of two variables. Do it once assuming that the student knows the graph of a linear equation of two variables is a line, and do it also without making that assumption. In any case, be clear about what you assume the student knows, and *make it as simple as possible.*

12. Let D be a dilation of the coordinate plane with center at the origin O and let L be a line whose slope is s. What is the slope of $D(L)$?

5 Some Word Problems

Here are some examples of word problems involving the solution of linear equations in one variable.

EXAMPLE 1 There are 39 coins made up of quarters and pennies, and they are worth \$4.47. How many quarters are there?

We follow the practice of §2 and simply transcribe the information faithfully into symbolic language before doing anything. So if there are Q quarters, then there are

$39 - Q$ pennies. In terms of cents, we have 447 cents, of which $Q \cdot 25$ cents come from the quarters and $39 - Q$ cents coming from the pennies. Obviously,

$$25Q + (39 - Q) = 447$$

This is a linear equation in one variable, so the technique of §3 allows us to solve this easily. Transposing, we have $25Q - Q = 447 - 39$, therefore $24Q = 408$, and therefore $Q = 17$. There are 17 quarter.

One should always check: 17 quarters lead to $17 \cdot 25 = 425$ cents. Added to $39 - 17 = 22$ cents (from the pennies) does give 447 cents.

Example 2 Find four consecutive odd integers so that the product of the second and fourth integers exceeds the product of the first and third integers by 64.

Let the smallest of the four odd integers be x. At the moment we do not worry about whether x is even or odd; we just transcribe the given information and wait to see what happens. No reason to do more than you have to! Thus the next three integers are $x+2$, $x+4$, and $x+6$. The given data is that $(x+2)(x+6)$ is bigger than $x(x+4)$ by 64. So

$$(x+2)(x+6) - x(x+4) = 64$$

The solution of this equation, which is not linear, begins with a simplification of the left side by the use of the distributive law. We get $x^2 + 8x + 12 - x^2 - 4x = 4x + 12$. Thus the equation becomes $4x + 12 = 64$, which is a linear equation in one variable after all. From $4x = 52$, we obtain $x = 13$. Thus the four integers are 13, 15, 17, 19. We check that $(15 \times 19) - (13 \times 17) = 285 - 221 = 64$.

By the way, the solution suggests that the initial assumption that the four consecutive integers be odd is irrelevant. All we need to know is that each integer is 2 more than the preceding one.

Example 3 Break 48 into two parts so that the smaller part is $\frac{2}{3}$ of the greater part.

Let s be the smaller part; then the greater part is $48 - s$. It is given that $s = \frac{2}{3}(48 - s)$. Thus $\frac{3}{2}s = 48 - s$, and $\frac{5}{2}s = 48$. It follows that $s = \frac{96}{5} = 19\frac{1}{5}$. We check that the greater part is $48 - \frac{96}{5} = \frac{144}{5}$, and $\frac{96}{5} = \frac{2}{3} \cdot \frac{144}{5}$.

The next few problems are about so-called *constant rates*: the constant rate of walking (which we call constant *speed*), the constant rate of water pouring into a tub, the constant rate of work, such as the number of square feet a lawn is mowed, etc. There is a fairly detailed discussion of this concept in §8 of Chapter 1 in the Pre-Algebra notes, and it is time to review that section. However, in view of the fact that not only is the concept of *rate* mangled in the standard materials, but the concept of *constancy*, which is central to the solution of this class of problems, is hardly ever clearly defined, we begin by recalling the needed precise definitions.

We will concentrate on speed; it will be seen that the extrapolation of the speed discussion to other kinds of rates is not difficult. Suppose Lisa takes a walk, and she walks *at a constant rate of* $2\frac{1}{2}$ *miles per hour*. What could the last phrase mean? To answer this question, we introduce the concept of Lisa's **average speed over a time interval from hour t_0 to hour t**, $t_0 < t$, to be the division

$$\frac{\text{total distance walked (in miles) from } t_0 \text{ hours to } t \text{ hours}}{t - t_0}$$

The unit used in this division is **miles per hour**, or more simply **mph**.[27] To say that Lisa walks at a **constant rate of $2\frac{1}{2}$ miles per hour** is to say, by definition, that her average speed over *any* time interval is $2\frac{1}{2}$ mph. Suppose Lisa starts walking at the origin O and we measure distance from O, so that at $t = 0$ she is 0 miles from O. At t hours, let us say she is y miles from O, which also means she has walked y miles from O after t hours. If she walks at a constant speed of $2\frac{1}{2}$ mph, then for any $t > 0$,

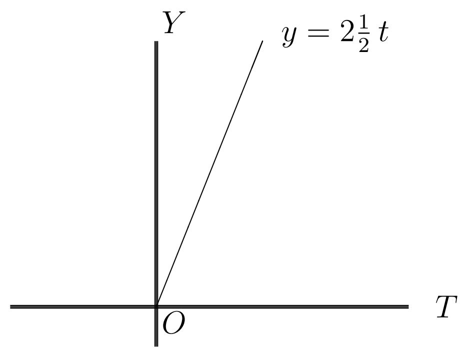
$$\frac{y}{t} = 2\frac{1}{2}$$

Equivalently,

$$y = (2\frac{1}{2})\, t$$

In terms of the symbols t and y, this is a linear equation in the two variables t and y, so that its graph is a straight line passing through the origin of the TY-plane.

[27] All it means is that we have identified the units on two separate number lines, one on the number line whose unit is 1 mile and the other on the number line whose unit is 1 hour. This identification makes possible the definition of average speed over a time interval because *division of numbers* is only defined for numbers on the same number line.



Notice that the (constant) speed, $2\frac{1}{2}$ mph, is the slope of the graph. Notice also the fact that we do not extend the graph to the left of the y-axis because we start from time 0, i.e., $t = 0$.

We observe, conversely, that if after t hours, for *any* $t > 0$, Lisa is y miles from O and y is related to t by the equation $y = 2\frac{1}{2}\,t$, then her average speed over any time interval from t_0 to t hours will be a constant, namely, $2\frac{1}{2}$ mph, for the following reason. Let us say she is y_0 miles away from O at t_0 hours, then by assumption, $y_0 = 2\frac{1}{2}t_0$. The distance she walks from t_0 hours to t hours is $y - y_0$ miles. Then her average speed over the time interval from t_0 to t hours is

$$\frac{y - y_0}{t - t_0} \;=\; \frac{2\frac{1}{2}\,t - 2\frac{1}{2}\,t_0}{t - t_0} \;=\; \frac{2\frac{1}{2}(t - t_0)}{t - t_0} \;=\; 2\frac{1}{2}\text{ mph}$$

Thus *the equation* $y = 2\frac{1}{2}\,t$ *for all* $t \geq 0$*, where* y *is always the distance Lisa walks in* t *hours, fully expresses the fact that she walks at a constant speed of* $2\frac{1}{2}$ *mph.*

In general, for a given motion, we define the **average speed over a time interval from hour t_0 to hour t**, $t_0 < t$, to be the division

$$\frac{\text{total distance traveled (in miles) from } t_0 \text{ hours to } t \text{ hours}}{t - t_0}$$

It is understood that the unit of distance (here in *miles*) and the unit of time (here in *hours*) can be any pre-assigned units. We say the motion has **constant speed v** (v being a fixed positive number) if its average speed over *any* time interval is always equal to v.

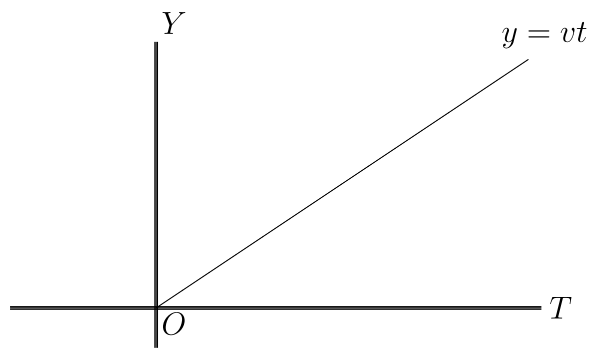
If we assume that the motion begins at the origin O at time $t = 0$, and the distance traveled after t hours, y miles, is measured from O, then the fact that the motion has constant speed v means

$$\frac{y}{t} = v \quad \text{for any } t > 0$$

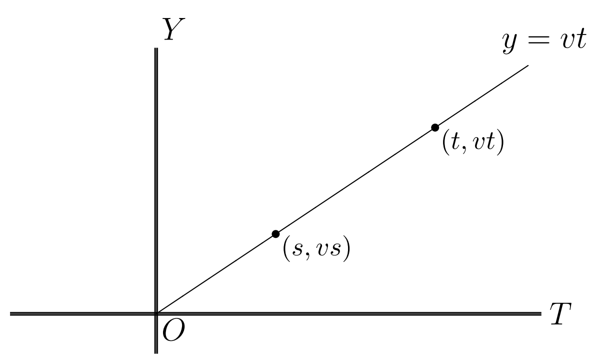
Equivalently,

$$y = v\,t \quad \text{for any } t > 0$$

As before, in terms of the symbols t and y, this is a linear equation in the two variables t and y, so that its graph is a ray issuing from the origin of the TY-plane (it does not extend to the left of the y-axis because the motion begins at $t = 0$). Furthermore, *the (constant) speed of v mph is exactly the slope of the graph.*



We now consider the converse statement. Let the motion starts from the origin at time 0, and suppose after t hours it moves y miles from O. If there is a constant v so that for all $t > 0$, y is related to t by the equation $y = v\,t$, then we claim that the motion has constant speed equal to v. Here is the reason. Consider the time interval from s to t, where $s < t$, and we will show that the average speed of the motion over the time interval from s to t is v.



Because the slope v of this graph (part of a line) can be computed by using *any* two points on the line, we take the two points (s, vs) and (t, vt) on the graph and obtain

$$\frac{vt - vs}{t - s} = v$$

But vt is the distance traveled from time 0 to time t, and vs is the distance traveled from time 0 to time s. Therefore the distance traveled from time s to time t is the numerator of the left side in the preceding equality, $vt - vs$. Obviously $t - s$ is the length of the time interval from time s to time t. Hence this equality, which is a geometric statement about the slope of a straight line, now becomes the statement that the average speed of the motion over the time interval from s to t is v. Putting the whole discussion together, we have:

> *a motion has constant speed v if, and only if, $y = vt$, where y is the distance traveled from time 0 to t hours and the motion starts from O at time 0.*

In the usual terminology of school texts, we have

$$\text{speed} = \frac{\text{distance traveled in a time interval of length } T}{T},$$

which is valid for any time interval of length T, where T is any number. This leads to the usual formulas: "speed is distance divided by time", "distance is speed multiplied by time", etc. There is a substantial difference between the standard treatment of these formulas and what we have done, however. The standard treatment does not explain what "speed" means,[28] much less what "constant speed" means. Therefore these formulas have to be memorized by brute force, which leads to the fact that rate problems are a terror among school students. On the other hand, once we have a precise definition of constant speed v as distance/time = v, the other formulas,

$$\text{distance} = v \times \text{time}$$

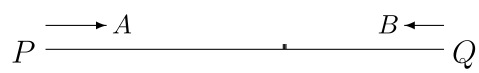
$$\text{time} = \text{distance}/v$$

[28] The concept of speed cannot be defined in school mathematics because it requires calculus; the only related concepts that are accessible to school mathematics are *average speed over a time interval* and *constant speed*.

follow from the definition of division (see §5 in Chapter 1 of the Pre-Algebra notes).

It should be very clear at this point that the term "speed", and more generally "rate", as used in school mathematics are in fact never precisely explained. In mathematics, whatever is not precisely explained should be ignored. What requires our real effort is the understanding the equation $y = vt$ in the way presented above. We illustrate with some examples.

Example 4 Two cars A and B move at constant speed. A starts from P to Q, 150 mile apart, at the same time that B starts from Q to P. They meet at the end of $1\frac{1}{2}$ hours. If A moves 10 mile per hour faster than B, what are their speeds?



Let the speed of A be v mph. Then the speed of B is $v - 10$ mph. In $1\frac{1}{2}$ hours, A covers $1\frac{1}{2}v$ miles while B covers $1\frac{1}{2}(v-10)$ miles. If they meet after $1\frac{1}{2}$ hours, then the distances they have covered, combined, would be the same as the distance between P and Q, which is 150 miles. Therefore

$$1\frac{1}{2}\ v + 1\frac{1}{2}\ (v - 10) = 150$$

There are many ways to solve this equation, one of them is to first multiply both sides by $\frac{2}{3}$ (noting that $1\frac{1}{2} = \frac{3}{2}$) to get $v + (v - 10) = 100$. Thus $2v = 110$, and $v = 55$. Therefore the speed of A is 55 mph, and the speed of B is 45 mph. We check: $\frac{3}{2} \cdot 55 + \frac{3}{2} \cdot 45 = \frac{165}{2} + \frac{135}{2} = 150$.

The discussion about speed can be carried over, *verbatim*, to the discussion of the rate of water flow or the rate of getting work done (painting a house, mowing a lawn, etc.), keeping in mind that we always idealize that these processes are evolving *at a constant rate*, in the sense that water flows out of the faucet at a constant rate, houses are painted at a constant rate, lawns are mowed at a constant rate, etc. In school mathematics, without the tool of calculus at our disposal, this is the only kind of problems we can handle. With this in mind, the following are some related examples.

Example 5 Water flows out of two faucets A and B at constant rate. Suppose the water flow from faucet A is 10 gallons per minute more than that from faucet B, and suppose a container has a capacity of 150 gallons. If both faucets are turn on at the same time and the container is filled in $1\frac{1}{2}$ minutes, what are the rates of the water flows in both faucets?

In accordance with the preceding discussion of constant speed, the concept of a constant-rate water flow (from a faucet) is that, for some fixed constant r, the amount of water coming out of the faucet after t minutes is rt gallons.[29] The number r is the so-called *rate of the water flow* and it is in terms of gallons per minutes. See the preceding interpretation of speed as a division of distance by time. Now let the rate from faucet A be x gallons per minute. The the rate from faucet B is $x - 10$ gallons per minute. After $1\frac{1}{2}$ minutes, the amount of water coming out of faucet A is $1\frac{1}{2}x$ gallons, and that of faucet B is $1\frac{1}{2}(x-10)$ gallons. Since after $1\frac{1}{2}$ minutes, the container is filled, the total amount of water from both faucets is equal to 150 gallons. Therefore

$$1\frac{1}{2}\,x + 1\frac{1}{2}\,(x-10) = 150$$

This is the same equation as before, and the answer is: the rate of faucet A is 55 gallons of water per minute and that of faucet B is 45 gallons of water per minute.

Example 6 Karen and Lisa paint houses at a constant rate. Suppose Karen paints 10 square meters more per hour than Lisa, and suppose a wall has an area of 150 sqare meters. If both Karen and Lisa paint this wall at the same time and they finish it in $1\frac{1}{2}$ hours, what are the rates at which each paints?

Let Karen paint x square meters per hour. Then Lisa paints $x - 10$ square meters per hour. After $1\frac{1}{2}$ hours, Karen has painted $1\frac{1}{2}x$ square meters and Lisa, $1\frac{1}{2}(x-10)$ square meters. So in $1\frac{1}{2}$ hours they have painted a combined area of $1\frac{1}{2}x+1\frac{1}{2}(x-10)$ square meters. Since the area of the wall is 150 square meters, we have

$$1\frac{1}{2}\,x + 1\frac{1}{2}\,(x-10) = 150$$

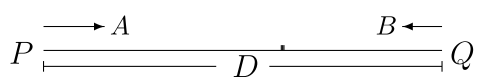
[29] This is the counterpart, for a motion of constant speed v, of *distance traveled after t hours* $=$ $v\,t$.

Same equation. So Karen paints 55 square meters per hour and Lisa 45 square meters per hour.

Examples 4, 5, and 6 put in clear evidence the fact that if we can understand one kind of rate problems, then we are in a position to do any other kind of rate problems.

Example 7 Two trains A and B run at constant speed. A goes from P to Q in 2 hours while B goes from P to Q in 3 hours. If A leaves P for Q at the same time that B leaves Q for P (on a separate but identical rail!), after how many hours will they meet?

This example is very similar to Example 4, although for this problem we do not know the distance between P and Q. Just call it D in order to finish the transcription of the given data into symbolic language.



As we saw in the solution of Example 4, we need the speed of both A and B to compute the distances they have traveled before they meet. By the definition of constant speed, the fact that A travels D miles in 2 hours means $\frac{D}{2}$ = speed of A. Similarly, the speed of B is $\frac{D}{3}$ mph. Let the two trains meet after x hours. Then in x hours, the distance traveled by A is $\frac{D}{2}x$ and that by B is $\frac{D}{3}x$. The total distance must be equal to the distance between P and Q, which is D. So

$$\frac{D}{2}\,x + \frac{D}{3}\,x = D$$

This seems to be an equation in both D and x until we realize that if we multiply both sides by $\frac{1}{D}$, we'd be left with

$$\frac{1}{2}x + \frac{1}{3}x = 1$$

Thus $(\frac{1}{2} + \frac{1}{3})x = 1$, and $x = \frac{6}{5}$ hours.

Example 8 Paul and Geneviève walk at a constant rate. Paul walks from their house to the train station in 30 minutes while Geneviève needs only 24 minutes to do the same. Geneviève gives Paul a head start of 4 minutes and then she starts off. Does she catch up with Paul, and if so, after how many minutes?

Since it takes Geneviève only 24 minutes to get to the station, it takes only $4 + 24 = 28$ minutes after Paul took off before she gets to the station. But 28 minutes after Paul took after, he is still on his way to the station. Therefore Geneviève overtakes him at some point on her way to the station. We have to find out exactly when.

We can do this problem at least two ways. First, suppose Geneviève catches up with Paul t minutes after *her* departure from the house. How far has she walked? We have only the time (t minutes) but not her speed; we do know it takes her 24 minutes to get to the station. *It is always safe to try doing the most obvious thing*, namely, if we let the distance from the house to the station be D miles, then we get immediately her speed: $\frac{D}{24}$ miles per minute (mpm). So in t minutes she has walked $\frac{D}{24}t$ miles. Paul's speed is $\frac{D}{30}$ mpm, and by the time Geneviève has walked t minutes, he would have walked $t+4$ minutes. Therefore when Geneviève catches up with him, he would have walked $\frac{D}{30}(t+4)$ miles. Since the two will have walked the same distance at that point,

$$\frac{D}{24}t = \frac{D}{30}(t+4)$$

This again looks like an equation in the two numbers D and t, but once more the D goes away as soon as we multiply both sides by $\frac{1}{D}$. So $\frac{1}{24}t = \frac{1}{30}(t+4)$ and therefore $\frac{30}{24}t = t+4$. This leads to $\frac{1}{4}t = 4$ and $t = 16$. So 16 minutes after Geneviève leaves the house, she catches up with Paul.

We can also do the same problem from the other end, i.e., the station instead of the house. Again, suppose Geneviève catches up with Paul t minutes after her departure from the house. Then in $(24-t)$ more minutes she would be at the station. Therefore the distance from where she catches with Paul to the station is $\frac{D}{24}(24-t)$ miles. For Paul, he will have walked $4+t$ minutes when Geneviève catches up with him, and therefore he has $30 - (4+t) = 26 - t$ minutes to go before he reaches the station. Since his speed is $\frac{D}{30}$ mpm, the remaining distance to the station can also be computed by $\frac{D}{30}(26-t)$. Thus

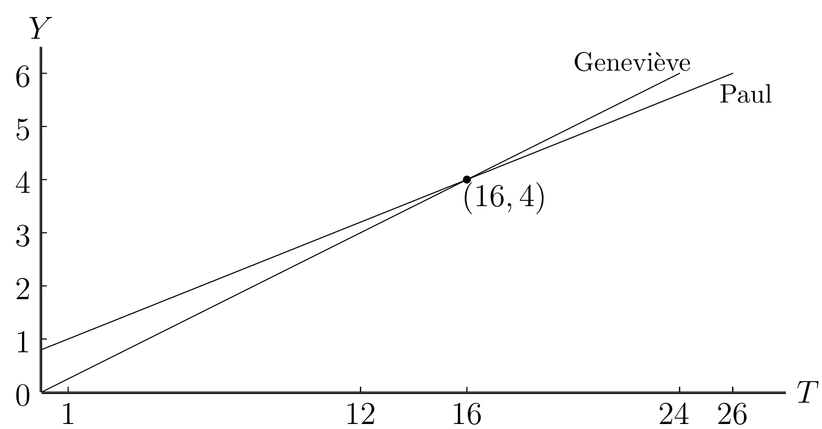
$$\frac{D}{24}(24-t) = \frac{D}{30}(26-t)$$

As before, we get $\frac{1}{24}(24-t) = \frac{1}{30}(26-t)$, which implies (upon multiplying both sides by $24 \cdot 30$) $30(24-t) = 24(26-t)$. Hence $6t = 24(30-26)$ and $t = 16$ once again.

It is instructive to look at the graphs of the equations describing the motion of Paul and Geneviève. We will use the first set of equations for illustration. For the sake of clarity , we need to give the distance D between the house and the station a definite value, say $D = 6$. If y denotes the distance from the house t minutes after Geneviève leaves the house, then for her, the equation is $y = \frac{6}{24}t = \frac{1}{4}t$. For Paul, the equation is $y = \frac{6}{30}(t+4) = \frac{1}{5}t + \frac{4}{5}$. We now graph both linear equations in two variables,

$$y = \frac{1}{4}\,t \qquad \text{and} \qquad y = \frac{1}{5}\,t + \frac{4}{5}\,,$$

on the same set of coordinate axes:



The y-intercept of the graph of Paul (which is $\frac{4}{5}$, as we saw above) now has a graphic interpretation: it gives his distance from the house when Geneviève leaves the house. The point of intersection of the two graphs, which is $(16,4)$, also has an interpretation: the x-coordinate tells the time when Geneviève catches up with Paul because at that instant, both are exactly the same distance (4 miles, the y-coordinate of the point) from the house.

This is the first time we discuss two graphs on the same set of axes simultaneously. We will pursue this discussion vigorously in the next section.

EXERCISES

1. A man has six hours at his disposal. How far can he ride in a car going at a constant speed of 25 mph if he has to ride a bicycle back at the rate of 6 mph?

2. A train loses $\frac{1}{6}$ of its passengers at the first stop, 25 at the second, 20% of the remainder at the third, three quarters of the remainder at the fourth; 25 remain, What was the original number?

3. Water flows out of two faucets A and B at constant rate. Faucet A fills a given container in 5 minutes, while faucet B fills it in 6 minutes. How long would it take to fill the container if both faucets are turned on at the same time?

4. The numerator of a fraction is 7 less than the denominator. If 4 is subtracted from the numerator and 1 added to the denominator, the resulting fraction equals $\frac{1}{3}$. What is the fraction?

5. A had twice as much money as B, but after giving B \$28, he has $\frac{2}{3}$ as much as B. How much did each have at first?

6. Find two numbers whose sum is $\frac{7}{6}$ and whose difference is $\frac{1}{6}$.

7. Lisa and Karen mow lawns at a constant rate. Lisa finishes mowing a certain lawn in 4 hours, but with Karen's help from the beginning, she does it in 3 hours. How long will it take Karen to mow it alone?

8. There are two heaps of coins, one containing nickels and the other dimes. The second heap is worth 20 cents more than the first, and has 8 fewer coins. Find the number in each heap.

9. If A has \$566 and B \$370, how much money must A give B in order that B may have $\frac{4}{5}$ as much as A?

10. A tank with a capacity of 150 gallons can be filled by one pipe in 15 minutes, and emptied by another in 25 minutes. After the first pipe has been open a certain number of minutes, it is closed, and the second pipe opened. The tank is then found to be empty 24 minutes after the first pipe was open. Assuming that water flow is always at a constant rate, how many minutes is each pipe open?

11. A train running at 30 mph requires 21 minutes longer to go a certain distance than does a train running at 36 mph. How great is the distance?

12. A woman drives a car for $3\frac{1}{2}$ hours and she finds that she has covered a distance of 130 miles. If she drives at a constant speed of 45 mph in the country and 20 mph within city limits, how many miles of her trip is in the country?

13. Paul can mow a certain lawn by all himself in 11 hours. After working for $2\frac{1}{2}$ hours, however, Paul is joined by Henry and the two together finish mowing the lawn in another 5 hours. Assume as always that both mow the lawn at constant rate, how long would it take Henry to mow the lawn alone? Explain clearly how you get the solution.

14. Aaron and Bill together can mow a lawn in 36 minutes. If Aaron mows it alone, it takes 58 minutes. Suppose the lawn is 280 square yards, and suppose both Bill and Aaron mow the lawn at a constant rate, how long would it take Bill to mow it alone?

15. Driving from A to B at a constant speed of 45 mph is 25 minutes faster than doing it at 39 mph. How great is the distance from A to B?

16. A solution consisting of water and alcohol has 70% alcohol. If 25 cc of water is added to the solution, how much alcohol must be added in order for the solution to still contain 70% alcohol?

17. Aaron, Bill, and Carl all mow a given lawn at a constant rate. If it is mowed individually, they can finish it in A, B, C hours, respectively. If Aaron and Bill mow it together, they finish mowing it in $2\frac{2}{9}$ hours, and if Aaron and Carl mow it together, they finish mowing it in $2\frac{2}{3}$ hours. However, if all three mow it together, they finish it in $1\frac{17}{23}$ hours. Write equations in terms of A, B, C to capture the above information, and explain how you arrive at these equations.

18. Fifteen minutes after Colin leaves for school, his mother discovers that he forgot to take his homework. She drives at a constant rate and it takes her 6 minutes to get to school. Colin walks to school at a constant rate and it takes him 24 minutes to get there. Use mental math to decides if Colin's mother can catch up with him, and if she does, compute how soon this happens after Colin leaves.

6 Simultaneous Linear Equations

In Example 8 of the preceding section, we found ourselves looking at the graphs of two linear equations in two variables. We saw that the point of intersection of the graphs corresponds to the solution of the problem. In this section, we inspect more closely the situation of two linear equations. We explain the meaning of the usual algebraic method of solution, analyze the nature of the solution, and call special attention to the interplay between the algebra and the geometry of such equations.

In the Appendix, we give a characterization of the perpendicularity of lines in terms of slope. The proof is an instructive exercise in the use of the geometric tools we have carefully assembled.

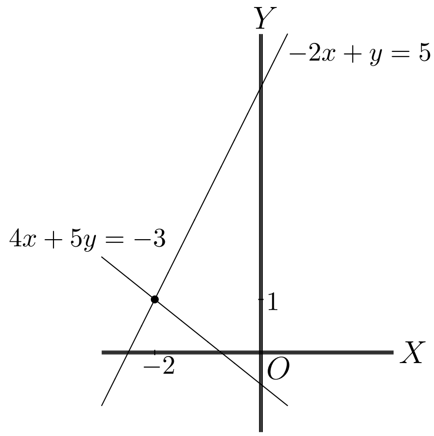
Solutions of linear systems and geometric interpretation

Recall that a solution to a linear equation such as $4x + 5y = -3$ is a pair of numbers (x_0, y_0) so that $4x_0 + 5y_0 = -3$. For example, $(0, -\frac{3}{5})$ is a solution. There are an infinite number of solutions to a linear equation of two variables and their totality constitutes the graph of the equation, which is a straight line. Still with $4x + 5y = -3$, suppose we also consider another equation $-2x + y = 5$ and ask if there could be a pair of numbers (x_0, y_0) so that it is a solution of *both* $4x + 5y = -3$ and $-2x + y = 5$. Indeed there is, for example, $(-2, 1)$, as it is easy to check. We say the pair of linear equations

$$\begin{cases} 4x & + & 5y & = & -3 \\ -2x & + & y & = & 5 \end{cases}$$

is a **system of linear equations**, or sometimes, **simultaneous (linear) equations** in the numbers x and y. To be precise, one would have to refer to such a pair of equations as **a linear system of two equations in two unknowns** or **two variables**. As in the case of a single equation in one variable, implicit in the writing down of such a system is the invitation to find out if there are pairs of numbers (x_0, y_0) which are solutions of both linear equations. This implicit statement will be taken for granted and will not be repeated in subsequent discussions. To **solve the system** is to find all the ordered pair of numbers (x_0, y_0) which are solution of both equations. Such an (x_0, y_0) is called a **solution of the system**. Sometimes we also call the collection of *all* these (x_0, y_0)'s **the** *solution of the system.* Thus $(-2, 1)$ is a solution of the above system. A priori, there may be others, but it will turn out that the solution of the system consists only of $(-2, 1)$. At present we are only concerned with systems consisting of two equations in two variables, but note that there will be occasion to consider systems of equations consisting of many equations in any number of unknowns.

Postponing for the moment the discussion of how to get a solution such as $(-2, 1)$ to the above system, let us first give a geometric interpretation of this $(-2, 1)$. Now $(-2, 1)$, being a solution of the first equation $4x + 5y = -3$, lies on the line defined by $4x + 5y = -3$. Similarly, $(-2, 1)$ also lies on the line defined by the second equation $-2x + y = 5$. This means *the solution* $(-2, 1)$ *is the point of intersection of the two lines defined by the equations in the linear system*, as shown:



Conversely, suppose (A, B) is not at the intersection of these two lines, then it cannot be a solution of the linear system because if (for example) the point (A, B) is not on the line defined by $-2x + y = 5$, then $-2A + B \neq 5$, and (A, B) is not a solution of both equations. Therefore, the point of intersection of the lines defined by the equations of the linear system is exactly the solution of the linear system.

This reasoning is perfectly general. Suppose we are given a linear system

$$\begin{cases} ax & + & by & = & e \\ cx & + & dy & = & f \end{cases}$$

where a, b, $\ldots f$ are constants. Let ℓ_1, ℓ_2 be the lines defined by the equations $ax + by = e$ and $cx + dy = f$, respectively. Then:

> *The solution of the system is* (*the set of points in*) *the intersection of the lines* ℓ_1 *and* ℓ_2.

We first show that if (x_0, y_0) is a solution of the system, it must lie on the intersection of ℓ_1 and ℓ_2. Since (x_0, y_0) is a solution of $ax + by = e$, the point (x_0, y_0) lies on ℓ_1. For the same reason, (x_0, y_0) lies on ℓ_2 as well. Therefore (x_0, y_0) is a point of intersection of ℓ_1 and ℓ_2. Conversely, if (x_0, y_0) lies in the intersection of ℓ_1 and ℓ_2, then being on ℓ_1 implies that it is a solution of $ax + b = e$ and being on ℓ_2 implies that it is a solution of $cx + d = f$. Thus (x_0, y_0) is a solution of the system, as desired.

Suppose the lines ℓ_1 and ℓ_2 are distinct nonparallel lines. Then we know they intersect at exactly one point. We have just given the precise reasoning why, if the lines defined by the two equations of a linear system of two linear equations in two unknowns are distinct nonparallel lines, then the solution of the linear system is the point of intersection of the two lines. This fact is usually decreed by fiat in standard texts, probably because the precise definition of the graph of an equation is rarely given or, if given, is not put to use. It is very important that you learn to make use of definitions in your teaching. In particular, please do not forget to explain *why* the solution of a linear system can be obtained from the intersection of the lines.

The algebraic method of solution

Next, we turn to he question of how to get a solution of a given linear system *algebraically* without looking at picture. We adopt the time-honored strategy of first *assuming* that there is a solution to the given linear system and use this information to find out what it has to be. Then we turn around to verify that the presumptive solution is indeed a solution of the linear system. The reasoning given below is perfectly general, but we choose to explain it in terms of the concrete linear system above, namely,

$$\left\{ \begin{array}{rcrcr} 4x & + & 5y & = & -3 \\ -2x & + & y & = & 5 \end{array} \right. \tag{1}$$

We want to shown that *if* there is an ordered pair of numbers (x, y) satisfying the system (1), then $\boldsymbol{(x, y) = (-2, 1)}$ in the sense that $x = -2$ and $y = 1$. We begin by multiplying both sides of the second equation in (1) by -2 with a view to making the coefficient of x in this equation equal to the coefficient of x in the first equation, i.e., 4. We get:

$$\left\{ \begin{array}{rcrcr} 4x & + & 5y & = & -3 \\ 4x & - & 2y & = & -10 \end{array} \right.$$

Now rewrite the system as

$$\left\{ \begin{array}{rcrcr} 4x & = & -5y & - & 3 \\ 4x & = & 2y & - & 10 \end{array} \right. \tag{2}$$

We emphasize that systems (1) and (2) are **equivalent** in the sense that every solution of one of them is a solution of the other; the validity of this assertion should be clear after a moment's reflection. Now, since both number expressions $-5y-3$ and $2y-10$ are equal to the number $4x$, we have

$$-5y-3=2y-10 \tag{3}$$

We can solve this linear equation in y, obtaining easily $y=1$. This leads to the important consequence that when $y=1$, (3) is true and therefore both equations in (2) become identical linear equations in x, or what is the same, *when* $y=1$*, both equations of the system in (1) are identical linear equations in* x, so that a solution of one equation (in the number x) in (1) is automatically a solution of the other equation (in x). We can be more explicit in the present case. When $y=1$, the system (1) becomes:

$$\left\{\begin{array}{rcr} 4x & = & -8 \\ -2x & = & 4 \end{array}\right. \tag{4}$$

and both equations are of course the same. So we get $y=1$ and $x=-2$, as claimed.

We should emphasize that thus far, what we have shown is merely that *if* a solution exists for the system in (1), it would have to be $(-2,1)$. This is not the same as saying that $(-2,1)$ is a solution of (1). Now we proceed to fill in this gap by showing that $(-2,1)$ *is* a solution. In one sense, this is trivial: letting $x=-2$ and $y=1$ in the equations in (1), we get $-3=-3$ and $5=5$, so that is that. But this is unsatisfactory because this simple numerical verification does not reveal *why* $(-2,1)$ is a solution and, more importantly, it does nothing to give us confidence that this method of obtaining a solution (via equations (1) through (4)) also works in other situations. So we do it differently.

We begin by analyzing the concept of a solution to a system. The meaning of $(-2,1)$ being a solution of the system (1) is that when $y=1$ in (1), then the two solutions of the two linear equations in x *turn out to be the same* (in fact, equal to -2, as we know). To drive home this point, if y is equal to 2 instead of 1, then the system (1) becomes

$$\left\{\begin{array}{rcrcr} 4x & + & 10 & = & -3 \\ -2x & + & 2 & = & 5 \end{array}\right.$$

The first equation yields $x = -\frac{13}{4}$ and the second equation yields $x = -\frac{3}{2}$. So the ordered pair $(-\frac{13}{4}, 2)$ solves the first equation of (1) but not the second, and the ordered pair $(-\frac{3}{2}, 2)$ solves the second equation but not the first. In short, there is no solution of (1) which has $y = 2$.

It remains to explain why, if $y = 1$, both equations in x of the system in (1) yield the same solution. This is because $y = 1$ is the solution of equation (3), and therefore both equations in (2) are identical when $y = 1$. Since systems (1) and (2) are equivalent, we see that both equations in (1) become identical linear equations in x when $y = 1$. Of course that solution is $x = -2$, as claimed.

We have written out the method of getting $(-2, 1)$ via the steps associated with (2)–(4) to facilitate the explanation of why $(-2, 1)$ is a solution. In practice, however, one achieves some simplificaton by skirting (2), as follows. Working directly with

$$\begin{cases} 4x & + & 5y & = & -3 \\ 4x & - & 2y & = & -10 \end{cases}$$

we subtract both sides of the second equation from the corresponding sides of the first (or, to conform with the basic principle enunciated in §3, we *add the negatives of both sides of the second equation to the corresponding sides of the first*), obtaining

$$5y - (-2y) = (-3) - (-10) \tag{5}$$

This leads to $y = 1$ as before. More important is the fact that equation (5), if we transpose $5y$ and -10 to the opposite sides, is the same equation as (3). Therefore, this way of "bringing the coefficient of the number x in both equations to be the same and then eliminate x by subtraction" achieves the same result as the method of solution in (2)–(4). This "subtraction method" is what is usually done to solve simultaneous equations.

We bring closure to this discussion by revisiting Example 8 of the last section. In retrospect, we had arrived at a linear system in the numbers t and y in that Example:

$$\begin{cases} y & = & \frac{1}{4}\,t & & \\ y & = & \frac{1}{5}\,t & + & \frac{4}{5} \end{cases} \tag{6}$$

This is a linear system the minute it is rewritten as

$$\left\{\begin{array}{rcrcl} -\frac{1}{4}t & + & y & = & 0 \\ -\frac{1}{5}t & + & y & = & \frac{4}{5} \end{array}\right.$$

When the system (6) is given as is, there is no question as to the fastest method to solve it: since both $\frac{1}{4}t$ and $\frac{1}{5}t + \frac{4}{5}$ are equal to the number y, we get

$$\frac{1}{4}t = \frac{1}{5}t + \frac{4}{5},$$

so that $\frac{1}{20} = \frac{4}{5}$, and $t = 16$. The first equation of (6) then gives $y = 4$. Thus $(16, 4)$ is the solution to (6), which was already obtained earlier by another method having nothing to do with linear systems. Moreover, the fact that the point $(16, 4)$ is the point of intersection of the two lines defined by the equations in (6) was already pointed out in Example 8.

Characterization of parallel lines in terms of slope

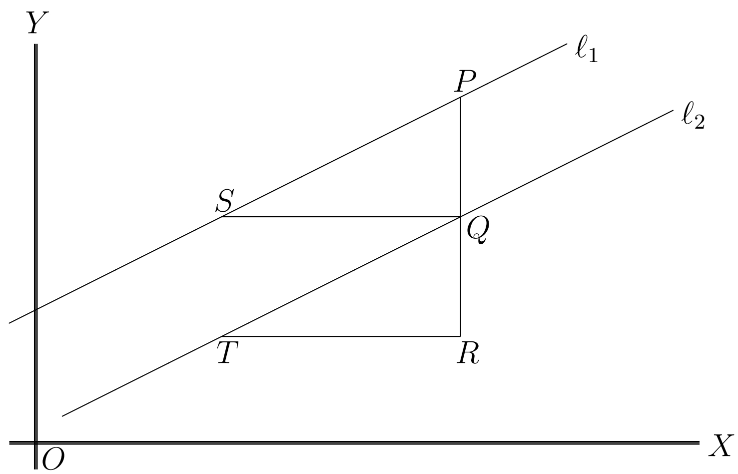
It is time to point out that not every linear system of two equations in two unknowns has a solution. Indeed, we proved above that the set of solutions of a linear system coincides with the set of points in the intersection of the two lines defined by the equations of the system. It follows that if these two lines are parallel, then they will have *no* intersection, and therefore the linear system will have no solution. For example, obviously the system

$$\left\{\begin{array}{rcrcl} x & + & 0 \cdot y & = & 1 \\ x & + & 0 \cdot y & = & 2 \end{array}\right. \tag{7}$$

can have no solution, and we understand this phenomenon from our present perspective by noting that the lines defined by the equations $x = 1$ and $x = 2$ are vertical lines and are thus parallel. To achieve a better understanding of this phenomenon in general, we will prove a basic property of a pair of lines in the plane.

Theorem 1 *Two distinct, non-vertical lines in the plane are parallel if and only if they have the same slope.*

Proof Let the lines be ℓ_1 and ℓ_2. We first assume that they are parallel and prove that they have the same slope. If either of ℓ_1 and ℓ_2 is horizontal, then since $\ell_1 \parallel \ell_2$, the other is also horizontal and both would have 0 slope. There would be nothing to prove in this case. So we may assume both ℓ_1 and ℓ_2 are not horizontal. Take a point P on ℓ_1 and let a vertical line through P intersect ℓ_2 at Q. (This vertical line must intersect ℓ_2 because the latter is not vertical.) Since the lines are distinct, $P \neq Q$. Go along this vertical line from P to Q and stop at a point R so that $|\overline{PQ}| = |\overline{QR}|$. From Q and R, draw horizontal lines which meet ℓ_1 and ℓ_2 at S and T, respectively.



Because $\triangle PQS$ and $\triangle QRT$ are right triangles with legs parallel to the coordinate axes, the slopes of ℓ_1 and ℓ_2 are

$$\frac{|\overline{PQ}|}{|\overline{SQ}|} \qquad \text{and} \qquad \frac{|\overline{QR}|}{|\overline{TR}|}$$

respectively. We have to show that these two numbers are equal. We do so by showing that $\triangle PQS$ and $\triangle QRT$ are congruent. Once this is done, then $|\overline{SQ}| = |\overline{TR}|$ because they are corresponding sides of congruent triangles. Since $|\overline{PQ}| = |\overline{QR}|$ by construction, we get

$$\frac{|\overline{PQ}|}{|\overline{QR}|} = \frac{|\overline{SQ}|}{|\overline{TR}|}$$

because both are equal to 1. By the cross-multiplication algorithm, this implies

$$\frac{|\overline{PQ}|}{|\overline{SQ}|} = \frac{|\overline{QR}|}{|\overline{TR}|}$$

which is what we are after. So it remains to prove that the triangles are congruent.

One way is to translate the whole plane in the direction from P to Q. Call this translation τ. So by definition, $\tau(P) = Q$. Now translation moves a line (not parallel to the direction of the translation) to a line parallel to itself. Thus $\tau(\ell_1)$ is a line passing through Q and parallel to ℓ_1. Since ℓ_2 is already such a line, the Parallel Postulate says $\tau(\ell_1) = \ell_2$. Now consider the line SQ. By construction, $SQ \parallel TR$. Since $\tau(Q) = R$ because by construction $|\overline{PQ}| = |\overline{QR}|$, the same reasoning therefore yields the fact that the line $\tau(SQ)$ coincides with the line TR. But S is the intersection of the lines ℓ_1 and SQ, while T is the intersection of the lines ℓ_2 and TR. Hence the fact that $\tau(\ell_1) = \ell_2$ and $\tau(SQ) = TR$ means that $\tau(S) = T$. This says the translation τ maps P to Q, Q to R, and S to T. By definition, $\triangle PQS \cong \triangle QRT$.

A more traditional argument is as follows. Observe that $\angle SPQ$ and $\angle TQR$ are equal because they are corresponding angles of the parallel lines ℓ_1 and ℓ_2 with respect to the transversal PR. Of course $\angle PQS$ and $\angle QRT$ are equal because they are both right angles. The equality $|\overline{PQ}| = |\overline{QR}|$ (which results from the construction OF R) then shows that $\triangle PQS \cong \triangle QRT$ because of the ASA criterion of congruence (Theorem G8 in Chapter 5 of the Pre-Algebra notes). In any case, this shows that nonvertical parallel lines have the same slope.

Conversely, suppose two distinct, non-vertical lines ℓ_1 and ℓ_2 have the same slope, and we have to show that they are parallel. We give two proofs: the first one being a direct continuation of the preceding line of geometric reasoning, and the second an algebraic one. First, if they have slope 0, then they are horizontal and are therefore parallel. We may therefore assume that they have nonzero slope so that they are both non-horizontal. We now perform the same construction as before to get triangles $\triangle PQS$ and $\triangle QRT$. The fact that ℓ_1 and ℓ_2 have the same slope then implies that

$$\frac{|\overline{PQ}|}{|\overline{SQ}|} = \frac{|\overline{QR}|}{|\overline{TR}|}$$

These quotients have equal numerators because we constructed the point R so that $|\overline{PQ}| = |\overline{QR}|$. The denominators are consequently equal as well, i.e., $|\overline{SQ}| = |\overline{TR}|$. Again, we let τ be the translation from P to Q, then $\tau(P) = Q$ by the definition of τ, and $\tau(Q) = R$ because $|\overline{PQ}| = |\overline{QR}|$. As above, the fact that $SQ \parallel TR$ and the Parallel Postulate imply that τ maps the line SQ to the line TR. Since also $|\overline{SQ}| = |\overline{TR}|$, we see that, necessarily, $\tau(S) = T$. Recall that already $\tau(P) = Q$.

This means τ maps the line ℓ_1 to the line ℓ_2. Because translation maps a line to another line parallel to itself, we conclude that $\ell_1 \parallel \ell_2$.

A more traditional proof would run as follows. Since $\angle PQS$ and $\angle QRT$ are equal because they are both right angles, the triangles $\triangle PQS$ and $\triangle QRT$ are congruent by the SAS criterion of congruence (Theorem G7 in Chapter 5 of the Pre-Algebra notes). Their corresponding angles $\angle SPQ$ and $\angle TQR$ are therefore equal. This implies $\ell_1 \parallel \ell_2$ because their corresponding angles relative to the transversal PR are equal. The proof of Theorem 1 is complete.

Here is a second algebraic proof. Since ℓ_1 and ℓ_2 are both nonvertical, say they have slope m. Then let the equations defining them be $y = mx+k$ and $y = mx+k'$, respectively, where $k \neq k'$ because by assumption the lines are distinct. Suppose they intersect at a point (x_0, y_0), then $y_0 = mx_0 + k$ and $y_0 = mx_0 + k'$, which then imply that $mx_0 + k = mx_0 + k'$, which in turn implies that $k = k'$. This is a contradiction to the earlier conclusion that $k \neq k'$. Thus the proof is concluded again.

Nature of the solution

Now we can analyze when a linear system of two equations in two unknowns has a solution. Let the system be

$$\left\{ \begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array} \right.$$

where a, b, $\ldots f$ are constants. Let ℓ_1, ℓ_2 be the lines defined by the equations $ax + by = e$ and $cx + dy = f$, respectively. There are several cases to consider.

Case 1. $b = d = 0$. Then by the definition of a linear equation of two variables, $a \neq 0$ and $c \neq 0$, and the system may be rewritten as

$$\left\{ \begin{array}{ccc} x & = & \frac{e}{a} \\ \\ x & = & \frac{f}{c} \end{array} \right.$$

In this case, the system has an infinite number of solutions if $\frac{e}{a} = \frac{f}{c}$, and the solutions are all ordered pairs of the form $(\frac{e}{a}, y)$ for *any* number y. The system has no solution if $\frac{e}{a} \neq \frac{f}{c}$. These assertions should be understood in terms of the graphs ℓ_1 and ℓ_2. These are two vertical lines passing through $(\frac{e}{a}, 0)$ and $(\frac{f}{c}, 0)$. If $\frac{e}{a} = \frac{f}{c}$, these lines are identical, and therefore any point on the line is a solution of both equations, and therefore of the system. If however $\frac{e}{a} \neq \frac{f}{c}$, then ℓ_1 and ℓ_2 are distinct vertical lines; they are parallel and therefore have no intersection. By an earlier remark in this section, the linear system has no solution.

Case 2. *Not both* $b = d = 0$. Let us say, $b \neq 0$. The consideration then breaks up into two sub-cases.

Case 2a. $d = 0$. Then the graph ℓ_2 is a vertical line. Since $b \neq 0$, ℓ_1 is not vertical and so ℓ_1 and ℓ_2 will intersect at one point. The system therefore has exactly one solution corresponding to the point of intersection.

Case 2b. $d \neq 0$. Then the system may be rewritten as

$$\begin{cases} y &= m_1 x + k_1 \\ y &= m_2 x + k_2 \end{cases}$$

where $m_1 = -\frac{a}{b}$, $k_1 = \frac{e}{b}$, $m_2 = -\frac{c}{d}$, and finally, $k_2 = \frac{f}{d}$. Now both ℓ_1 and ℓ_2 are non-vertical line. Recall from assertion (i) near the end of §4 that the slope of ℓ_1 is m_1 and the slope of ℓ_2 is m_2. Thus if $\frac{a}{b} \neq \frac{c}{d}$, the lines ℓ_1 and ℓ_2 are not parallel according to Theorem 1, and they will intersect so that the system will have exactly one solution. If $\frac{a}{b} = \frac{c}{d}$, then the lines ℓ_1 and ℓ_2 are parallel or identical, according to whether $\frac{e}{b} \neq \frac{f}{d}$ or $\frac{e}{b} = \frac{f}{d}$. If the former, ℓ_1 and ℓ_2 do not intersect and the system has no solution. If the latter, then the lines ℓ_1 and ℓ_2 coincide and the system will have an infinite number of solutions corresponding to each point on the line.

We can summarize all this into one theorem.

Theorem 2 *Given a linear system*

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

where a, b, $\ldots f$ are constants. Then:
(i) *If $b = d = 0$, the system has an infinite number of solutions if $\frac{e}{a} = \frac{f}{c}$, and has no solution if $\frac{e}{a} \neq \frac{f}{c}$.*
(ii) *If $b \neq 0$ but $d = 0$, the system has exactly one solution.*
(iii) *If $d \neq 0$ but $b = 0$, the system has exactly one solution.*
(iv) *If $b \neq 0$ and $d \neq 0$, then:*

the system has exactly one solution if $\frac{a}{b} \neq \frac{c}{d}$, *the system has no solution if* $\frac{a}{b} = \frac{c}{d}$, *but* $\frac{e}{b} \neq \frac{f}{d}$, *and*

the system has an infinite number of solutions if both $\frac{a}{b} = \frac{c}{d}$ *and* $\frac{e}{b} = \frac{f}{d}$.

Partial fractions and Pythagorean triples

We now give two applications of simultaneous equations. The first one is to express certain rational expressions in a number x as a sum of simpler rational expressions also in x. Consider the simple sum:

$$\frac{5}{x-2} + \frac{4}{x+3} = \frac{5(x+3)+4(x-2)}{(x-2)(x+3)}$$

After simplifying the numerator of the right side and multiplying out $(x-2)(x+3) = x^2+x-6$, we get the identity

$$\frac{5}{x-2} + \frac{4}{x+3} = \frac{9x+7}{x^2+x-6} \tag{8}$$

This is straightforward. Things get interesting, however, if you happen not to know identity (8) ahead of time but ask whether $\frac{9x+7}{x^2+x-6}$ can be expressed as a sum of (constant) multiples of the simple rational expressions $\frac{1}{x-2}$ and $\frac{1}{x+3}$. In general terms this question may be understood as part of our overall desire to express complicated objects in terms of simpler ones (think of the prime decomposition of a whole number, for example). On a more concrete level, this question arises naturally in calculus, and is a special case of *the partial fraction decomposition of a rational expression*. The answer to this question is by no means obvious, for two reasons. One is that even knowing $x^2+x-6 = (x-2)(x+3)$ ahead of time, one would be inclined to believe that $\frac{9x+7}{x^2+x-6}$ is a sum of more complicated rational expressions such as $\frac{ax+b}{x-2}$ and $\frac{cx+d}{x+3}$, for some appropriately chosen constants a, b, c, d, rather than just a sum of $\frac{5}{x-2}$ and $\frac{4}{x+3}$ without any x's in the numerators. The other reason is that even if you believe that such an expression is possible, there remains the question of how to get the precise values of the numerators, i.e., 4 and 5.

In order to answer this question, we have to first quote two facts without proof; the proofs are not difficult but they do take up time and space that we cannot afford at this point.

(A) Let (a_1x+b_1), (a_2x+b_2), $\ldots$ (a_nx+b_n) be n linear polynomials in x (n is a positive integer) so that none is a constant multiple of another. Let $p(x)$ be a polynomial in x of degree less than n. Then there are constants c_1, c_2, $\ldots$ c_n so that

$$\frac{p(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{c_1}{a_1x+b_1} + \frac{c_2}{a_2x+b_2} + \cdots + \frac{c_n}{a_nx+b_n}$$

(B) Suppose the following two n-th degree polynomials in x (n is a postive integer) are equal for *all* x:

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \;=\; b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$$

Then the coefficients of the polynomials are pairwise equal: $a_n = b_n$, $a_{n-1} = b_{n-1}$, $\ldots a_0 = b_0$.

Using Fact (A), we see that there must be constants a and b so that

$$\frac{9x+7}{(x-2)(x+3)} = \frac{a}{x-2} + \frac{b}{x+3}$$

which is valid no matter what x may be. We now use Fact (B) to recover equation (8), i.e., to obtain the values of the constants a and b as 5 and 4, respectively. By the addition of rational expressions,

$$\begin{aligned} \frac{a}{x-2} + \frac{b}{x+3} &= \frac{a(x+3)+b(x-2)}{(x-2)(x+3)} \\ &= \frac{(a+b)x+(3a-2b)}{(x-2)(x+3)} \end{aligned}$$

Combining the two equalities, we get

$$\frac{9x+7}{(x-2)(x+3)} = \frac{(a+b)x+(3a-2b)}{(x-2)(x+3)}$$

and therefore the numerators must be equal. Hence,

$$9x+7 = (a+b)x + (3a-2b)$$

for all x. From Fact (B), we know that the coefficients $a + b$ and $3a - 2b$ must be equal to 9 and 7, respectively. In other words, we have the following simultaneous linear equations in a and b:

$$\left\{ \begin{array}{rcrcl} a & + & b & = & 9 \\ 3a & - & 2b & = & 7 \end{array} \right.$$

We can solve this system by simply multiplying the first equation by 2 and then adding it to the second equation. This yields $5a = 25$ and therefore $a = 5$. The first equation now gives $b = 4$, as claimed.

The method is clearly sufficient to deal with the general situation.

We now give the second application of simultaneous linear equations. We say three positive *integers* a, b, c form a **Pythagorean triple** $\{a, b, c\}$ if $a^2 + b^2 = c^2$. In other words, a, b and c are the lengths of three sides of a right triangle. *Note that the third member of a Pythagorean triple is, by definition, the length of the hypotenuse of the right triangle.* It goes without saying that the key point of the definition of a Pythagorean triple is that all three numbers are *positive integers*. Everybody knows that 3, 4, 5 form a Pythagorean triple; some may even know that $\{5, 12, 13\}$ is another Pythagorean triple, or even that $\{8, 15, 17\}$ is yet another example. But are there others?

Our purpose is to produce Pythagorean triples at will by solving an extremely simple linear system of equations. It will be obvious that we will get an infinite number of Pythagorean triples by this method. It is even true that the method produces *all* the Pythagorean triples, though we will not prove this fact here. One would like to say that this method is due to the Babylonians some thirty-eight centuries ago, circa 1800 B.C. (*Babylon*, about sixty miles south of Baghdad in present day Iraq), but a more accurate statement would be that it is the *algebraic rendition* of the method one infers from a close reading of the celebrated cuneiform tablet, **Plimpton 322**, which lists fifteen Pythagorean triples.[30] See Eleanor Robson, Neither Sherlock Holmes nor Babylon: A reassessment of Plimpton 322, *Historia Mathematica*, 28 (2001), 167-206.

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[30] Lest you entertain for even a split second the idle thought that people couldn't have known such advanced mathematics thirty-eight centuries ago and that these triples were probably hit upon by trials and errors, let it be noted that the largest triple in Plimpton 322 is $\{12709, 13500, 18541\}$.

Let us first perform a conceptual simplification. Take $\{3,\ 4,\ 5\}$, for example. Once we are in possession of this triple, we will in fact be in possession of an infinite number of Pythagorean triples, namely, $\{6,\ 8,\ 10\}$, $\{9,\ 12,\ 15\}$, $\{12,\ 16,\ 20\}$, and in general, $\{3n,\ 4n,\ 5n\}$ for any positive integer n. Clearly, if you already have the Pythagorean triple $\{3,\ 4,\ 5\}$, there is not much glory in claiming that you also have another Pythagorean triple, namely, $\{6,\ 8,\ 10\}$. Accordingly, we define a Pythagorean triple $\{a,\ b,\ c\}$ to be **primitive** if the integers a, b, and c have no common divisor other than 1 (i.e., if k is a positive integer that divides all three a, b and c, then $k = 1$), and will henceforth concentrate on getting primitive Pythagorean triples. We say a Pythagorean triple $\{a,\ b,\ c\}$ is a **multiple** of another Pythagorean triple $\{a',\ b',\ c'\}$ if there is a positive integer n so that $a = na'$, $b = nb'$, and $c = nc'$. In this terminology, a given Pythagorean triple is either primitive, or is a multiple of a primitive Pythagorean triple.[31] Therefore whenever a Pythagorean triple is given, we lose nothing by replacing it with the primitive Pythagorean triple of which the first Pythagorean triple is a multiple. For example, instead of dealing with $\{15,\ 36,\ 39\}$, we will replace it by $\{5,\ 12,\ 13\}$.

We will give a proof of the following theorem. Observe that its statement makes use of the fact that any two fractions can be written as two fractions with the same denominator (FFFP).[32]

Theorem 3 *Let (u, v) be the solution of the linear system*

$$\left\{\begin{array}{ccccc} u & + & v & = & \dfrac{t}{s} \\ \\ u & - & v & = & \dfrac{s}{t} \end{array}\right.$$

where s, t are positive integers with $s < t$. If we write u and v as two fractions with the same denominator, $u = \frac{c}{b}$ and $v = \frac{a}{b}$, then $\{a,\ b,\ c\}$ form a Pythagorean triple.

There is extra incentive in providing a proof of this theorem, not only because the proof is very simple, but also because it actually tells us why the solution (u, v)

[31] The reason for this assertion will be left as an exercise.
[32] See Chapter 1, §2 of he Pre-Algebra notes.

of the linear system furnishes a Pythagorean triples. So with (u, v) as the solution of the linear system, we multiply the corresponding sides of the two equations in the theorem to get $(u+v)(u-v) = \frac{t}{s} \cdot \frac{s}{t}$, or $u^2 - v^2 = 1$. So with $u = \frac{c}{b}$ and $v = \frac{a}{b}$, we have

$$\left(\frac{c}{b}\right)^2 - \left(\frac{a}{b}\right)^2 = 1$$

Multiplying through both sides of this equality by b^2 gives $c^2 - a^2 = b^2$ and therefore,

$$a^2 + b^2 = c^2$$

We have our Pythagorean triple and the proof of Theorem 3 is complete.

It is easy to explain how Theorem 3 came about, but before doing that, let us put it to use to produce some new Pythagorean triples.

Example 1 Consider

$$\begin{cases} u + v = 2 \\ u - v = \frac{1}{2} \end{cases}$$

Adding the equations gives $2u = \frac{5}{2}$ so that $u = \frac{5}{4}$. From the second equation, we get $v = u - \frac{1}{2} = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$. Thus we have retrieved the grandfather of all Pythagorean triples, $\{3,\ 4,\ 5\}$.

Example 2 Consider

$$\begin{cases} u + v = \frac{3}{2} \\ u - v = \frac{2}{3} \end{cases}$$

Adding the equations gives $2u = \frac{13}{6}$ so that $u = \frac{13}{12}$. From the second equation, we get $v = u - \frac{2}{3} = \frac{13}{12} - \frac{2}{3} = \frac{5}{12}$. By Theorem 3, $\{5,\ 12,\ 13\}$ is a Pythagorean triple, which of course we already know.

Example 3 Consider

$$\begin{cases} u + v = \frac{4}{3} \\ u - v = \frac{3}{4} \end{cases}$$

Adding the equations gives $2u = \frac{25}{12}$ so that $u = \frac{25}{24}$ and the second equation gives $v = \frac{25}{24} - \frac{3}{4} = \frac{7}{24}$. By Theorem 3, $\{7,\ 24,\ 25\}$ is a Pythagorean triple. Since this is new to most people, one should check directly that $7^2 + 24^2 = 25^2$.

Example 4 Consider

$$\begin{cases} u + v = \frac{69}{2} \\ u - v = \frac{2}{69} \end{cases}$$

Adding the two equations gives $2u = \frac{4765}{138}$ so that $u = \frac{4765}{276}$. From the second equation, we get $v = \frac{4765}{276} - \frac{2}{69} = \frac{4757}{276}$. This time we get an unfamiliar Pythagorean triple $\{276,\ 4757,\ 4765\}$. Although Theorem 3 guarantees that this is indeed a Pythagorean triple, it would be good for your soul to directly check that $276^2 + 4757^2 = 4765^2$ is in fact true.

Observe that thus far, every single Pythagorean triple has been primitive. Now consider:

Example 5 Consider

$$\begin{cases} u + v = 5 \\ u - v = \frac{1}{5} \end{cases}$$

Adding the equations, we obtain $2u = \frac{26}{5}$ and multiplying both sides by $\frac{1}{2}$ gives $u = \frac{26}{10}$. From the second equation of the system, we get $v = \frac{26}{10} - \frac{1}{5} = \frac{24}{10}$. By Theorem 3, $\{10,\ 24,\ 26\}$ is a Pythagorean triple. This is not a primitive triple because it is a multiple of $\{5,\ 12,\ 13\}$, which we already know and which is clearly primitive.

Observe, however, that if we had taken the trouble to reduce $u = \frac{26}{10}$ to its lowest terms, then we would obtain $u = \frac{13}{5}$, and then we would get $v = \frac{13}{5} - \frac{1}{5} = \frac{12}{5}$, and the primitive triple $\{5,\ 12,\ 13\}$ would be the result. Thus we see that different values of s and t do not always lead to distinct primitive Pythagorean triples.

We now explain the genesis of Theorem 3. We will follow the time-honored method of *assuming* that we already have a Pythagorean triple $\{a,\ b,\ c\}$, and proceed to find out what it must be. By assumption, $a^2 + b^2 = c^2$, and by dividing through by b^2, we get $(a/b)^2 + 1 = (c/b)^2$. We now adopt the convention of letting

$$u = \frac{c}{b} \quad \text{and} \quad v = \frac{a}{b} \tag{9}$$

Thus u and v are both fractions (i.e., positive rational numbers) and, by our convention of letting c be the length of the hypotenuse of the right triangle, we always have $u > v$. Now $v^2 + 1 = u^2$ and therefore,

$$u^2 - v^2 = 1$$

Since $u^2 - v^2 = (u+v)(u-v)$, we get $(u+v)(u-v) = 1$. But we are assuming a, b, c to be known quantities, so both u and v are also known quantities and therefore so are $u+v$ and $u-v$. Let s, t be positive integers with $s < t$ so that $u + v = \frac{t}{s}$. (We are letting $u+v$ be a fraction bigger than 1 because $(u+v)(u-v) = 1$ and therefore one of $u+v$ and $u-v$ is greater than 1 and the other is less than 1. Clearly, $u+v > u-v$, so $u+v > 1$.) Because $(u+v)(u-v) = 1$, necessarily, $u - v = \frac{s}{t}$. Therefore we have:

$$\begin{cases} u + v = \dfrac{t}{s} \\ \\ u - v = \dfrac{s}{t} \end{cases} \tag{10}$$

where, we recall, the s and t are positive integers with $s < t$. We may regard this system as a system of linear equations in the variables u and v, and it is exactly the system in the statement of Theorem 3. From this point of view, Theorem 3 becomes inevitable.

We give a refinement of Theorem 3 by directly solving system (10). Adding the two equations, we get $2u = \frac{t}{s} + \frac{s}{t} = \frac{t^2+s^2}{st}$, and therefore

$$u = \frac{t^2 + s^2}{2st}$$

From the second equation of (10), we then obtain

$$v = u - \frac{s}{t} = \frac{s^2+t^2}{2st} - \frac{s}{t} = \frac{s^2+t^2}{2st} - \frac{2s^2}{2st} = \frac{t^2 - s^2}{2st}$$

so that

$$v = \frac{t^2 - s^2}{2st}$$

Since $u^2 - v^2 = 1$, we have

$$\left(\frac{t^2+s^2}{2st}\right)^2 - \left(\frac{t^2-s^2}{2st}\right)^2 = 1$$

Multiplying both sides by $(2st)^2$, we get

$$(t^2+s^2)^2 - (t^2-s^2)^2 = (2st)^2$$

or,

$$(s^2 - t^2)^2 + (2st)^2 = (s^2 + t^2)^2$$

(Compare problem 2 in the Exercises of Section 1.) This shows that if s, t are positive integers and $t > s$, then

$$\{2st,\ t^2 - s^2,\ t^2 + s^2\} \quad \textit{is a Pythagorean triple}$$

We have therefore presented two ways of obtaining Pythagorean triples: by giving values of s and t into the preceding formula, or by using Theorem 3 and solving the linear system there. Of course the former is a consequence of the latter, but for school mathematics, the latter is more instructive.

With a little more work, one can prove that *if s and t are relatively prime* (i.e., no common divisor other than ± 1)*, and if one of them is even and the other odd, then the triple* $\{2st,\ t^2 - s^2,\ t^2 + s^2\}$ *is primitive.* With more work still, it can be shown that every primitive Pythagorean triple is represented in terms of suitable s and t in this manner.

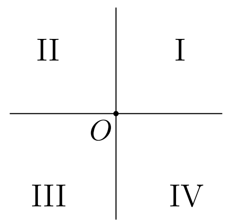
Appendix

In Theorem 1, we characterized parallelism in terms of slope. A companion theorem is the characterization of perpendicularity in terms of slope. We give the latter here, not only for reason of completeness, but also because it is now becoming common in textbooks to adopt the absurd practice of *defining* perpendicularity in terms of slope. The absurdity comes from the fact that perpendicularity has already been defined long ago in terms of the degree of an angle (see §1 of Chapter 5 in the Pre-Algebra notes). What we need is therefore *the proof of a theorem*, not a second definition.

Theorem 4 *Two nonvertical lines are perpendicular if and only if the product of their slopes is equal to* -1.

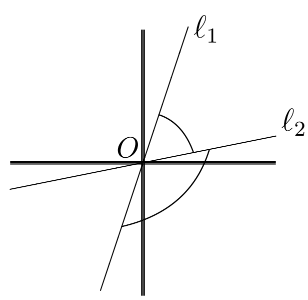
Proof Let ℓ_1 and ℓ_2 be the two given lines. First suppose they are perpendicular, and we will prove that the product of their slopes is -1.

Because ℓ_1 and ℓ_2 are nonvertical and are perpendicular to each other, they are also nonhorizontal. We begin by treating the special case where both lines pass through the origin O; later we will use Theorem 1 to extend the argument to the general case. So let ℓ_1 and ℓ_2 pass through O. To describe the relative positions of the lines, observe that the four right angles[33] formed by the positive and negative coordinate axes, with vertex at the origin O, are usually called the **four quadrants** of the coordinate system and are labeled **I, II, III** and **IV**, as shown.



Since ℓ_1 and ℓ_2 are non-vertical and non-horizontal, they must lie (with the exception of O) completely inside either quadrants I and III, or quadrants II and IV, as shown.

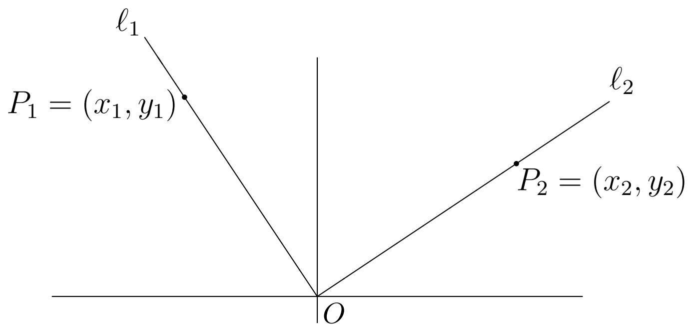
If both ℓ_1 and ℓ_2 lie in quadrants I and III, the degree of the angle between the rays on the lines is either greater than 90° or less than 90°, and ℓ_1 cannot be perpendicular to ℓ_2, as shown.



[33]Remember that an agle is a region in the plane rather than two rays issuing from a common vertex.

For a similar reason, not both ℓ_1 and ℓ_2 can lie in quadrants II and IV. We may therefore assume that ℓ_1 lies in quadrants II and IV and ℓ_2 lies in quadrants I and III.

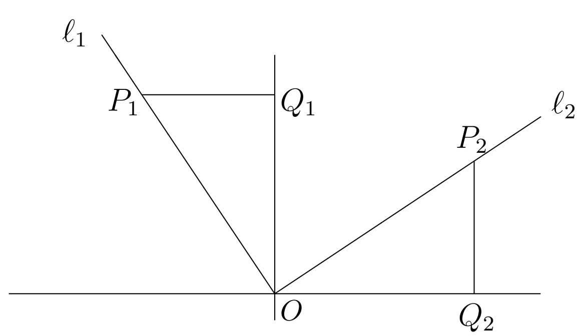
We choose points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ on ℓ_1 and ℓ_2, respectively, so that both lie above the x-axis. Then P_1 lies in quadrant II and P_2 lies in quadrant I. It follows that $y_1,\ y_2 > 0$, but $x_1 < 0$ and $x_2 > 0$.



The slope of ℓ_1 computed using points P_1 and O is y_1/x_1, which is negative, while the slope of ℓ_2 computed using points P_2 and O is y_2/x_2, which is positive. Therefore, the perpendicularity of ℓ_1 and ℓ_2 implies that the product of their slopes must be negative. It remains to check that the *absolute value* of the product of slopes is 1.

Let P_1, P_2 be chosen so that they are equidistant from O, i.e., $|OP_1| = |OP_2|$. Because the rotation ϱ of 90 degrees around the origin O carries ℓ_2 to ℓ_1, the fact that ϱ is a congruence implies that ϱ carries P_2 to P_1. Let the vertical line from P_2 meet the x-axis at Q_2, and let $Q_1 = \varrho(Q_2)$. Now Q_1 lies on the y-axis and, as ϱ is a congruence, $P_1Q_1 \perp x$-axis, and

$$|P_1Q_1| = |P_2Q_2| \quad \text{and} \quad |OQ_1| = |OQ_2|$$

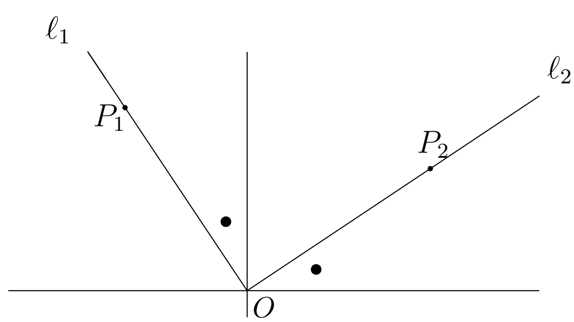


By the way a coordinate system is set up (see the discussion in the first subsection of §4), we know that $|P_1Q_1|$ and $|OQ_1|$ are the absolute values of the x and y-coordinates of the point P_1. Thus computing the slope of ℓ_1 using the points P_1 and O, we see that the absolute value of this slope is $|OQ_1|/|P_1Q_1|$. The absolute value of the slope of ℓ_2 is of course $|P_2Q_2|/|OQ_2|$. Thus, taking into account the previous equalities, the product of the absolute values of the slopes of ℓ_1 and ℓ_2 is

$$\frac{|OQ_1|}{|P_1Q_1|} \cdot \frac{|P_2Q_2|}{|OQ_2|} \quad = \quad \frac{|OQ_2|}{|P_2Q_2|} \cdot \frac{|P_2Q_2|}{|OQ_2|} \quad = \quad 1$$

This completes the proof that the product of the slopes of two nonvertical perpendicular lines which pass through the origin O must be equal to -1.

Still assuming that two lines ℓ_1 and ℓ_2 pass through the origin, how shall we approach the proof of the converse, namely, that if the product of the slopes of ℓ_1 and ℓ_2 is -1, then $\ell_1 \perp \ell_2$? In other words, if P_2 and P_1 are two points on ℓ_2 and ℓ_1, respectively, we want to prove $|\angle P_1OP_2| = 90°$.



Since we are already given that $\angle Q_1OQ_2$ is a right angle and $|\angle Q_1OQ_2| = |\angle P_2OQ_2| + |\angle Q_1OP_2|$, it means that if we can show

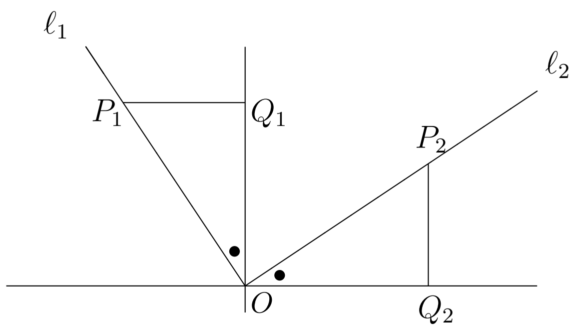
$$|\angle P_1OQ_1| = |\angle P_2OQ_2|$$

then we would have

$$|\angle P_1OP_2| = |\angle P_1OQ_1| + |\angle Q_1OP_2| = |\angle P_2OQ_2| + |\angle Q_1OP_2| = 90°$$

So how can we show $|\angle P_1OQ_1| = |\angle P_2OQ_2|$? We resort to the standard reasoning of identifying these angles as corresponding parts of similar or congruent triangles. For conceptual simplicity, we will only make use of congruent triangles.

Suppose now we have two lines ℓ_1 and ℓ_2 passing through O so that the product of their slopes is -1. We must prove that $\ell_1 \perp \ell_2$. Since the slopes of L_1 and L_2 have opposite signs, the above discussion shows that we may assume ℓ_2 lies in quadrants I and III, and ℓ_1 lies in quadrants II and IV. Let P_2 be some point on ℓ_2 lying in quadrant I. Drop a vertical line from P_2 so that it meets the x-axis at Q_2.



Let Q_1 be the point on the positive y-axis so that $|OQ_1| = |OQ_2|$ and let a horizontal line from Q_1 meet the ℓ_1 at P_1. If we can prove that $\triangle P_1OQ_1 \cong \triangle P_2OQ_2$, then we would have $|\angle P_1OQ_1| = |\angle P_2OQ_2|$, so that

$$
\begin{aligned}
|\angle P_1OP_2| &= |\angle P_1OQ_1| + |\angle Q_1OP_2| \\
&= |\angle P_2OQ_2| + |\angle Q_1OP_2| \\
&= |\angle Q_1OQ_2| = 90^\circ
\end{aligned}
$$

In other words, $\ell_1 \perp \ell_2$.

It remains to prove that $\triangle P_1OQ_1 \cong \triangle P_2OQ_2$. Since the product of the slopes of L_1 and L_2 is -1, the product of the absolute values of slopes of L_1 and L_2 is equal to 1. By a reasoning that is familiar to us by now, this means

$$
\frac{|P_2Q_2|}{|OQ_2|} \cdot \frac{|OQ_1|}{|P_1Q_1|} \;=\; 1
$$

Since $|OQ_1| = |OQ_2|$, we have

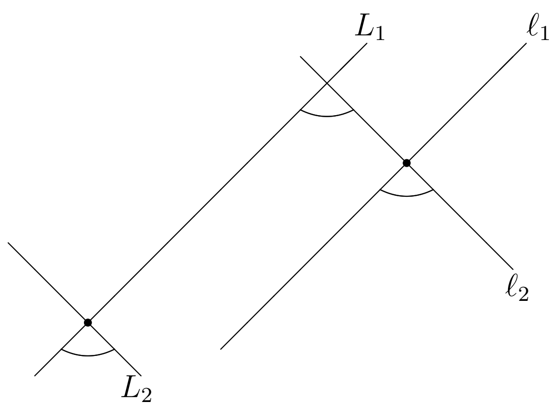
$$\frac{|P_2Q_2|}{|P_1Q_1|} = 1$$

and therefore $|P_2Q_2| = |P_1Q_1|$. Since $\angle P_2Q_2O$ and $\angle P_1Q_1O$ are right angles, the SAS criterion of congruence (Theorem G7 in Chapter 5 of the Pre-Algebra notes) implies that $\triangle P_1OQ_1 \cong \triangle P_2OQ_2$, as desired. This proves that if two lines ℓ_1 and ℓ_2 passing through O are such that the product of their slopes is -1, then $\ell_1 \perp \ell_2$.

We finish the proof of Theorem 4 by dealing with the general case that the two given lines ℓ_1 and ℓ_2 do not pass through the origin. Let L_1 and L_2 be lines passing through the origin O so that $\ell_1 \parallel L_1$ and $\ell_2 \parallel L_2$. We need the following simple lemma.

Lemma *Let ℓ_1 and ℓ_2 be intersecting lines, and let lines L_1 and L_2 be parallel to ℓ_1 and ℓ_2, respectively. Then the perpendicularity of ℓ_1 and ℓ_2 is equivalent to the perpendicularity of L_1 and L_2.*

The proof is an immediate consequence of the considerations of corresponding angles of parallel lines, as shown. The details can be left as an exercise.



Now suppose $\ell_1 \perp \ell_2$. By the lemma, also $L_1 \perp L_2$. Since L_1 and L_2 pass through O, the product of the slopes of L_1 and L_2 is -1. By Theorem 1, the slopes of ℓ_1 and L_1 are equal, as are the slopes of ℓ_2 and L_2. Hence the product of the slopes of ℓ_1 and ℓ_2 is -1. Conversely, suppose the product of the slopes of ℓ_1 and ℓ_2 is -1. By Theorem 1, the product of the slopes of L_1 and L_2 is also -1. Since L_1 and L_2 pass through O, $L_1 \perp L_2$. By the lemma, $\ell_1 \perp \ell_2$. The proof of Theorem 4 is complete.

Side remark We have been talking about lines that *slant this way* / *or that way* \ informally without being precise. Now precision is finally possible. We say **a line slants this way /** if the line passing through the origin and parallel to it lies in quadrants I and III , and similarly, we say **a line slants this way ** if the line passing through the origin and parallel to it lies in quadrants II and IV. It follows from Theorem 1 that a non-vertical and non-horizontal line slanting this way / has positive slope, and one slanting this way \ has negative slope.

EXERCISES

1. Solve:

(a) $\begin{cases} 7x - 3y &= 10 \\ 3x - 5y &= -5 \end{cases}$ (b) $\begin{cases} 7x - 9y &= 15 \\ 8y - 5x &= -17 \end{cases}$

(c) $\begin{cases} \frac{2}{5}x - \frac{5}{6}y &= -\frac{1}{2} \\ \frac{1}{6}x + \frac{5}{9}y &= \frac{5}{2} \end{cases}$ (d) $\begin{cases} 12x + 11y &= 172 \\ 28x - 17y &= 60 \end{cases}$

(e) $\begin{cases} 0.08x + 0.9y &= 0.46 \\ 0.1x - 0.04y &= 0.16 \end{cases}$ (f) $\begin{cases} 5x - \frac{3}{4}y &= 2 \\ x + 2y &= \frac{11}{6} \end{cases}$

2. Solve:

(a) $\begin{cases} \frac{6}{x} + \frac{12}{y} &= -1 \\ \frac{8}{x} - \frac{9}{y} &= 7 \end{cases}$ (b) $\begin{cases} \frac{9}{x} - 3y &= 4 \\ \frac{3}{x} + 2y &= \frac{10}{3} \end{cases}$

3. Prove that a Pythagorean triple is either primitive, or is a multiple of a primitive Pythagorean triple.

4. Alan's age is $\frac{6}{5}$ of Bill's, and 15 years ago, his age was $\frac{13}{10}$ of Bill's. Find their ages.

5. If 3 is added to the numerator of a fraction and 7 subtracted from the denominator, its value is $\frac{6}{7}$. But if 1 is subtracted from the numerator and 7 added to the denominator, its value is $\frac{2}{5}$. Find the fraction.

6. The contents of one barrel is $\frac{5}{6}$ wine, and of another $\frac{8}{9}$ wine. How many gallons must be taken from each to fill another barrel whose capacity is 24 gallons, so that the mixture may be $\frac{7}{8}$ wine?

7. Express $\frac{8x+2}{x^2-1}$ as a sum of integer multiples of $\frac{1}{x+1}$ and $\frac{1}{x-1}$.

8. Express $\frac{-5(x+1)}{3(x^2+x-12)}$ as a sum of integer multiples of $\frac{1}{x+4}$ and $\frac{1}{x-3}$.

9. In each of the following, you are asked to solve the linear system in Theorem 3 with the given values of s and t to obtain Pythagorean triples. You may use a *scientific calculator*, especially for (i) and (j) below.

(a) $s = 2,\ t = 5$.
(b) $s = 4,\ t = 5$.
(c) $s = 1,\ t = 4$.
(d) $s = 1,\ t = 3$.
(e) $s = 3,\ t = 13$.
(f) $s = 1,\ t = 12$.
(g) $s = 3,\ t = 13$.
(h) $s = 1,\ t = 6$.
(i) $s = 54,\ t = 125$.
(j) $s = 8,\ t = 9907$.

10. In (j) of the the last problem, the largest number in the Pythagorean triple has 8 digits. Suppose you have a calculator with only a 12-digit display on the screen. Explain how you can use such a calculator to directly verify that the triple of numbers so obtained *is* a Pythagorean triple.

11. The second digit of a two-digit number is $\frac{1}{3}$ of the first digit. If the number is divided by the difference of the digits, the quotient is 15 and the remainder is 3. Find the number.

12. Write out a detailed prof of the Lemma after Theorem 4.

7 Functions and Their Graphs

A major concern of algebra, and in fact of all of mathematics, is the concept of a function. In this section, we give the definition of this concept and that of its graph, and single out the graphs of so-called *real-valued functions of one variable* for emphasis. The remainder of these notes will be devoted to the study of functions.

The basic definitions

A **function** from a set A to a set B is a rule (i.e., a precise prescription) that assigns (or associates) to each element of A an element of B. To be precise, we should emphasize that a function, by definition, assigns to each element of A only *one* element of B. If the function is denoted by f, then $\boldsymbol{f : A \to B}$ is the correct notation to capsulize this information. However, when A and B are understood, the function f is often denoted generically by $\boldsymbol{f(x)}$; this is not a good notation, but is one that appears in most textbooks and so you may as well get used to it. If f assigns the element b of B to an element a of the set A, we write

$$f(a) = b$$

and say b is the **value of $\boldsymbol{f}$ at $\boldsymbol{a}$**. If A and B are subsets of the real numbers, such an f is called a **real function of one variable**, or more correctly, **real-valued function of one variable**.

The set A of a function $f : A \to B$ is called the **domain** of f; there is no universally accepted terminology for the set B. Some call it the **target**, others call it **range**, and yet others call it **co-domain**. You have to be alert to this ambiguity in the literature. This monograph tries to minimize any reference to B precisely because of the ambiguity.

Because these notes have put so much emphasis on the precision of language, you may be startled to find that the definition of a function above speaks of "a rule that assigns an element to another element" when the meaning of a "rule" is not completely transparent. Let it be noted that this slight ambiguity is intentional rather than an intrinsic flaw. To arrive at a completely precise definition of a function $f : A \to B$, one way is to think of a in A and $b = f(a)$ in B as an ordered pair (a, b); then instead of saying f assigns b to a, we say in self-explanatory language that "the ordered pair (a, b) belongs to f". To make explicit the fact that f assigns only one element of B to a, we say that for a function f, if both ordered pairs (a, b) and (a, b') are in f (where b and b' are elements of B), then necessarily $b = b'$. In this precise language, the *abstract* definition of a function $f : A \to B$ is that it is a collection of ordered pairs (a, b) where a is in A and b is in B, so that if (a, b) and (a, b') are in this collection, then $b = b'$. Having said that, one has to recognize that such (stilted) precision is not likely to be appreciated by school students, or even by students who do not plan on being math majors in college. So the hope is that the above informal definition of a function is an acceptable compromise between clarity and accessibility.

In a course on introductory algebra, the kind of functions one encounters are usually real functions of one variable. Often, such functions can be described symbolically by formulas. Thus, the function

$$F : \{\text{real numbers}\} \to \{\text{real numbers}\}$$

which assigns to each number its square can be succinctly given as $F(x) = x^2$ for each number x. We note for emphasis that $F(x)$ is always ≥ 0 for any x, so that if we write instead

$$F : \{\text{real numbers}\} \to \{\text{all real numbers} \geq 0\}$$

then we would also be correct. For the moment, however, we want to give a general discussion and would not want to restrict ourselves, yet, to these common cases. For example, if G is the function

$$G : \{\text{a deck of cards}\} \to \{\text{club, diamond, heart, spade}\}$$

which assigns to each card its suit, then what G does to each card would be difficult to describe in symbols. One can illustrate by giving some examples, such as

G(King of diamonds) = diamond

G(Two of spades)= spade

G(Queen of hearts) = heart, etc.

Yet another example of a function is a person's age, or more precisely, the association of each person with his/her age . What we have is a function $K : \{\text{people}\} \to \{\text{whole numbers}\}$ so that if x is a person, $K(x) =$ the age of x. Once you get this idea, you begin to see many examples of functions in real life. For example, writing down a person's gender is in effect a function $H : \{\text{people}\} \to \{\text{M, F}\}$, where the latter means the set consisting of the two letters "M" and "F". And so on.

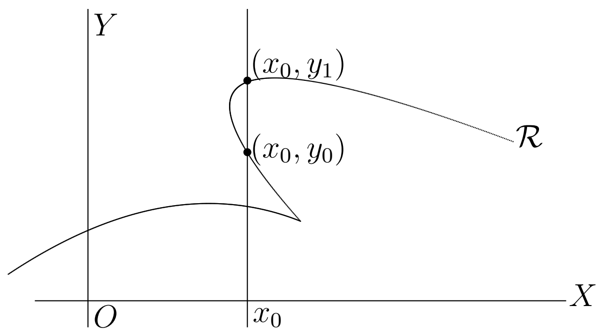
An effective way to get to know the concept of a function is to look at many examples of real functions of one variable and to examine their *graphs*. We now define this concept.[34] Let f be a function from a set of numbers A to a set of numbers B, $f : A \to B$. Then the **graph of f** is the set of *all* the points $(x, f(x))$ in the plane, where x is an element of A. In general, of course, the set A is infinite, so that the graph of f is an infinite set as well. Although it is impossible to literally get hold of the *whole* graph of any function, it is usually the case that plotting a finite number of well-chosen points in the graph is enough to reveal the essential features of the graph, and therefore of the function itself. In a later subsection, we will give some standard examples of graphs of functions. It should be noted that plenty of practice in plotting points on a graph is an essential component in the learning about graphs and functions. So please remember: don't let your students use the graphing calculator until they have achieved fluency in plotting points.

The graph G of a real function of one variable f is thus a subset of the plane. We should point out two things about the graphs of functions. One is that the graph is in fact what we called above the *abstract* definition of a function, i.e., a collection of ordered pairs of numbers G with a special property about the second coordinate of its points, namely, if two points (x_0, y_0) and (x_0, y_1) (which have the same first coordinate) are in G, then necessarily $y_0 = y_1$. This is because, by definition, (x_0, y_0) being a point of G means $y_0 = f(x_0)$. Similarly, $y_1 = f(x_0)$. Thus f assigns

[34] Although we are defining the graph of a function only for real functions of one variable here, the same definition is valid for any function. But of course when the function is not a real function of one variable, the picture of a graph becomes less tangible.

to x_0 the value y_0 as well as the value y_1. But a function assigns only one value to each element in the domain, so $y_0 = y_1$. Second, this property of the graph G of such a function f has a geometric interpretation: the intersection of the graph G with a vertical line $x = x_0$ (which is the set of all the points of the form (x_0, y), where y is any number) is either empty or consists of *one* point. This is called the **vertical line rule**: a vertical line can intersect the graph of a function at at most one point.

Naturally, a general subset $\mathcal{R}$ of the plane does not necessarily satisfy this special property, i.e., it can happen that (x_0, y_0) and (x_0, y_1) are both in $\mathcal{R}$ but $y_0 \neq y_1$. This would be the case if $\mathcal{R}$ is itself a vertical line, for example, or something like the following:



A subset of the plane is called a **relation**. In advanced mathematics, some relations of significance, which are not graphs of functions, do arise every now and then, but not often. Most of the time, it is functions that take center stage and therefore it is the graphs of functions (rather than relations) that are of interest. You should know what a relation is (as you do now), but should not waste precious classroom time on relations. You will come across textbooks that devotes many pages to various aspects of relations, including definitions of the *domain* and *range* of a relation. You will even come across standardized tests that demand that your students know what the range of a given relation is. As far as mathematics is concerned, such information is of negligible import. So just ignore relations in your teaching.

We now introduce one last definition in this subsection. Suppose we are given a function H from the plane to the set of all numbers, i.e., for any point (x, y) in the plane, H assigns to it a number $H(x, y)$. Such a function H is called a **function of**

two variables, or more precisely, a **real-valued function of two variables**. Let c be any fixed number. Then the **graph of the equation $\boldsymbol{H = c}$** is the set of all the points (x', y') in the plane so that $H(x', y') = c$.

We have now defined the concept of the graph of a real function of one variable, and for any function of two variables H, we have also defined the graph of an equation $H = c$ for a fixed constant c. Both are subsets of the plane. Where these concepts of graphs come together is in the following situation. Let f be a real function of one variable:

$$f: \{\text{all numbers}\} \rightarrow \{\text{all numbers}\}$$

Define a function of two variables H_0 by $H_0(x, y) = y - f(x)$ for all x and y. Then

$$\{\text{the graph of } f\} = \{\text{the graph of the equation } H_0 = 0\}.$$

This is because (x, y) being in the graph of $H_0 = 0$ is equivalent to $y - f(x) = 0$, which is equivalent to $y = f(x)$, which is then equivalent to $(x, y) = (x, f(x))$. The last is of course equivalent to (x, y) being in the graph of f itself.

Activity Let a real function of two variables H be defined by $H(x, y) = x - y$. What is the graph of the equation $H = 0$? What is the graph of the equation $H = 1$? What is the graph of the equation $H = -25$? How are these graphs related?

Why functions?

The simple answer to this question is that nature and human activities are never static and they are always changing. To describe change, we need functions. The following three examples give a general idea.

We first make an observation: arithmetic is the tool to study a static situation. For example, suppose you have just brewed a cup of coffee and are waiting for it to cool down. You may formulate an arithmetic problem of the following type:

> *Let us say the coffee is* 195° (*Fahrenheit*) *at the beginning, but after 4 minutes it is* 143°. *What is its average rate of cooling in the first four minutes?* (See the discussion of average speed and average rate in §8 of Chapter 1 in the Pre-Algebra notes.)

The answer is of course

$$\frac{\text{total change in temp. from 0 to 4 minutes}}{\text{4 minutes}} = \frac{195 - 143}{4} = \frac{52}{4} = 13 \text{ deg/min.}$$

More important than the answer is the framework that undergirds the formulation of this problem: Take two snapshots of the evolving state of your cup of coffee, once at the beginning and a second time at the 4 minute mark, and freeze those two moments in time. Then looking only at those two static moments, we come up with the above arithmetic problem. But what if we ask a more realistic question, one that is perhaps of pressing concern: how long do you have to wait before the cup of coffee becomes drinkable? It is obvious that such arithmetic problems would bear little relevance to this question.

What we need, for starters, a way to describe the temperature of the coffee at various times. A primitive response to this need may be to create a table:

time after brewing	temperature
0	195
1	180
2	165
3	153
4	143
5	135

However, if you want to know the temperatures at the half-minute marks, then you'd need a bigger table:

time after brewing	temperature	time after brewing	temperature
0	195	3	153
0.5	187	3.5	148
1	180	4	143
1.5	172	4.5	139
2	165	5	135
2.5	159	5.5	132

Now if you also want the temperatures at the quarter minute marks, you'd need a table that is even bigger. Clearly there is no end in sight of the size of the table you need if you want a complete profile of the whole evolving situation, and you soon realize that what you really need is not a table of enormous size but a *function* $f(t)$, where

$$f : \{\text{all numbers} \geq 0\} \rightarrow \{\text{all numbers}\}$$

$f(t) =$ the temperature of the coffee t minutes after it is brewed.

Once we have the right concept in the form of a function (and not a table), the next step is determine this function in the form of a reasonable formula. The rest of the story is related to *Newton's law of cooling*; the long and short of it is that there is such a formula using concepts in calculus:

$$f(t) = 70 + 125\left(\frac{110}{125}\right)^t$$

which assumes an ambient temperature of 70 degrees Fahrenheit. What is important for us is the realization that without the concept of a function to describe the change in temperature, such scientific progress would not be possible.

Consider a second example: a man drives to the airport which is 25 miles away. He plans to leave his house two hours before departure time. If we want to see how far he is from the airport, clearly one number won't get the job done because this distance depends on the time when the distance is measured. Our experience with the coffee problem suggests that we make use of a function F for this description, such that

$F(t) =$ his distance (in miles) from the airport t minutes after he leaves his house

Thus $F(0) = 25$. In general, F assigns to each number $t \geq 0$ another number which is his distance in miles from the airport at time t. Even a skeletal description of this function in terms of a few values of t can tell a story, as for instance:

t	$f(t)$	t	$f(t)$
0	25	30	13
5	24	35	19
10	22.5	40	24
15	21	43	25
20	16	44	25
25	10	45	24.5
26	9	55	13
27	10	60	7.5
28	11	67	0

We can see that he has to start his trip slowly probably because of city traffic, so that after 10 minutes he only travels two and a half miles. Around the 26th minute after he leaves home, he turns around as the values of $f(27)$ and $f(28)$ and those of subsequent minutes show that he is driving away from the airport. He forgets to bring his photo-ID (a guess!). He manages to get home at 43 minutes after his departure and it takes him only about a minute to get the necessary document. Then he speeds a bit as he makes it to the airport in 23 minutes ($67 - 44 = 23$); not trivial considering the traffic condition these days. He has a few minutes to spare.

As a final example, consider the problem of the temperature of the city of Berkeley on a certain day. To say that Berkeley is 67° (Fahrenheit) makes no sense, strictly speaking. Is the temperature taken in the early dawn or in the afternoon? In Berkeley, this could mean a 25° difference. And where is the temperature measured: at the top of the hill (about 1500 feet high), downtown, or by the Bay? The difference here could be another 15°. If we start measuring the time t in hours from midnight, then $0 \leq t \leq 24$. To specify the geographic location, we need two more numbers which may be thought of as the idealized x and y coordinates. Berkeley being a small city, 5 miles from the city center in any direction would include everything. Therefore, a scientifically usable description of the temperature of Berkeley would make use of a function T, so that, if $\mathcal{S}$ is the region in 3-space consisting of all ordered triples of numbers (x, y, t) so that x and y satisfy $|x|,\ |y| \leq 5$ (miles), and $0 \leq t \leq 24$ (hours), then[35]

$$\begin{aligned} F:\ \mathcal{S} \rightarrow&\ \{\text{all numbers}\} \\ F(x,y,t) =&\ \text{the temperature of Berkeley, } t \text{ hours past midnight at a spot} \\ &\ \ \text{specified by the } x \text{ and } y \text{ coordinates} \end{aligned}$$

Incidentally, such a function F is said to be a **function of three variables**, because three numbers x, y, and t are involved in its definition. It may not have escaped your attention that we give F this name as an afterthought, without holding forth on the philosophical implications of what a "variable" is. This is as it should be. (Compare

[35] Notice also the use of absolute value to describe the physical extent of the city. It means of course that $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$. See §6 of Chapter 2 in the Pre-Algebra notes.

the discussion of the term "variable" given at the beginning of §§1 and 2.)

Quite apart from the description of change, functions have already forced their way into our work whether we know it or not. Transformations of the plane, including rotations, reflections, and translations, are functions which assign to each point of the plane another point of the plane, i.e., these are examples of function $\mathcal{T}$, so that

$$\mathcal{T}: \ \{\text{the plane}\} \to \ \{\text{the plane}\}$$

For example, the translation T which moves every point of the plane 2 units to the left horizontally is precisely given by:

$$T(x, y) = (x - 2, y)$$

These are among the simplest examples of how functions *naturally* arise. Of course functions are everywhere as soon as you look around. You will be seeing many more from now on. The purpose of this discussion is to make it plain that the concept of a function is not something artificially concocted for the purpose of giving students a hard time. Rather, it is a tool, created out of necessity, to succinctly describe the phenomena around us, be they sociological or mathematical. It is indispensable.

Some examples of graphs

There is little or no mystery to the concept of the graph of a function, at least not when the function is a real-valued function of one variable. Just follow the precise definition and plot as many points as possible in the coordinate plane to get a feeling of what the graph looks like. For the school classroom, we repeat:

> *The importance of actually plotting points on the graph by hand cannot be over-emphasized, and this is especially true in an age of affordable graphing calculators.* ***Be sure to insist on it in your classroom.***

Let us start with a linear function of one variable, $f(x) = ax+b$, where a and b are constants. If one is approaching this problem for the first time, there is no substitute for graphing a simple function like $g(x) = 2x - 3$ by plotting a few points of its graph and observing that they all line up in a straight line, e.g.,

$$(0, -3), \quad (1, -1), \quad (2, 1), \quad (3, 3), \quad (4, 5), \quad (5, 7), \quad (6, 9), \quad (7, 11)$$

But let us address the problem of graphing $f(x) = ax + b$ in general, for a particular reason.

We observed at the end of the subsection The basic definitions that the graph of $f(x) = ax + b$ is the graph of the equation $y - (ax + b) = 0$ which, from §4, is a line. Thus all graphs of linear functions of one variable are lines. Consider now a slightly different problem: for a fixed constant c, what is the set of *all* the points (x', y') so that $y' - (ax' + b) = c$? This set is called a **level set** of the function of two variables $H(x, y) = y - (ax + b)$, and is denoted by $\boldsymbol{\{H = c\}}$. Of course, if $c = 0$, this would be the same question as asking for the graph of f. Now (x', y') being in $\{H = c\}$ is equivalent to $y' - (ax' + b) = c$, which in turn is equivalent to $(-a)x' + y' = b + c$, which is equivalent to (x', y') being a solution of the linear equation in two variables $(-a)x + y = b + c$. The conclusion is that a level set $\{H = c\}$ is always a line, namely, the graph of the equation $(-a)x + y = b + c$.

> *The reason for the terminology of "level set" for $\{H = c\}$ come from the fact that if we graph $H(x, y)$ in 3-space, then $\{H = c\}$ is exactly the intersection of this graph with the plane $z = c$, which is considered to be "level" with the xy-plane.*

Activity Let $H(x, y) = 3x - y$. Describe the level sets $\{H = 0\}$, $\{H = 1\}$, $\{H = 2\}$, and $\{H = -4\}$.

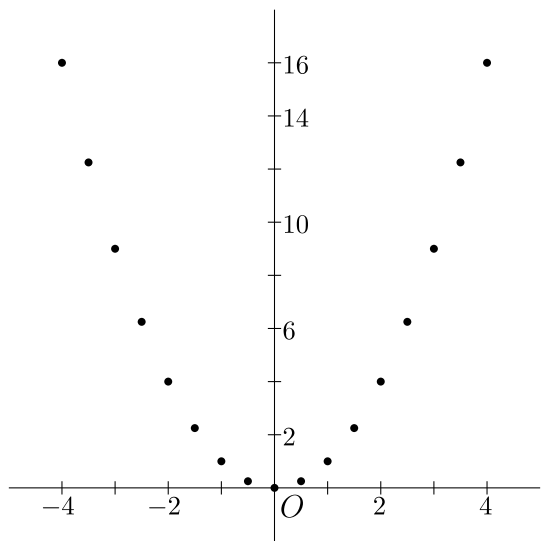
Let us try a function which is not linear. For example, take the square function $s : \mathbf{R} \to \mathbf{R}$, $s(x) = x^2$ for all numbers x. The graph of s consists of all the points of the form (x, x^2), where x is arbitrary. Since $(-x)^2 = x^2$, we see that the graph includes both (x, x^2) and $(-x, x^2)$, no matter what x may be. The point $(0, 0)$ is an obvious point on the graph. We can put in values of $x = \pm 1, \pm 2, \pm 3, \pm 4$ to get the points

$$(\pm 1, 1) \qquad (\pm 2, 4) \qquad (\pm 3, 9) \qquad (\pm 4, 16)$$

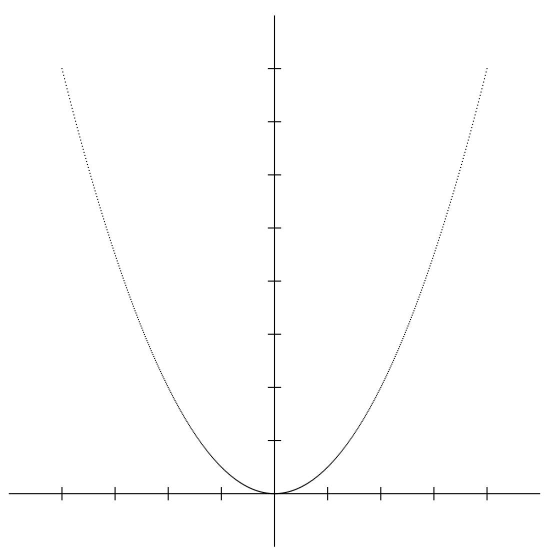
Let us also throw in

$$(\pm 0.5, 0.25) \qquad (\pm 1.5, 2.25) \qquad (\pm 2.5, 6.25) \qquad (\pm 3.5, 12.25)$$

for good measure, and we get the following sequence of points on the graph of s. Note that in order to make the picture manageable, we have shrunk the scale of the y-axis by half.



It is not difficult to extrapolate from these points to envision the graph as the following curve:



This curve is an example of what is called a *parabola*. Parabolas are discussed more fully in §§11 and 12.

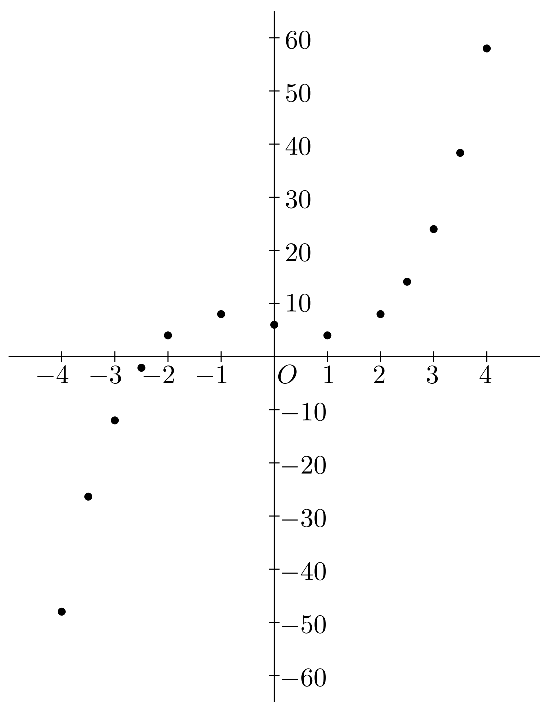
With the availability of scientific calculators, there should be no hesitation in asking students to graph quite sophisticated functions, e.g., a function such as $x \mapsto \frac{x^4-7x+5}{x^2+2}$. Let us illustrate with a slightly sophisticated one such as $G : \mathbf{R} \to \mathbf{R}$ given by $G(x) = x^3 - 3x + 6$. Recall from §1 that this is called a *cubic polynomial*, or more simply, a **cubic**. Since we have no idea what to expect, we try some obvious numbers, e.g., $G(0)$, $G(\pm 1)$, $G(\pm 2)$, $G(\pm 3)$, $G(\pm 4)$, getting the following points on the graph of G:

$$\begin{array}{lllll} (-4,-46) & (-3,-12) & (-2,4) & (-1,8) & (0,6) \\ (1,4) & (2,8) & (3,24) & (4,58) & \end{array}$$

Because the jumps in the values of G between the values of x at 2 and 3, 3 and 4, -2 and -3, and -3 and -4 are so great, we also get the additional points on the graph of G:

$$(-3.5,-26.375) \quad (-2.5,-2.125) \quad (2.5,14.125) \quad (3.5,38.375)$$

By compressing the y-axis, we can exhibit these points as follows:



The graph seems to cross the x-axis between -3 and -2. Suppose it crosses the x-axis at $(x_0, 0)$, then $0 = G(x_0)$ *by definition of the graph of* G. This means $x_0^3 - 3x_0 + 6 = 0$. Such an x_0 is called a **root** of the cubic polynomial equation $x^3 - 3x + 6 = 0$. The roots of a polynomial equation are of great interest in mathematics. For this reason, one may try to get a better estimate of this x_0. We have $G(-2.5) = -2.125$, $G(-2.4) = -0.624$, $G(-2.3) = 0.733$, so it is intuitively clear that x_0 is between -2.4 and -2.3. By experimenting with $G(-2.31)$, etc., we can get even better estimates of this root.

Notice that the graph has a bump above (roughly) -1, and has a trough above (roughly) 1. It is a known fact, proved in advanced courses, that *the graph of a cubic polynomial can have at most one bump and one trough.* Therefore we are very fortunate that with the choice of the nine obvious points on this graph, we already know that there are no further troughs and bumps below $x = -4$ and above $x = 4$. So the graph will continue to go down as we go to the left of the x-axis, and continue to go up as we go to the right of the x-axis.

If we were less fortunate and the chosen points happen not to reveal the bump and the trough of the graph, then we would have to plot more points, since these features may not have appeared yet or may not even exist (look at the graph of $g(x) = x^3$, for example).

Let us graph the function h given by $h(x) = \frac{1}{x}$.

We first address an issue concerning this h that we have not confronted thus far, and it is this. Would it be correct to say that h is a function from all numbers to all numbers? In other words, would it be correct to say that

$$h : \{\text{all numbers}\} \to \{\text{all numbers}\}$$

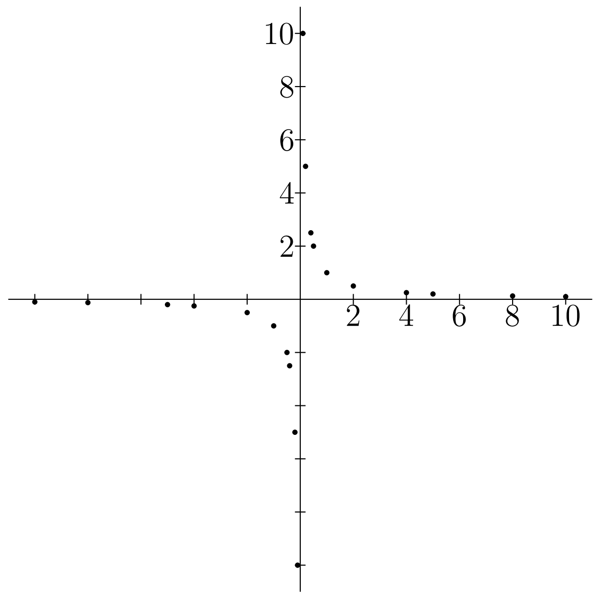
The answer is no, because h cannot assign any number to 0, as infinity is not a number. So the correct statement is that

$$h : \{\text{all nonzero numbers}\} \to \{\text{all numbers}\}$$

That said, we start plotting points. Again, there are two obvious points: $(1, 1)$ and $(-1, -1)$. Beyond that we take some random values of x and compute $\frac{1}{x}$, and we will remark on the following choices in due course:

(0.1, 10)	(0.2, 5)	(0.4, 2.5)	(0.5, 2)	(2, 0.5)	(4, 0.25)
(5, 0.2)	(8, 0.125)	(10, 0.1)	$(-0.1, -10)$	$(-0.2, -5)$	$(-0.4, -2.5)$
$(-0.5, -2)$	$(-2, -0.5)$	$(-4, -0.25)$	$(-5, -0.2)$	$(-8, -0.125)$	$(-10, -0.1)$

The corresponding picture is then:



Notice that there are two separate curves here, and they are called the *two branches of a hyperbola*. Regrettably, we will not pursue the study of hyperbolas.

The plotted points above exhibit a pattern: if $0 < a < b$ or $a < b < 0$, then $\frac{1}{a} > \frac{1}{b}$. (In an exercise, you are asked to prove this in general.) This pattern justifies that this limited choice of the points on the graph is enough to reveal the general behavior of the graph: it tells us that as the upper right curve extends to the right end of the positive x-axis, all it does is to get closer and closer to the x-axis, and as it approaches 0 from the positive x-direction, all it does is to get closer and closer to the positive y-axis. A similar statement also applies to the lower left curve.

EXERCISES

1. Prove that if $0 < a < b$ or $a < b < 0$, then $\frac{1}{a} > \frac{1}{b}$.

2. *Plot enough points* in the graph of each of the following functions to get an accurate picture of the graph: (*i*) $x^2 - 2x + 5$, (*ii*) x^3, (*iii*) $x^3 + 2$, (*iv*) $(x - 5)^3$.

(Use a scientific calculator.)

3. *Plot enough points* in the graph of each of the following functions to get an accurate picture of the graph: (*i*) $x^2 - x$, (*ii*) $3x^2 - 4x + 1$, (*iii*) $x^3 - x^2 - 4x + 4$, (*iv*) $2x^3 - 4x^2 + x + 6$. (Use a scientific calculator.)

4. *Plot enough points* in the graph of each of the following functions to get an accurate picture of the graph: $f_1(x) = 2x^2$, $f_2(x) = 2x^2 + 3$, $f_3(x) = 2(x-1)^2$, and $f_4(x) = 2(x-1)^2 + 3$. How are they related? Why?

5. Let $f(x) = ax^2$ and $g(x) = a(x-b)^2 + c$, where a, b, c are constants. Describe how the graphs of f and g are related. Why?

6. (a) Let H be the function of two variables defined by $H(x,y) = \frac{2}{3}x - \frac{1}{4}y + 5$. Describe the level sets $\{H = 1\}$ and $\{H = -2\}$ individually and how they are related. (b) Let H be the function of two variables defined by $H(x,y) = ax + by + d$, where a, b, d are constants with $b \neq 0$. Let c and c' be distinct constants. Describe the level sets $\{H = c\}$ and $\{H = c'\}$ individually and how they are related.

8 Linear functions and proportional reasoning

With the concept of a function available, we are now in a position to revisit the earlier discussion of rates and constant rates in Section 5, especially Examples 4–8 and make better sense of that discussion. We also take this opportunity to critically examine the concept of proportional reasoning.

Constant rate and linear functions

For definiteness, let us consider the case of motion. An object in motion is described by a function f, where

$$f(t) = \text{ the distance traveled from time 0 to time } t$$

Let us say t is measured in hours and $f(t)$ in miles. We adopt the usual convention of letting time 0 be the starting time and that $f(0) = 0$ by definition. Formally introduce the concept of **average speed** during the time interval from time t_1 to time t_2 $(0 \leq t_1 < t_2)$ as the distance traveled from time t_1 to time t_2 divided by the length of the time interval (which is of course $t_2 - t_1$). In terms of f, we therefore have:

$$\text{average speed from } t_1 \text{ to } t_2 = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

Average speed therefore measures miles per hour (*mph*). Finally, we say the motion has **constant speed v** mph if the average speed over *any* time interval is the fixed number v. Here is the basic observation.

Theorem *Notation as above, the fact that the motion has constant speed v mph is equivalent to the fact that f is given by $f(t) = vt$ miles for all $t \geq 0$.*

Proof First suppose the motion has constant speed v. Then by definition,

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} = v$$

for any t_1 and t_2 so that $0 \leq t_1 < t_2$. By letting $t_1 = 0$ and let t_2 be arbitrary, we get

$$\frac{f(t_2)}{t_2} = v$$

which is the same as

$$f(t_2) = vt_2$$

Since t_2 is arbitrary, we have $f(t) = vt$ for all t. Conversely, suppose $f(t) = vt$. By definition,

$$\begin{aligned} \text{average speed from } t_1 \text{ to } t_2 \quad &= \quad \frac{f(t_2) - f(t_1)}{t_2 - t_1} \\ &= \quad \frac{vt_2 - vt_1}{t_2 - t_1} \end{aligned}$$

$$
\begin{aligned}
&= \frac{v(t_2 - t_1)}{t_2 - t_1} \\
&= v
\end{aligned}
$$

This then proves the theorem.

For your beginning students in algebra, it would be good to point out that the cancellation of $(t_2 - t_1)$ in the last calculation depends on the fact that it is a nonzero number, which explains why we were so careful to specify that $t_1 < t_2$.

Naturally, an entirely similar discussion can be given for water flow at a constant rate, work done at a constant rate, etc. For example, in case of water flow from a faucet (let us say), let F be the function so that $F(t)$ is the amount of water (in gallons) coming out of the faucet from time 0 to time t (in minutes). The rate of flow is constant, by definition, if there is a fixed number r so that during any time interval from t_1 to t_2 $(0 \le t_1 < t_2)$ the *total amount of water coming out of the faucet in this time interval* divided by *the length of the time interval* is always equal to r gallons per minute. In other words,

$$\frac{F(t_2) - F(t_1)}{t_2 - t_1} = r$$

for all t_1 and t_2 so that $0 \le t_1 < t_2$. One then proves in exactly the same way that the rate of water flow being a constant r gallons per minute is equivalent to $F(t) = rt$ gallons for all $t \ge 0$.

For work such as painting a house, let H be the function so that $H(t)$ is the number of square feet painted from time 0 to time t (in minutes). The rate of painting is constant, by definition, if there is a fixed number c so that during any time interval from t_1 to t_2 $(0 \le t_1 < t_2)$ the *total number of square feet painted in this time interval* divided by *the length of the time interval* is always equal to c square feet per minute. Thus,

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1} = c$$

for all t_1 and t_2 so that $0 \le t_1 < t_2$. One then proves as before that the rate of painting the house being c square feet per minute is equivalent to $H(t) = ct$ square feet.

We now do a prototypical "rate problem" from the perspective of linear functions. This problem can be done without algebra, so it is the reasoning behind the solution based on algebra that will be the main interest here. In the ensuing discussion, one can also appreciate the importance of being able to transcribe verbal information into symbolic expressions (Section 2).

EXAMPLE Joshua, Li and Manfred are going to paint a house together. It is estimated that, individually, it will take them 18 hours, 15 hours, and 16 hours, respectively, to paint the whole house. Assuming that each person paints at a constant rate, how long will it take them to do it together?

By the constant rate assumption, we know that there are numbers j, ℓ, and m so that the number of square feet that each of Joshua, Li and Manfred paints in t hours are, respectively,

$$\begin{aligned} J(t) &= jt \\ L(t) &= \ell t \\ M(t) &= mt \end{aligned}$$

We can determine each of these constants j, ℓ and m. Let A be the the number of square feet of the house that need painting. Since it take Joshua 18 hours to paint the house, we see that $J(18) = A$. Thus $18j = A$ and $j = \frac{A}{18}$. In like manner, we get $\ell = \frac{A}{15}$ and $m = \frac{A}{16}$. If all three paint together, then in t hours, each of Joshua, Li and Manfred paints, respectively, $\frac{A}{18}t$, $\frac{A}{15}t$, and $\frac{A}{16}t$ square feet. Therefore all three together paint

$$\frac{A}{18}t + \frac{A}{15}t + \frac{A}{16}t = \left(\frac{1}{18} + \frac{1}{15} + \frac{1}{16}\right) At$$

square feet in t hours. If they paint A square feet (i.e., the whole house) in t_0 hours, then

$$\left(\frac{1}{18} + \frac{1}{15} + \frac{1}{16}\right) At_0 = A$$

Multiplying both sides by $\frac{1}{A}$, we get

$$\left(\frac{1}{18} + \frac{1}{15} + \frac{1}{16}\right) t_0 = 1$$

and therefore

$$t_0 \;=\; \frac{1}{\frac{1}{18}+\frac{1}{15}+\frac{1}{16}} \;=\; 4\frac{32}{37} \quad \text{hours.}$$

Note two things. The preceding argument shows that, when the three work together, they paint at a constant rate of $\frac{A}{18}+\frac{A}{15}+\frac{A}{16}$ square feet per hour. A second thing of note is that, if one is used to the use of symbols, then the preceding solution is entirely straightforward and is devoid of subtlety. Compare with any solution that does not use algebra.

Before we leave this section, we introduce a terminology. In a linear function $f(x)=ax+b$, the number b is called the **constant term** of the function. If $b=0$, then we say the function $f(x)=ax$ is **without constant term**. It remains to observe that the linear functions that arise from rate problems are linear functions without constant term. Such functions are the subject of the next subsection.

Proportional reasoning

Consider the following problem (taken from *NCTM Standards*, p. 83):

> *A group of 8 people are going camping for three days and need to carry their own water. They read in a guide book that* 12.5 *liters are needed for a party of 5 persons for 1 day. How much water should they carry?*

This problem is supposed to illustrate the importance of a concept called "proportional reasoning" in middle school mathematics. Yet as it stands, *the problem cannot be done without the additional assumption that each person drinks the same amount of water each day.* What is noteworthy is that this assumption is missing from the problem.

The classical (traditional) approach to such a problem is to *set up a proportion*: if w is the number of liters of water 8 people need per day, then 12.5 liters is to 5 people as w is to 8 people, so

$$\frac{12.5}{5}=\frac{w}{8}$$

Use this equation to solve for w, and the answer to the problem is therefore $3w$ liters. Unfortunately, *there is no logical reasoning that can be used to explain this approach*, at least none that is based on what is given in the problem. In mathematics, if an assumption is not made explicitly, then it cannot be used. This WYSIWYG characteristic cuts both ways: it is a burden on the teachers because one must be careful of what one says, but it is exactly this quality that makes mathematics learnable to students. They don't need to worry about having the know things they have not been told in order to solve problems.

A more recent approach is to isolate the concept of "proportional reasoning" and use it as a way to understand problems of this type. It would seem that this concept does not have a precise meaning. According to the volume *Adding It Up*, p. 241, it is about "understanding the underlying relationships in a proportional situation and working with these relationships", but this begs the question of what a "proportional relAtionship" means. One guess is that "proportional reasoning" means the ability to fluently apply the concept of a linear function without constant term. In order for "proportional reasoning" to be applicable in this sense, one has to begin by *making explicit* the assumption that if $f(x)$ is the amount of water (in liters) x people need each day, then $f(x)$ *is a linear function in x without constant term.* (We proceed to allow x to be an arbitrary number rather than just whole numbers; see the discussion of the profit function in Section 7.) Therefore we may write

$$f(x) = cx$$

for some fixed number c. Since $f(1) = c$, the definition of f means that c is the amount of water one person needs per day. What is given is that $f(5) = 12.5$. Thus $5c = 12.5$, and $c = 2.5$. It follows that $f(8) = 8 \times 2.5 = 20$, and the answer is $3f(8) = 60$ liters.

Incidentally, with the availability of the linear function $f(x) = cx$, we can at last understand what "setting up a proportion" is all about. We know that $\frac{f(x)}{x} = c$ no matter what x may be, so for arbitrary x_1 and x_2, we will always have $\frac{f(x_1)}{x_1} = \frac{f(x_2)}{x_2}$, because both quotients are equal to c. The preceding proportion of $\frac{12.5}{5} = \frac{w}{8}$ is now seen to be nothing other than the statement that

$$\frac{f(5)}{5} = \frac{f(8)}{8}$$

Therefore one can begin to make sense of this venerable device of "setting up a proportion" from the standpoint of a linear function without constant term.

Let us revisit the original problem. It requires students to infer from the given data that every person in the camping trip drinks the same amount of water per day, namely, k liters. This is not reasonable. Indeed, even young kids can see that some people drink lots of water and others very little. How then could one formulate a problem having at least some contact with the "real world" along this line? A more responsible, and mathematically more accurate, formulation of the problem might read something like this:

> A group of 8 people are going camping for three days and need to carry their own water. They read in a guide book that 12.5 liters are needed for a party of 5 persons for 1 day. If one infers from the guide book that these figures provide a rough estimate of the amount of water consumed by a party of any size on any day, roughly how much water should they carry?

It goes without saying that the words, "provide a rough estimate", are nothing but code words for which students need precise and explicit explanations. One expects, therefore, that *students would receive instruction on reasoning of the following kind.* The key words in the problem are those describing the amount of water consumed by a party of *"any size on any day"*. These are words that convey the *generality* of the message of the guide book. If *any* 5 persons drink 12.5 liters per day, then students need to be made aware of the commonly accepted interpretation of this statement to mean that *any* person drinks roughly $\frac{12.5}{5} = 2.5$ liters on any day. Hence, for any positive integer n, n persons would drink, again roughly, $n \times 2.5 = 2.5n$ liters on a given day. Therefore it makes sense to define a function f so that $f(n)$ is roughly the amount of water n persons consume on a given day, and we saw that we had an expression for this f: $f(n) = 2.5n$ liters.

While the preceding discussion succeeds in making more sense of the conclusion that each person drinks 2.5 liters per day, the more important message is the need to make explicit, in one way or another, the underlying linear relationship in problems related to proportional reasoning. *One must set some ground rules so that students* *are not required to* ***guess*** *the linear relationship underlying the problem but that the linear relationship is made clear in some fashion.* We do not wish to enforce the rigid requirement that a linear function be handed to students in each problem of this type; such a requirement would be anti-educational. Rather, there should be universal recognition that linear relationships cannot be taken for granted, and students need explicit instructions as to when they take place.

Consider another problem (taken from *Balanced Assessment for the Mathematics Curriculum*):

> *John's grandfather enjoys knitting. He can knit a scarf 30 inches in 10 hours. He always knits for 2 hours each day.*
>
> 1. *How many inches can he knit in 1 hour?*
> 2. *How many days will it take Grandpa to knit a scarf 30 inches long?*
> 3. *How many inches long will the scarf be at the end of 2 days? Explain how you figured it out.*
> 4. *How many hours will it take Grandpa to knit a scarf 27 inches long? Explain your reasoning.*

Think through and do this problem yourself before comparing your solution with the suggested solution in *Balanced Assessment*, which is as follows:

1. 3 inches, by division: $30 \div 10$.
2. 5 days, by division: $10 \div 2$.
3. 12 inches. Give explanation such as: In one day he knits $3 \times 2 = 6$ inches. In 2 days he knits 2×6 inches.
4. 9 hours. Give explanation such as: To knit 27 inches takes $27 \div 3$ hours.

As a problem in mathematics, with the exception of Part 2, *this problem is not solvable as it stands* because what is given cannot support any kind of logical reasoning for its solution. The missing assumption is that grandfather knits at a constant rate. Without this assumption, *how can one begin to think about such a problem?* Where to begin? This problem is symptomatic of a generic failure of mathematics education at the moment, namely, students are asked to develop conceptual understanding of a given topic and yet the topic is not taught in a manner that meets the minimum requirements of mathematics. For the case at hand, students are asked to develop an understanding of a multiplicative process but they cannot do so because the given assumptions are not sufficient to support the needed reasoning. Think of WYSIWYG again.

What must be made explicit is the fact that, if $g(t)$ is the number of inches grandfather can knit in t hours, then $g(t)$ *is a linear function without constant term.* The given data is that $g(10) = 30$. Thus if we write

$$g(t) = \ell t$$

for some constant ℓ, then from $g(10) = 30$, we get $10\ell = 30$, and $\ell = 3$. The answers to the four parts are then, in succession: $f(1) = 3$, $\frac{10}{2} = 5$, $f(4) = 12$, and finally, the value of t_0 so that $f(t_0) = 27$, which results in $t_0 = 9$.

To drive home the point of the need to understand linear functions without constant term in all such "proportional reasoning" problems, we make one more comment on the proposed solution to Part 4 in *Balanced Assessment*. A little reflection would reveal that this method of solution is merely a repackaging of the proportion

$$\frac{30}{10} = \frac{27}{h}$$

where h is the number of hours it takes grandfather to knit 27 inches. From our vantage point, we know that $\frac{g(t)}{t}$ is always equal to the constant ℓ no matter what t may be. In particular, if h has the same meaning as before and we are given $g(h) = 27$, then we have

$$\frac{g(10)}{10} = \frac{g(h)}{h},$$

and this is exactly the proportion implicitly used in *Balanced Assessment* to solve the problem.

Note that there is a real difference between the camping problem and the knitting problem. The camping problem is a so-called *discrete* problem: there is a natural unit for people, namely, 1 person, and there is no way to further break down this unit. For example, there is no such thing as 0.1 person. On the other hand, the knitting problem is a standard rate problem which involves time, and there is no natural "smallest" unit for time. *Any* time interval can be broken down into smaller time intervals. Thus, if one tries to say constant speed means that the distance traveled is the same in any time interval of one minute's duration, one can ask whether the same is true when measuring the distances traveled in time intervals of a second's duration, or in a millisecond's duration, etc. The lack of a natural unit for measuring time makes the knitting problem harder. This is the difference between the discrete and the continuous.

In one way or another, students must be given the information that a certain function is a linear function without constant term, *because this is part of the basic assumption of constant rate*. See the discussion of the preceding subsection. Mathematics education should not engage in the practice of *not* telling students that there is such a linear function without constant term, and then make judgement about students' conceptual understanding (or lack thereof) simply because they cannot *guess* that there is such a linear function. Mathematics is not concerned with making wild guesses about a hidden agenda.

It is not a matter of what kind of conceptual understanding students should have about proportional reasoning, but rather a matter of being able to clearly and unambiguously articulate, in textbooks and in the classroom, *and in a way that is grade appropriate*, what a linear function is and when a linear function is being assumed in word problems. As we have seen in the preceding two examples, until we make explicit the fact that the functions f and g are linear functions without constant terms, students have no way of doing the problems except by *guessing*. It makes no sense whatsoever to demand that students have the conceptual understanding that those functions are linear. *Without an explicit assumption, there is no reason why they should be linear.* We must never forget the WYSIWYG characteristic of mathematics.

Further discussion of the role of proportional reasoning in the school mathematics curriculum is given at the end of Section 4 in *What is so difficult about the preparation of mathematics teachers?*

http://www.cbmsweb.org/NationalSummit/Plenary_Speakers/wu.htm

EXERCISES

1. Suppose Jessica can do a piece of work in 5 days (think of it as painting a house) and Jessica and Helena together can do it in 3 days. Assuming that each works at a constant rate, in how many days can Helena do the work alone?

2. A man walks from point A to point B at a constant rate. If he walks at the rate of 1 yard per second, then it takes him $5\frac{1}{2}$ minutes more to get to point B than if he walks at the rate of 4 yards per 3 seconds. How far is point A from point B?

3. A freight train runs 6 miles an hour less than a passenger train. It runs 80 miles in the same time that the passenger trains runs 112 miles. Assuming that both trains run at a constant rate, find the rate of each train.

4. A train leaves A for B, 112 miles apart, at 9 am, and one hour later a train leaves B for A; they met at 12 noon. If the second train had started at 9 am and the first at $9:50$ am, they would also have met at noon. Assume that each train runs at a fixed constant speed, find their speeds.

5. A can do a piece of work in $\frac{2}{3}$ as many days as B, and B can do it in $\frac{4}{5}$ as many days as C. Together they can do it in $3\frac{7}{11}$ days. Assuming constant rate of work, in how many days can each each do it alone?

6. Two faucets pour into a tub. The first faucet alone can fill the tub in 18 minutes, and the second faucet alone can fill the tub in 22 minutes. Assume the constancy of the rates of the water flow as usual. Let the first faucet be turned on for 4 minutes before the second faucet is turned on, and t minutes later the tub is filled. What is t?

7. Two people A and B walk straight towards each other at constant speed. A walks $2\frac{1}{2}$ times as fast as B. If they are 2000 feet apart initially, and if they meet after $3\frac{1}{3}$ minutes, how fast does each walk?

8. Joshua, Li and Manfred mow lawns at a constant rate. How long would it take the three of them to mow a lawn if, for the same lawn, it takes Joshua and Li 2 hours to mow it together, Li and Manfred 3 hours to mow it together, and Joshua and Manfred 4 hours to mow it together?

9. The following is the kind of puzzle one finds in magazines for easy reading: "If 3 people can paint 4 houses in 5 hours, how long will it take 7 people to paint 9 houses?" (a) What additional assumptions must you make in order to make the problem solvable? (b) Solve it; your explanation has to be absolutely clear.

10. The following is a favorite problem in middle school mathematics: "On a certain map, the scale indicates 3 centimeters represents the actual distance of 8 miles. Suppose the distance between two cities on this map measures 1.7 centimeters. What is the actual distance between these two cities?" (a) Suppose you are the teacher. What additional explanation must you give your students about this problem before they can solve it? (b) Solve it.

11. Consider this problem: "*If 25 cows consume 400 lb. of hay in a week, how long will 300 lb. of hay last for 12 cows?*" (a) What other assumptions do you need to add to make the problem solvable? (b) Solve it.

9 Linear Inequalities and Their Graphs

So far we have only discussed equations because *school algebra* is primarily about equations. But school algebra also includes everything related to number computations and, as such, inequalities are an integral part of the algebra curriculum because they arise naturally in various mathematical contexts as well as in real life. In this section, we give careful definitions of the basic concepts and prove the most rudimentary facts related to inequalities.

How do inequalities arise in real life?

Real life numerical data tend to appear as inequalities rather than as equalities. After all, it is rare that two measurements are exactly the same, e.g., people's heights, weights, exam scores, etc. Equalities, such as identities, are the exceptions. Thus we have to learn to handle inequalities. For the consideration of linear inequalities, the following problem is a prototypical one.

> [Manufacturing Problem] A video game manufacturer is invited to a game show, and is told that she can bring up to 50 games. She has two games, A and B, and has up to \$6000 to spend on manufacturing costs. Game A costs \$75 to manufacture and will bring in a net profit of §125, while Game B costs \$165 to manufacture but will bring in a net profit of \$185. Assuming that she sells every game she brings, how many games of each kind should she manufacture if she wants to maximize her profit?

It is clear that in this case there is no *equation* to solve, because the answer to the problem consists of a pair of number, a certain number of A games and a certain number of B games, so that this combination brings in a profit bigger than any other possible combination. The emphasis here is on the words *bigger than*, i.e., inequality. One way to understand what is involved in a problem of this nature is to approach it in a naive way and get to see why naivety doesn't pay. For example, a casual glance at the data would suggest that it is more profitable to sell Game A than Game B, in the following sense. Suppose you have \$165. Then if you manufacture one B game, you only make \$185, but if you use the same amount to manufacture two A games (each costing \$75), you'd not only make \$250 $(= 2 \times 1215)$ but would have \$15 to spare.

> *A precise way to think about this is to notice that each Game A brings in a profit that is* $1\frac{2}{3}$ *of its manufacturing cost* (*because* $\frac{125}{75} = 1\frac{2}{3}$), *but each Game B only brings in a profit that is only about* $1\frac{1}{8}$ *of its manufacturing cost* (*because* $\frac{185}{165} = 1\frac{4}{33}$, *which is about* $1\frac{4}{32} = 1\frac{1}{8}$).

One's first impulse is therefore to say that the manufacturer should bring only A games to the show. *But*, \$6000 is good for manufacturing 80 A games as it takes only \$75 to manufacture one A game, whereas she is allowed to bring only 50 games. So if she brings 50 A games, she would be using only $50 \times 75 = 3750$ dollars for manufacturing, thereby wasting \$2250 $(= 6000 - 3750)$ of her \$6000 budget. We see right away that this may be a bad strategy in terms of maximizing profit, and we can easily confirm it, as follows. With 50 A games, she would make $50 \times 125 = 6250$ dollars, but with 48 A games and 2 B games instead (which still add up to 50 games), she would make $(48 \times 125) + (2 \times 185) = 6370$ dollars, and $6370 > 6250$. It may be pointed out that the cost of manufacturing 48 A games and 2 B games is

$$(48 \times 75) + (2 \times 165) = 3930$$

dollars, which would be well within her \$6000 budget.

She can also approach this problem from the opposite end, namely, knowing that each Game B brings a profit of \$185 whereas each Game A brings a mere \$125, she could rightly decide to concentrate on Game B and forget about Game A. The problem now is that she cannot bring 50 B games to the show because her budget of \$6000 wouldn't allow it: the cost of manufacturing 50 B games is $50 \times 165 = 8250$ dollars, which is more than \$6000. Again, this would suggest that bringing all B games to the show is a poor strategy for maximizing profit. For confirmation, notice that a budget of \$6000 can produce at most 36 B games because $\frac{6000}{165} = 36\frac{4}{11}$, and her profit from

36 B games would be $36 = 6660$ dollars, whereas the same budget can equally well produce 34 B games and 5 A games (because $(34 \times 165) + (5 \times 75) = 5985 < 6000$) and these 39 games now bring a profit of

$$(34 \times 185) + (5 \times 125) = 6915$$

dollars, which is \$255 more than \$6660.

It is now clear that there is an inherent push-pull in this problem: bringing too many A games would under-utilize the \$6000 manufacturing budget because of the 50-game quota, and bringing too many B games would under-utilize the 50-game quota because of the \$6000 manufacturing budget. Neither of these options would bring in the maximum profit. Intuitively, the combination of A games and B games that brings in the maximum profit must be a kind of "equilibrium" between the number of A games and the number of B games. What we need to understand in mathematical terms is how to negotiate the push-pull in order to arrive at this equilibrium. The main theme of this chapter is about the mathematical understanding of this push-pull which, as adumbrated above, is grounded on an understanding of inequalities.

> *One more comment before we proceed. While we are trying to promote the need to better understand inequalities, this manufacturing problem may suggest that we forget about inequalities and just get a solution by guess-and-check instead. And why not? Consider the pair of whole numbers* $(m, 50 - m)$, *where* m (*respectively,* $50 - m$) *is the number of A games* (*respectively, B games*) *the manufacturer produces. As* m *runs from 0 to 50, the 51 possible profits of* $\{125m + 185(50 - m)\}$ *dollars exhaust all possibilities and one of these 51 numbers is then the solution. This is correct. However, we use small numbers here* (*50, 6000, 75, etc.*) *only for ease of illustration. Similar problems coming from industry would involve far bigger numbers and far more choices than just two, namely, the choice between A games and B games. In such situations, guess-and-check would be entirely impractical and a general understanding of inequalities becomes mandatory.*

The symbolic transcription

The need for a better understanding of inequalities would be more in evidence if we begin with a transcription of the given data of the Manufacturing Problem into symbolic language. (Review §2 at this point if necessary.)

Suppose the manufacturer produces x A games and y B games. Then the resulting profit $P(x, y)$ is

$$P(x, y) \;=\; 125x + 185y,$$

We want to find values x_0 and y_0 so that the profit $P(x_0, y_0)$ is a maximum. Notice that we are regarding P as a function defined on a pair of whole numbers. Now x and y are not arbitrary but are under **constraints** (a technical terms for "restrictions") that come with the problem. Because the game manufacturer can bring at most 50 games, x and y are constrained by the inequality

$$x + y \leq 50$$

Her manufacturing budget imposes another constraint in that she has at most \$6000 to spend on the production:

$$75x + 165y \leq 6000$$

There are also two other obvious but indispensable constraints: $x \geq 0$ and $y \geq 0$. In summary then, the problem becomes:

> *We want to maximize the profit* $P(x, y) = 125x + 185y$ *among all* x *and* y *which satisfy:*
>
> $$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 50 \\ 75x + 165y \leq 6000 \end{cases}$$

The reader may have noticed that this formulation of the problem is imprecise in at least one respect: we did not specify that x and y have to be integers. Consequently, $P(x, y)$ is now regarded as a function defined on the plane and, as such, we shall refer to P as the **profit function** on the plane. We will deal with this intentional imprecision only when the need for doing so arises later, but for now, we concentrate on constructing a mathematical framework to understand the problem. What we propose to do is to ignore for the time being the fact that x and y are integers and interpret these four inequalities as constraints on a point (x, y) in the xy-plane where x and y are just numbers, integers or otherwise. The collection of all the points (x, y) satisfying these four constraints is a certain region $\mathcal{R}$ in the plane. In a terminology that will be formally introduced in the next subsection, $\mathcal{R}$ is called the **graph of these inequalities** (in the plane). The profit function $P = 125x + 185y$ can now be thought of as a function on this $\mathcal{R}$. The Manufacturing Problem now become a purely mathematical problem *independent of context*:

Among the points in the graph $\mathcal{R}$ of the inequalities

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 50 \\ 75x + 165y \leq 6000 \end{cases}$$

at which point does the profit function $P(x, y) = 125x + 185y$ *assign the maximum value?*

The virtue of this reformulation of the Manufacturing Problem is that it points to clearly defined mathematical tasks:

(i) What does the graph $\mathcal{R}$ of a collection of inequalities look like?

(ii) Can we achieve enough of an understanding of the profit function $P(x, y) = 125x + 185y$ to predict where it might assignment its maximum value in $\mathcal{R}$?

In the following subsections, we will do the necessary spade work for analyzing graphs of inequalities.

Basic facts about inequalities and applications

Our first concern is with the behavior of inequalities with respect to arithmetic operations. This topic has been treated in §6 of Chapter 2 in the Pre-Algebra notes, but we will briefly recall the relevant facts (mostly without proof).

Let x, y, z, w be arbitrary numbers in the following discussion. Recall that the inequality $\boldsymbol{x < y}$ (or written differently as $\boldsymbol{y > x}$) means, by definition, that x is to the left of y on the number line. Recall also the

> **Trichotomy law** *Given two numbers x and y, then one and only one of the three possibilities holds: $x = y$, or $x < y$, or $x > y$.*

There is also a weaker notion of inequality in the form of $\boldsymbol{x \leq y}$, which means x is less than or equal to y, or in symbols, $x < y$ or $a = b$. For emphasis, we may sometimes explicitly refer to "$\leq$" as **weak inequality**. Observe the following simple consequences of the definitions which will be used in subsequent discussions without comment (x, y, z are numbers):

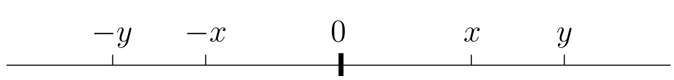
$$x < y, \quad y < z \Longrightarrow x < z$$
$$x \leq y, \quad y \leq x \Longrightarrow x = y$$

The second one uses the Trichotomy law, of course.

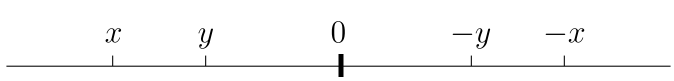
In the following five assertions about inequalities, (A) – (F), we state everything in terms of $<$, but they will all remain valid if "$<$" is replace by "$\leq$" throughout for the same reasons. Moreover, while each of these six inequalities has been proved only for *rational numbers* x, y, z, w in the Pre-Algebra notes, they remain valid when x, y, z, w are arbitrary numbers, thanks to FASM. .

(A) For any $x, y \in \mathbf{Q}$, $x < y \iff -x > -y$.

(Recall that the symbol "$\iff$" means "is equivalent to".) The reason is simply that if $0 < x < y$, then we have

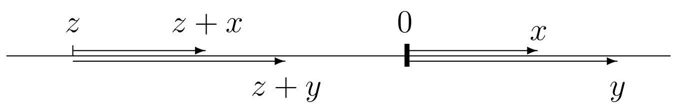


On the other hand, if $x < y < 0$, then we have



(B) For any $x, y, z \in \mathbf{Q}$, $x < y \iff x + z < y + z$.

The intuitive content of (B) can be seen from the following picture, which shows the case where $x > 0$ and $y > 0$ (whether z is positive or negative is irrelevant):

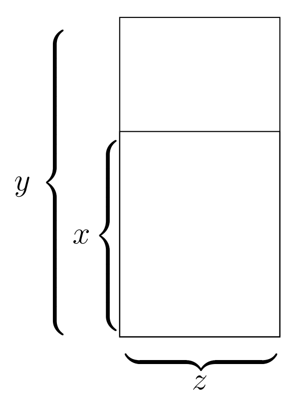


(C) For any $x, y, \in \mathbf{Q}$, $x < y \iff y - x > 0$.

For example, $(-4) < (-3) \implies 4 + (-4) < 4 + (-3)$ by (B), so that $(-4) < (-3) \implies 0 < (-3) - (-4)$. (Recall, the symbol "$\implies$" means "implies".) Conversely, $(-3) - (-4) > 0 \implies (-3) - (-4) + (-4) > 0 + (-4)$, i.e., $(-3) - (-4) > 0 \implies (-3) > (-4)$.

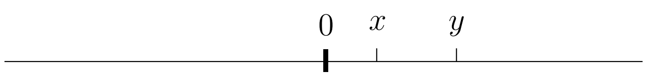
(D) For any $x, y, z \in \mathbf{Q}$, if $z > 0$, then $x < y \iff xz < yz$.

Briefly, if $x < 0 < y$, then $xz < 0$ and $yz > 0$ and there would be nothing to prove. Therefore we need only consider the cases where x and y have the same sign (which, we recall, means they are both positive or both negative). By (A), we may simply let $x, y > 0$. Then x, y, and z are fractions and xz and yz are areas of rectangles with sides of length x, z and y, z, respectively. Since $x < y$, clearly the rectangle corresponding to yz contains the rectangle corresponding to xz and therefore has a greater area. Hence $yz > xz$.

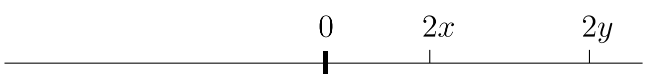


(E) For any $x, y, z \in \mathbf{Q}$, if $z < 0$, then $x < y \iff xz > yz$.

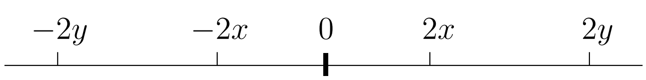
The following is an intuitive way to understand why, if $z < 0$, then multiplying an inequality by z will reverse that inequality. Consider the special case where $0 < x < y$ and, for definiteness, let $z = -2$. So we want to understand why $(-2)y < (-2)x$. We know $(-2)y = -(2y)$ and $(-2)x = -(2x)$. Thus we want to see, intuitively, why $-2y < -2x$. From $0 < x < y$, we get the following picture:



Then the relative positions of $2x$ and $2y$ do not change as each of x and y is pushed further to the right of 0 by the same factor of 2. (Of course, if z were $\frac{1}{2}$, then x and y would be pushed closer to 0 by the same factor of $\frac{1}{2}$, so their relative positions would still be the same.)



If we reflect this picture across 0, we get the following:



We see that $-2y$ is now to the left of $-2x$, so that $-2y < -2x$, as claimed.

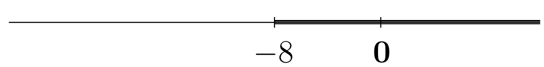
(F) For any $x \in \mathbf{Q}$, $x > 0 \iff \frac{1}{x} > 0$.

This is because $x(\frac{1}{x}) = 1 > 0$, so that x and $\frac{1}{x}$ must be both positive or negative.

We will make frequent use of (A)–(F) in the succeeding discussion, sometimes without any explicit reference. Here is one simple application.

Example 1. Exhibit all the numbers x on the number line which satisfy $(5-2x)+12 \ > \ 4-(3x-5)$.

The set of all these x's is called the **graph of** $\boldsymbol{(5-2x)+12 > 4-(3x-5)}$ **on the number line**, and Example 1 is usually expressed as: *Graph the inequality* $(5-2x)+12 > 4-(3x-5)$ *on the number line.* As in the case of solving linear equations, one simply **isolates the variable** x **in the inequality**, in the sense of transposing all the x's to one side of the inequality by making repeated use of (B). Thus $(5-2x)+12 > 4-(3x-5)$ is equivalent to $(5-2x) > 4-(3x-5)-12$. Now $4-(3x-5)-12 = -8-(3x-5) = -8-3x+5 = -3-3x$. So $(5-2x)+12 > 4-(3x-5)$ is equivalent to $5-2x > -3-3x$, which, by (B) again, is equivalent to $5-2x+3x > -3$, and is equivalent to $-2x+3x > -3-5$, i.e., $x > -8$. Thus we see that $(5-2x)+12 > 4-(3x-5)$ is equivalent to the inequality $x > -8$. This means all the x's which satisfy $(5-2x)+12 > 4-(3x-5)$ are exactly the same as those which satisfy $x > -8$. These x's therefore can be represented by the thickened semi-infinite line segment below.



Things get a bit more interesting when absolute value appears in inequalities. Recall (again, see §6 in Chapter 2 of the Pre-Algebra notes) that for any number x, the **absolute value** $\boldsymbol{|x|}$ of x is by definition:

$$|x| = \text{ the distance of } x \text{ from } 0$$

Thus

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example, $|-\frac{7}{6}| = \frac{7}{6}$, $|1728| = 1728$, and $|-62.9| = 62.9$. Note in particular that

$$|x| \geq 0 \text{ for every number } x \text{, and } |x| = 0 \text{ is equivalent to } x = 0.$$

The other basic properties of absolute value are:

$|xy| \; = \; |x| \cdot |y|$ for all numbers x and y.

(*Triangle inequality*) $|x+y|$ *leq* $|x|+|y|$ for all numbers x and y.

A key point about absolute value is that the inequality $|x| < b$ for numbers x and b ($b > 0$) can be expressed directly in terms of ordinary inequalities. Introduce for this purpose the **double inequality**

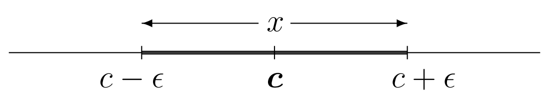
$$\boldsymbol{a \leq b \leq c}$$

for the inequalities:

$$a \leq b, \quad \text{and} \quad b \leq c$$

where a, b, c are numbers. Then we have:

Let x, c be arbitrary numbers and let ϵ be a positive number. Then $|x - c| \leq \epsilon$ is equivalent to the double inequality $c - \epsilon \leq x \leq c + \epsilon$.



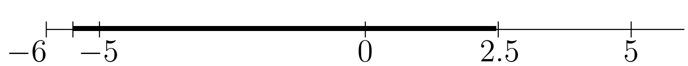
It follows that

For two numbers x and y, $|x - y|$ is the distance between x and y.

The following examples give a good indication of how to handle inequalities which contain absolute values by making use of these facts.

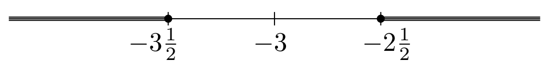
EXAMPLE 2 Graph $|2x+3| - 6 < 2$ on the number line.

The inequality is equivalent to $|2x+3| < 8$, so that it is equivalent to $-8 < 2x + 3 < 8$. The left inequality is $-8 < 2x + 3$, which is equivalent to $-11 < 2x$, which is equivalent to $-5.5 < x$. The right inequality is $2x + 3 < 8$, which is equivalent to $2x < 5$, which is equivalent to $x < 2.5$. Thus $|2x+3| - 6 < 2$ is equivalent to $-5.5 < x < 2.5$, and the graph of $|2x+3| - 6 < 2$ is as shown:



EXAMPLE 3 Graph $|6+2x|>1$ on the number line.

The inequality is the same as $\frac{1}{2}|6+2x| > \frac{1}{2}\cdot 1$, which in turn is the same as $|3+x| > \frac{1}{2}$ (cf. Problem 6 (*iv*) of the following Exercises). Since $|3+x| = |x-(-3)|$, the inequality is therefore the same as $|x-(-3)| > \frac{1}{2}$. This means we have to find all the points x so that its distance from -3 is bigger than $\frac{1}{2}$. From the picture,



we see that the graph is the union of two semi-infinite segments: the segment to the left of $-3\frac{1}{2}$ and including $-3\frac{1}{2}$, and the segment to the right of $-2\frac{1}{2}$ and including $-2\frac{1}{2}$.

Graphs of inequalities in the plane

We can now begin to tackle question (*i*) of the preceding subsection, i.e., what does the graph of a collection of inequalities look like in the plane? First, we need a general definition of the **graph of a linear inequality of two variables** $ax+by \geq c$ (where a, b, c are given constants): it is the set of *all* the points (x', y') in the plane whose coordinates x' and y' satisfy this inequality. i.e., $ax'+by' \geq c$. The **graph** of $ax+by > c$ is defined in like manner. Similarly, one defines the graphs of $ax+by \leq c$ and $ax+by < c$. It is customary in mathematics to denote the graph of an inequality such as $ax+by \geq c$ by the notation $\boldsymbol{\{ax+by \geq c\}}$, and we will use this notation below.

The **graph of a collection of inequalities** is by definition the set of all the points which satisfy each of the inequalities in the collection. It follows that the graph of a collection of inequalities is the intersection of all the graphs of the individual inequalities.

Note that the concept of the *graph of an inequality* or *a collection of inequalities* is usually used without an explicit definition in textbooks and professional development materials. When this happens, no mathematical reasoning is possible in any discussion concerning graphs of linear inequalities.

To describe the graph of an inequality in greater detail, we will show, with the availability of a coordinate system in the plane, that every line has the following *separation property* (see §1 of Chapter 5 in the Pre-Algebra notes).

Theorem 1 (Plane separation) *A line L divides the plane into two* non-empty *subsets, L^+ and L^-, called* **half-planes**. *These half-planes have the following properties*:

> (i) *The plane is the* **disjoint union** *of L, L^+, and L^-*, i.e., the union of L, L^+, and L^- is the whole plane and no two of these sets have any point in common.
>
> (ii) *If two points A and B in the plane belong to the same half-plane, then the line segment AB does not intersect L.*
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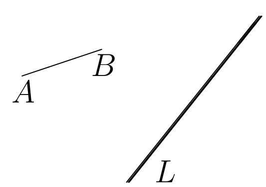
> 
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>
> (iii) *If two points A and B in the plane belong to different half-planes, then the line segment AB must intersect the line L.*
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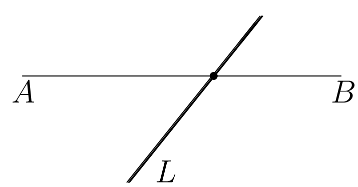
> 
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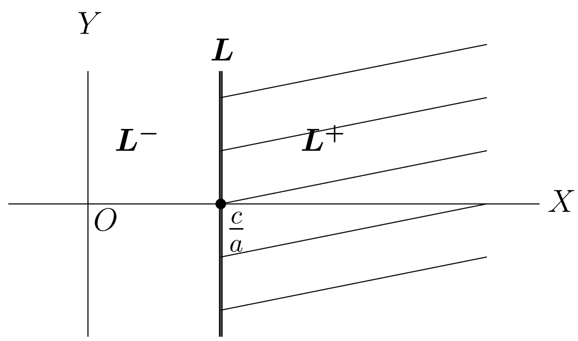
In the course of the proof, we will specify, in terms of coordinates, what L^+ and L^- are.

Proof Let the equation of L be $ax + by = c$, where a, b, c are constants and a, b are not both zero. There are two cases to consider: L is vertical and L is nonvertical.

Suppose L is vertical, then $b = 0$ and L is defined by $x = \frac{c}{a}$. By definition, L^+ is the set of all the points on the "positive" side of L, i.e., all the points (x', y') so that $x' > \frac{c}{a}$ (see shaded region below), and L^- is the set of all the points on the "negative" side of L, i.e., all the points (x', y') so that $x' < \frac{c}{a}$.



Clearly there is no point in common between any two of L^+, L^-, and L, and equally clearly, each is nonempty and their union is the whole plane. This shows property (i) holds. The fact that (ii) and (iii) also hold when L is vertical is simple to prove and will be left as an exercise.

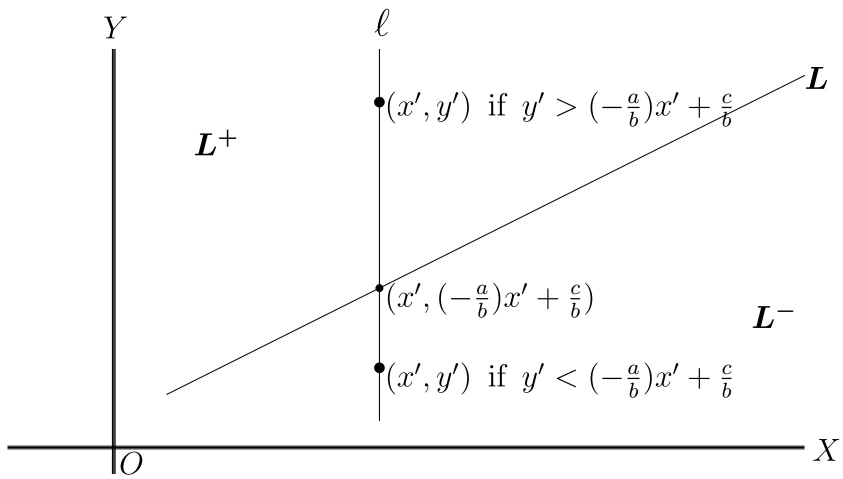
Activity If L is the line defined by $x = 2$, is $(3.1, 1.5)$ in L^+? $(3.1, -1)$? $(-1, 2)$? $(1.5, -\frac{1}{2})$? $(2, 2)$? $(0, 2)$?

Now suppose L is not vertical. Then $b \neq 0$ in $ax + by = c$. We may write the equation of L as

$$y = (-\frac{a}{b})\, x + \frac{c}{b}$$

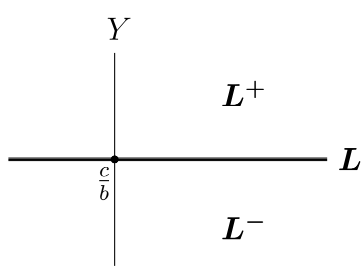
Take any point (x', y') not in L, and let ℓ be the vertical line passing through (x', y'). Because L is not vertical, L and ℓ are not parallel so that they meet at a point on ℓ and therefore the x-coordinate of the point of intersection has to be x'. Because this point also lies on L, the y-coordinate must be $(-\frac{a}{b})x' + \frac{c}{b}$. Thus ℓ and L intersect at the point $(x', (-\frac{a}{b})x' + \frac{c}{b})$.

We now define the half-planes L^+ and L^-, as follows. If the point (x', y') **lies above the point of intersection of ℓ and L**, in the sense that $y' > (-\frac{a}{b})x' + \frac{c}{b}$, then we say (x', y') lies in L^+. If on the other hand, (x', y') **lies below the point of intersection of ℓ and L**, in the sense that $y' < (-\frac{a}{b})x' + \frac{c}{b}$, then we say (x', y') lies in L^-. Since the point (x', y') is arbitrary, our definition of L^+ and L^- is complete.



Again, it is obvious that there is no point in common between any two of L^+, L^-, and L, that each is nonempty, and that their union is the whole plane. We have therefore proved that L^+, L^-, and L satisfy property (i) of the theorem.

We will informally refer to L^+ as the **points above $\boldsymbol{L}$** or the **upper half-plane of $\boldsymbol{L}$**, and L^- as the **points below $\boldsymbol{L^-}$** or the **lower half-plane of $\boldsymbol{L}$**. If the line L is horizontal, $L = \{by = c\}$, then the terminology of "above" and "below" becomes even more appropriate.



Now to prove that L^+, L^-, and L satisfy also properties (ii) and (ii) of the theorem, we will need the following lemma about linear functions of one variable. We say a function f of one variable is **increasing** if for every s and t in the domain of f so that $s < t$, we have $f(s) < f(t)$. Similarly we define f to be **decreasing** if for every s and t in the domain of f so that $s < t$, we have $f(s) > f(t)$.

Lemma 1 *Let $f(x) = ax + b$ be a linear function so that $a \neq 0$. If $a > 0$, then f is increasing, and if $a < 0$, then f is decreasing.*

Proof Suppose $a > 0$, then $s < t$ implies $as < at$, and therefore

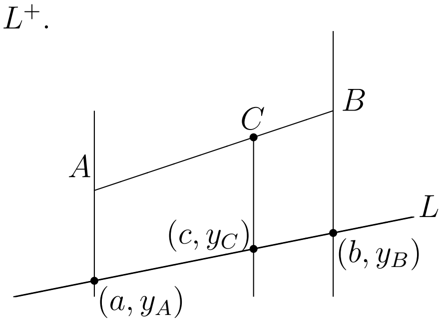
$$f(s) \ = \ as + b \ < \ at + b \ = \ f(t)$$

On the other hand, if $a < 0$, then $s < t$ implies $as > at$, and therefore

$$f(s) \ = \ as + b \ > \ at + b \ = \ f(t)$$

The lemma is proved.

To return to the proof of Theorem 1, we now prove that L^+ satisfies property (ii). Suppose two points A and B lie in L^+. We want to show that any point C lying in the segment AB is also in L^+.



Let $A = (a, a')$ and $B = (b, b')$. By the definition of L^+, if the vertical line passing through A intersects L at (a, y_A), then $a' > y_A$. Similarly, if the vertical line passing through B intersects L at (b, y_A), then $b' > y_B$. If the x-coordinate of C is c, so that $C = (c, c')$ for some number c', and if the vertical line passing through C intersects L at a point (c, y_C), we have to prove that

$$c' \ > \ y_c$$

If the segment AB is vertical, then the whole segment will be above L since its endpoints are. In particular, C would be above L and therefore lies in L^+. We may therefore assume that AB is not vertical and that $a < b$. Let the equation of the line containing AB be

$$y \ = \ mx + k \quad \text{for some constants } m \text{ and } k$$

In this context, observe that both A, B, C lie on this line means

$$ma + k \;=\; a'\,, \quad mb + k \;=\; b'\,, \quad \text{and} \quad mc + k \;=\; c'$$

Recall the equation of L is

$$y \;=\; (-\frac{a}{b})\,x + \frac{c}{b}$$

so that we likewise have

$$(-\frac{a}{b})\,a + \frac{c}{b} \;=\; y_A, \quad (-\frac{a}{b})\,b + \frac{c}{b} \;=\; y_b \quad \text{and} \quad (-\frac{a}{b})\,c + \frac{c}{b} \;=\; y_C$$

Notice that the cruz of the matter is the comparison of the y-coordinate of a point on the line containing AB with the y-coordinate of the point on L "immediately below it". It is therefore natural to consider the linear function which is the difference of the y-coordinates between points on these lines:

$$h(x) \;=\; \left((-\frac{a}{b})\,x + \frac{c}{b}\right) - (mx + k)$$

We are given:

$$h(a) \;=\; y_A - a' \;<\; 0$$

$$h(b) \;=\; y_B - b' \;<\; 0$$

and we want to know whether or not

$$h(c) \;=\; y_C - c' \;<\; 0$$

Rewrite h as

$$h(x) \;=\; (-\frac{a}{b} - m)\,x + (\frac{c}{b} - k)$$

so that it is clearly displayed as a linear function in x. Now if the coefficient $-\frac{a}{b} - m$ of x is 0, then h is a constant function and therefore $h(c) = h(a)$, and since $h(a) < 0$, we would have $h(c) < 0$ and we are done. We may therefore assume that $-\frac{a}{b} - m$ is nonzero. Then Lemma 1 implies h is either increasing or decreasing. But we have $a < c < b$, so either

$$h(a) \;<\; h(c) \;<\; h(b)$$

or

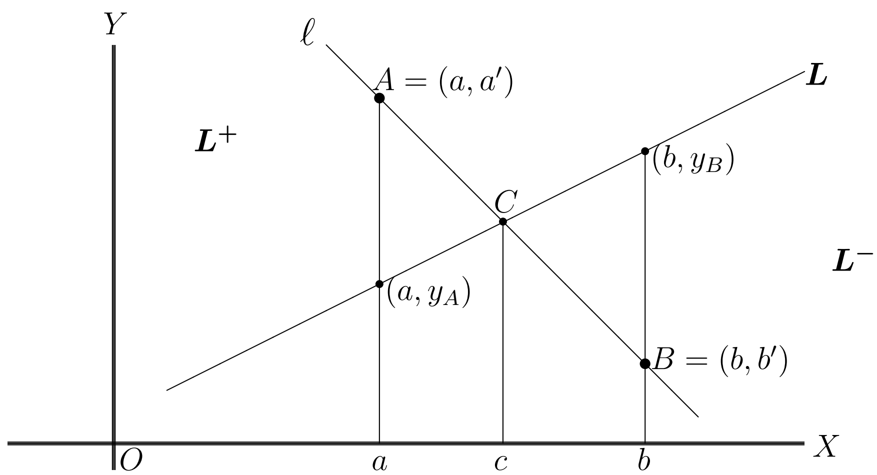
$$h(a) \;>\; h(c) \;>\; h(b)$$

In the first case, the fact that $h(b) < 0$ implies $h(c) < h(b) < 0$. In the second case, the fact that $0 > h(a)$ implies that $0 > h(a) > h(c)$. So $h(c) < 0$ in any case. This completes the proof that L^+ satisfies property (ii).

The proof that L^- also satisfies property (ii) is entirely similar.

It remains to prove that L^+, L^-, and L satisfy property (iii). Let $A = (a, a')$ be a point of L^+ and $B = (b, b')$ be a point of L^-. We have to prove that the segment AB intersects L. As before, let the vertical line through A intersect L at (a, y_A) and let the vertical line through B intersect L at (b, y_B). By the definition of L^+ and L^-, we have

$$a' > y_A \quad \text{and} \quad b' < y_B$$



Now if the segment AB is vertical, then $a = b$ and $(a, y_A) = (b, y_B)$, so that $b' < y_A < a'$. This implies that the segment AB contains the point (a, y_A) of L and we are done. We may therefore assume that AB is not vertical and that $a < b$, and as before, we may write the equation of the line ℓ containing AB as

$$y = mx + k \quad \text{for some constants } m \text{ and } k$$

Recall that the equation of L is

$$y = (-\frac{a}{b})\, x + \frac{c}{b}$$

We again look at the function h defined by the difference of the y-coordinates; precisely, $h(t)$ is the y coordinate of the point of intersection of the vertical line $x = t$

with L minus the y coordinate of point of intersection of the vertical line $x = t$ with ℓ. Thus,

$$h(x) \;=\; \Big((-\frac{a}{b})\,x + \frac{c}{b}\Big) - (mx + k)$$

We have $h(a) < 0 < h(b)$ because

$$h(a) \;=\; y_A - a' \;<\; 0 \;<\; y_B - b' \;=\; h(b)$$

As before, we can also write h as

$$h(x) \;=\; (-\frac{a}{b} - m)\,x + (\frac{c}{b} - k)$$

If the coefficient $-\frac{a}{b} - m$ of x is 0, then h would be a constant function so that $h(a) = h(b)$, which contradicts $h(a) < h(b)$. Thus $-\frac{a}{b} - m \neq 0$ and therefore

$$-\frac{a}{b} \;\neq\; m$$

This means the lines L and ℓ have different slope and are therefore not parallel (Theorem 1 of §6). Let their point of intersection be $C = (c, c')$. Notice that $h(c) = 0$ because the vertical line $x = c$ now intersects both L and ℓ at the same point C. We claim that $a < c < b$ so that C lies on the *segment* AB (rather than just lying on the line ℓ) and the proof of Theorem 1 would be complete. To prove this claim, assume the contrary. Then either $c < a < b$ or $a < b < c$. Consider the first case, $c < a < b$. By Lemma 1, h is either increasing or decreasing, so that either

$$h(c) \;<\; h(a) \;<\; h(b)$$

or

$$h(c) \;>\; h(a) \;>\; h(b)$$

Knowing $h(a) < h(b)$, the only possibility is $h(c) < h(a) < h(b)$. But as we have seen, $h(c) = 0$, so we get $0 < h(a)$, which contradicts $h(a) < 0$. Similarly, we use $h(b) > 0$ to show $a < b < c$ is also impossible. Theorem 1 is proved.

The characterization of the half-planes in Theorem 1 is intuitive enough, but it is somewhat clumsy in applications (such as solving the Manufacturing Problem). The following alternate characterization of the half-planes complements Theorem 1.

For the statement of the theorem, let a line L be given so that its defining equation is $ax + by = c$. We will use ℓ to denote the linear function of two variables $\ell : \{\text{plane}\} \to \{\text{real numbers}\}$ so that for every point (x', y'), $\ell(x', y') = ax' + by'$. In this notation, L is just the level set $\{\ell = c\}$ (see the end of the first subsection in §7). Given any number k, the graph of the inequality $ax + by > k$ can now be denoted more simply as $\boldsymbol{\{\ell > k\}}$, and the graph of the inequality $ax + by < k$ will likewise be denoted by $\boldsymbol{\{\ell < k\}}$. Thus, $\{\ell > k\}$ is the set of all the points (x', y') in the plane so that $ax' + by' > k$.

Theorem 2 *Let a line L be given and let its equation be $ax + by = c$, where a, b, c are constants and at least one of a and b is not equal to 0. With ℓ as the function on the plane so that $\ell(x', y') = ax' + by'$ for all (x', y'),*

> *(i) the sets $\{\ell < c\}$ and $\{\ell > c\}$ are the two half-planes L^+ and L^- defined by L as in Theorem 1.*
>
> *(ii) If $b > 0$, then $L^+ = \{\ell > c\}$ and $L^- = \{\ell < c\}$. If $b < 0$, then $L^+ = \{\ell < c\}$ and $L^- = \{\ell > c\}$.*

Part (ii) of this theorem makes part (i) more precise, but *in practice, there is no need for such a precise statement.* There are, after all, only two half-planes of L, and if it is a matter of deciding which of the two is $\{ax + by > c\}$, it can be done very simply, as follows. Take a point (x', y') of L^+ and we check whether the inequality $ax' + by' > c$ is true or not. If it is, then (x', y') belongs to $\{ax + by > c\}$ by definition, and necessarily, $\{ax + by > c\} = L^+$. If it is not, then (x', y') does not belong to $ax' + by' > c$ so that $\{ax + by > c\}$ cannot be L^+. But since Theorem 2 guarantees that $\{ax + by > c\}$ must be a half-plane, we conclude that $\{ax + by > c\} = L^-$.

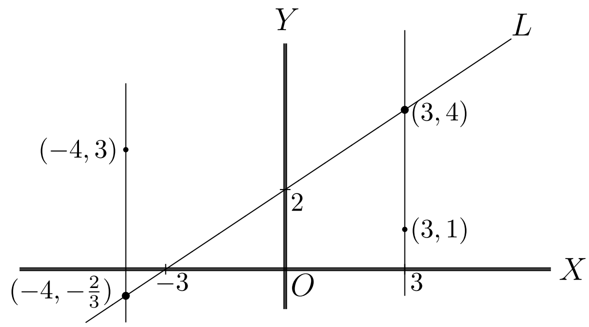
Let us begin with an example.

Example 4. Let L be the line defined by $2x - 3y = -6$.

Now $\ell(x, y) = 2x - 3y$. We want to see whether $\{\ell > -6\}$ is L^+ or L^-. Take a point $(-4, 3)$. One can directly check that

$$\ell(-4, 3) \;=\; 2(-4) - 3(3) \;=\; -17 \;<\; -6$$

and therefore $(-4,3)$ lies in $\{\ell < -6\}$. For the purpose of understanding the *proof* of Theorem 2, however, a more clumsy computation of this fact will be more insightful.



So let the vertical line $x = -4$ passing through $(-4,3)$ intersects L at a point, and this point is easily seen to be $(-4,-\frac{2}{3})$. First of all, since $3 > -\frac{2}{3}$, we see that $(-4,3)$ lies in L^+. Now because the coefficient -3 of y in ℓ is negative, we see that the inequality $3 > -\frac{2}{3}$ leads to

$$(-3)3 \; < \; (-3)(-\frac{2}{3})$$

Hence:

$$\ell(-4,3) \; = \; 2(-4) + (-3)(3) \; < \; 2(-4) + (-3)(-\frac{2}{3}) \; = \; -6$$

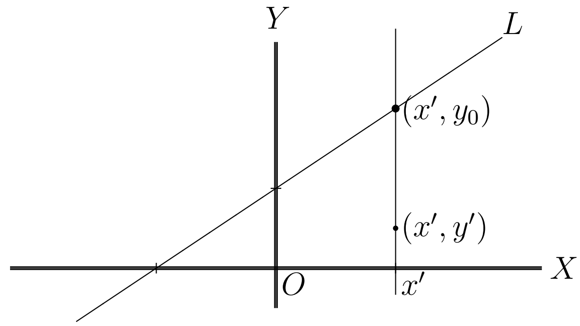
where the last step is because $(-4,-\frac{2}{3})$ lies on L. So $\ell(-4,3) < -6$ after all and $(-4,3)$ belongs to $\{\ell < -6\}$, as before.

Consider the point $(3,1)$. The vertical line $x = 3$ passing through it intersects L at $(3,4)$ and since $1 < 4$, we see that $(3,1)$ lies in L^-. We now perform the same clumsy computation to verify that $(3,1)$ belongs to $\{\ell > -6\}$. Indeed, $1 < 4$ implies $(-3)1 > (-3)4$, so that

$$\ell(3,1) \; = \; 2(3) + (-3)(1) \; > \; 2(3) + (-3)4 \; = \; -6$$

where again, the last step is the result of $(3,4)$ being on L. Thus $\ell(3,1) > -6$, as desired.

Proof of Theorem 2 Part (*ii*) implies part (*i*), so we will prove part (*ii*). For definiteness, let us assume $b < 0$ in $ax + by = c$. Take a point (x', y') and let the vertical line $x = x'$ passing through (x', y') intersect L at (x', y_0). Suppose (x', y') is in L^-, so that $y' < y_0$. Because $b < 0$, we get $b\,y' \; > \; b\,y_0$.



Therefore, (x', y') belongs to $\{\ell > c\}$ because

$$\ell(x', y') \;=\; ax' + by' \;>\; ax' + by_0 \;=\; \ell(x', y_0) \;=\; c$$

where the last step is because (x', y_0) lies on L. Similarly, if (x', y') is in L^+, then $\ell(x', y') < c$ so that (x', y') belongs to $\{\ell < c\}$. Of course for any point (x', y') on L, $\ell(x', y') = c$.

Since for any point (x', y') in the plane, $\ell(x', y')$ is either equal to c, or $> c$, or $< c$, we see that (x', y') is either on L, or in $\{\ell > c\}$, or in $\{\ell < c\}$. Consequently, L^- is not just a part of $\{\ell > c\}$ but is equal to $\{\ell > c\}$. Likewise, $L^+ = \{\ell < c\}$.

The proof of the theorem in case $b > 0$ is similar. Theorem 2 is proved.

The occasion will arise when more precision regarding half-planes is needed, for the following reason. The half-planes L^+ and L^- do not include L, but we shall see presently that there is a need to also consider half-planes together with L itself. We will therefore refer to $L^+ \cup L$ and $L^- \cup L$ as the two **closed half-planes** of L.[36] The two closed half-planes are not disjoint as they have L in common. If there is any fear of confusion, we will refer to L^+ and L^- as the two **open half-planes** of L for emphasis.

Theorem 2 allows us to see why the concept of a closed half-plane is relevant. Indeed, suppose we want to know the graph of the *weak* inequality $ax + by \leq c$. By Theorem 2, we know $\{\ell < c\}$ is one of L^+ and L^-. Let us say for definiteness that $\{\ell < c\} = L^+$. It follows that the graph of $ax + by \leq c$ is just L^+ together with the graph of $ax + by = c$, which is L. Therefore, in self-explanatory notation, $\boldsymbol{\{\ell \leq c\}}$

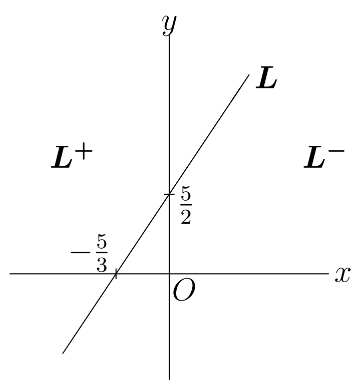
[36]The terminology of a *closed* half-plane is given in §1 of Chapter 5 in the Per-Algebra notes.

is simply the closed half-plane $L^+ \cup L$.

The following examples illustrate how to make use of Theorem 2.

Example 5. Graph $3x - 2y > -5$ in the plane.

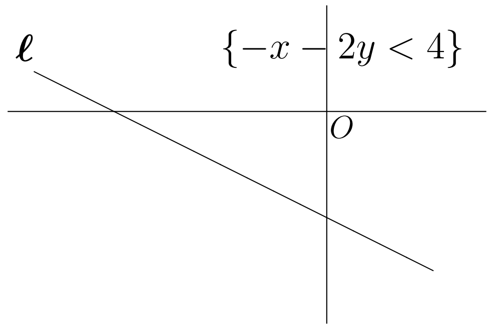
The line L defined by $3x - 2y = -5$ is shown below.



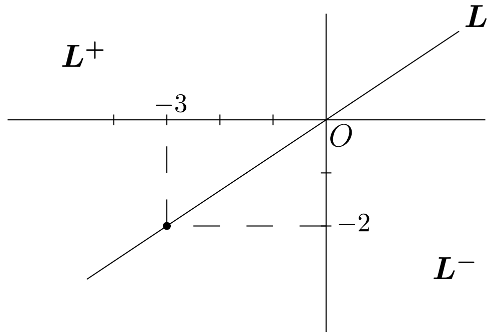
The coefficient of y being -2 and therefore negative, Theorem 2(ii) says the graph of $3x - 2y > -5$ is L^-. This fact can also be deduced by the more mundane method of checking which half-plane O belongs. Visibly, $(0, 0)$ belongs to L^-, but it also belongs to $\{3x - 2y > -5\}$ as $0 > -5$. Since Theorem 2 says $\{3x - 2y < -5\}$ must be either L^+ or L^-, we know that it is L^-.

Example 6. Find the graph of the pair of inequalities $-x - 2y < 4$ and $-2x + 3y > 0$.

This example asks for the intersection of the graphs of the individual inequalities. Let ℓ be the line $-x - 2y = 4$. Now $(0, 0)$ belongs to $-x - 2y < 4$ because $0 < 4$, so the graph of $-x - 2y < 4$ is the upper half-plane ℓ^+ of ℓ, as shown below.

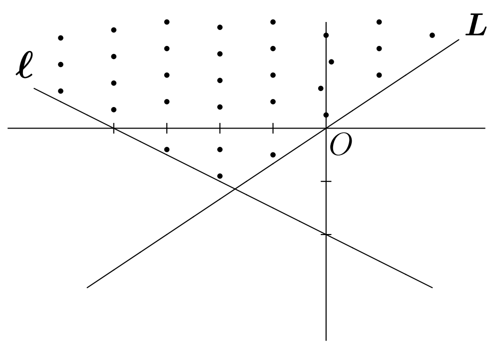


It remains to determine the graph of $-2x + 3y > 0$. Let L be the line defined by $-2x + 3y = 0$. Then the picture is the following:



Since the coefficient of y in $-2x + 3y$ is 3 and $3 > 0$, Theorem 2(ii) implies that $\{-2x + 3y > 0\}$ is also the upper half-plane L^+ of L. Another way is to check that the point $(-3, 0)$ is in L^+ because $(-3, -2)$ is on L, and it is also in $\{-2x + 3y > 0\}$ because $(-2)(-3) + 3 \cdot 0 > 0$. Thus by Theorem 1, $\{-2x + 3y > 0\} = L^+$.

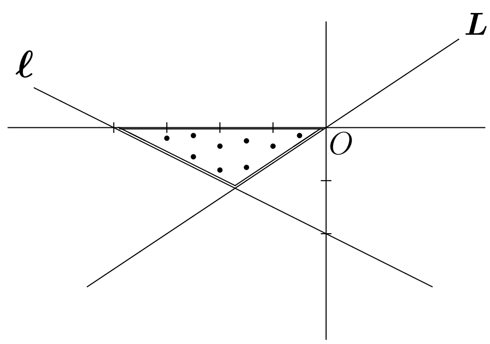
The graph of the pair $-x - 2y < 4$ and $-2x + 3y > 0$ is therefore the intersection $L^+ \cap \ell^+$, which is the dotted region below *without* the two semi-infinite boundary line segments.



The graph in Example 6 is an "infinite region" in a sense that is self-explanatory (although "infinite" in this context can be precisely described in advanced mathematics). In applications such as the Manufacturing Problem of this section, however, the graph would tend to be a polygon with the edges included. As an illustration of such a polygon, let us see how we can obtain one by a slight elaboration on Example 6. The graph of the two weak inequalities $-x - 2y \le 4$ and $-2x + 3y \ge 0$ is the intersection of the closed half-planes $L^- \cup L$ and $\ell^+ \cup \ell$, and is the same dotted region as above *plus* the two semi-infinite boundary line segments. Call this region $\mathcal{S}$. Now consider not just the graph of this pair but the graph of this pair plus a third inequality, namely, the graph of the three weak inequalities

$$-x - 2y \le 4, \quad -2x + 3y \ge 0, \quad \text{and} \quad y \le 0$$

is the intersection of $\mathcal{S}$ with the closed lower half-plane of the x-axis, and is therefore the following dotted triangular region together with the three edges.

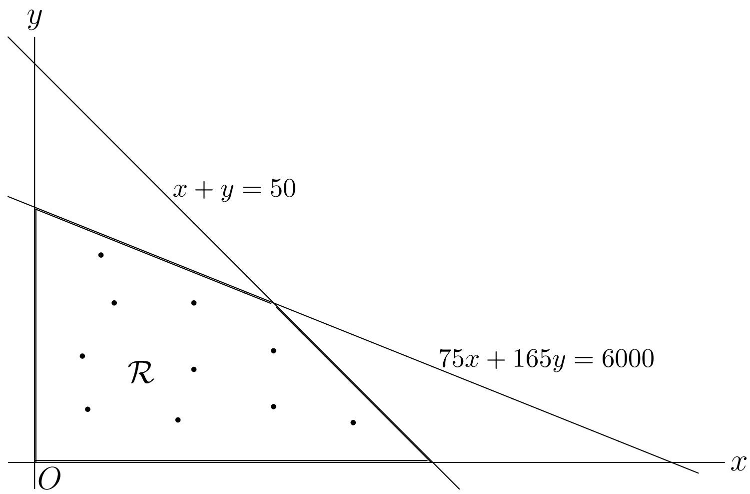


Behavior of linear functions in the plane

At this point, we have all the needed information to tackle task (*ii*) above, i.e., to understand the behavior of the profit function $P(x, y) = 125x + 185y$ of the Manufacturing Problem. Recall that we have introduced the region $\mathcal{R}$ as the graph of the following four inequalities:

$$\begin{cases} x \ge 0 \\ y \ge 0 \\ x + y \le 50 \\ 75x + 165y \le 6000 \end{cases}$$

$\mathcal{R}$ is called the **feasibility region** of the problem. It is the dotted region below:



Our reformulation of the Manufacturing Problem now asks at which point (x_0, y_0) of $\mathcal{R}$ the profit function $P(x, y) = 125x + 185y$ **attains** (or **achieves**) **its maximum in** $\mathcal{R}$, in the sense that if (x', y') is any other point of $\mathcal{R}$, then the profit P at (x_0, y_0) (which is $125x_0{+}185y_0$) is at least as big as the profit at (x', y') (which is $125x'{+}185y'$). In other words,

$$125x_0 + 185y_0 \geq 125x' + 185y'$$

for *any* point (x', y') in $\mathcal{R}$. The point (x_0, y_0) is called a **maximum point** of the profit function. We now pause to tie up a loose end by pointing out that, in the context of the Manufacturing Problem, it makes sense to talk about the profit at a point (x, y) (i.e., $125x + 185y$) only if x and y are both *whole numbers.* After all, there is no such thing as the profit derived from 1.27 A games and $\frac{13}{7}$ B games. However, as a *mathematical* problem, it makes perfect sense to ask for the value of $125x + 185y$ regardless of whether or not they are whole numbers. What we need to be careful about is to make sure that, *at the end, we come up with a solution* (x_0, y_0) *of the Manufacturing Problem where the coordinates* x_0 *and* y_0 *are whole numbers.* In the meantime, by freeing x and y from the constraint of being only whole numbers, we can look at the profit function $P(x, y) = 125x + 185y$ as an assignment to any point (x, y) in the region $\mathcal{R}$ a number $125x + 185y$. Such an assignment is something that mathematics can handle with ease. While it is true that the profit function $P(x, y) = 125x{+}185y$ for whole numbers x and y is something concrete and down-to-earth and the assignment to each point (x, y) of $\mathcal{R}$ the number $125x + 185y$

becomes something more abstract, part of learning algebra includes learning when to take an abstract approach to a problem. The extension of the concept of "profit" to include $125x + 185y$ for *any* numbers x and y is a good example of the needed abstraction for the solution of many problems in algebra, including the Manufacturing Problem.

To continue the discussion, we will see that there is no advantage to gain by studying the specific profit function $P = 125x + 185y$ instead of something more general. Accordingly, we let a, b, e be three fixed numbers, with $a \neq 0$ or $b \neq 0$, and consider the linear function of two variables $P(x, y) = ax + by + e$. The concepts of a linear function attaining (or achieving) a maximum in $\mathcal{R}$, and a point in $\mathcal{R}$ being a maximum point of P in $\mathcal{R}$ would then be defined similarly. For example, we say $P(x, y) = ax + by + e$ **achieves a maximum at a point** (x_0, y_0) **in** $\mathcal{R}$ if for any point (x', y') of $\mathcal{R}$, we always have $ax_0 + by_0 + e \geq ax' + by' + e$. And of course, we would say (x_0, y_0) is a **maximum point of** P **in** $\mathcal{R}$. The corresponding concepts for a **minimum** in place of a maximum is similarly defined.

With such a linear function $P(x, y) = ax+by+e$ understood, take a point (x_0, y_0) and let $P(x_0, y_0) = k$, i.e., $ax_0 + by_0 + e = k$. Recall that the level set $\{P = k\}$ is a line L defined by the equation $ax + by = (k - e)$, and (x_0, y_0) lies on L. We can now rephrase Theorem 2(ii) in the language of the level set of a linear function $P(x, y) = ax + by + e$.

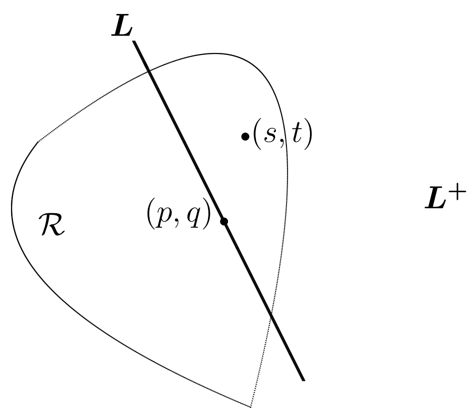
Theorem 3 *Given a linear function $P(x, y) = ax + by + e$ with $b \neq 0$. Let (x_0, y_0) be an arbitrary point in the plane and let P assign to it the value k. Let L be the level set $\{P = k\}$ containing (x_0, y_0). Then the half-plane L^+ consists of all the points to which P assigns numbers which are $> k$ if $b > 0$, and consists of all the points to which P assigns numbers which are $< k$ if $b < 0$. The half-plane L^- consists of all the points to which P assigns numbers which are $< k$ if $b > 0$, and consists of all the points to which P assigns numbers which are < 0 if $b > 0$.*

Theorem 3 immediately leads to our first conclusion concerning where a linear function $P(x, y) = ax + by + e$ can achieve its maximum in a region $\mathcal{R}$.

> With $b > 0$, if a linear function $P(x, y) = ax+by+e$ achieve its maximum in $\mathcal{R}$ at a point (p, q) of $\mathcal{R}$, then $\mathcal{R}$ lies completely in the closed half-plane

$L^{-} \cup L$ of the level set L of P containing (p, q). If on the other hand P achieves its minimum in $\mathcal{R}$ at (p, q), then $\mathcal{R}$ lies completely in the closed half-plane $L^{+} \cup L$.

This is because, by Theorem 3, if $\mathcal{R}$ contains a point (s, t) of L^{+}, then P assigns to (s, t) a value exceeding the value it assigns to (p, q), and therefore P does not achieve its maximum at (p, q), as shown:



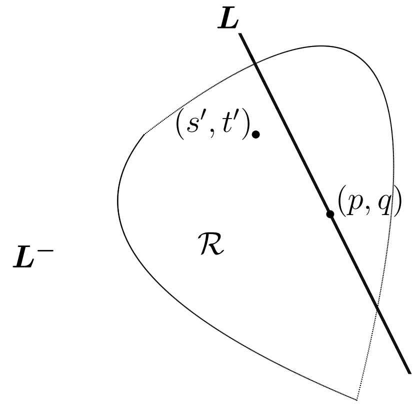
The statement about minimum is proved in the same way.

In a similar manner, we have:

If $b < 0$, if a linear function $P(x, y) = ax + by + e$ achieve its maximum in $\mathcal{R}$ at a point (p, q) of $\mathcal{R}$, then $\mathcal{R}$ lies completely in the closed half-plane $L^{+} \cup L$ of the level set L of P containing (p, q). If on the other hand P achieves its minimum at (p, q), then $\mathcal{R}$ lies completely in the closed half-plane $L^{-} \cup L$.

It is instructive to give a proof of this assertion which makes use of the assertion we have just proved for the case $b > 0$. Because the coefficient b is now negative rather than positive, we consider instead the linear function $Q(x, y) = -(ax + by + e)$. Then $Q(x, y) = (-a)x + (-b)y - e$, and the coefficient of y in Q is now positive, and we can apply to Q what we have learned in case $b > 0$. Observe that if P achieves its maximum in $\mathcal{R}$ at a certain point (p, q), then Q achieves its minimum in $\mathcal{R}$ at this (p, q) because: $ap + bq + e \geq ax' + by' + e$ for all (x', y') in $\mathcal{R}$ means

$-(ap+bq+e) \leq -(ax'+by'+e)$ for all (x',y') in $\mathcal{R}$, by assertion (ϵ) on inequalities, which is then precisely the statement that Q achieves its minimum in $\mathcal{R}$ at (p,q). Now suppose P achieves its maximum in $\mathcal{R}$ at (p,q), and yet $\mathcal{R}$ contains a point (s',t') of L^-, as shown. We want to show that this is impossible.



Let P assign to (p,q) the value k as usual. Then L is the level set $\{Q=-k\}$ of Q. Therefore, we know that L^- consists of all the points to which Q assigns values $< -k$.[37] In particular then, Q assigns to (s',t') a value smaller than $-k$, i.e., $-(as'+bt'+e) < -k$. By assertion (ϵ) on inequalities again, we have $as'+bt'+e > k$, and this contradicts the given data that P achieves its maximum in $\mathcal{R}$ at (p,q). Hence we have reached the desired conclusion that no such (s',t') exists. The statement about minimum is proved in the same way.

We can summarize the foregoing discussion as follows. Let us call a point (p,q) an **interior point** of $\mathcal{R}$ if (p,q) is in $\mathcal{R}$ and is disjoint from the boundary of $\mathcal{R}$. Then:

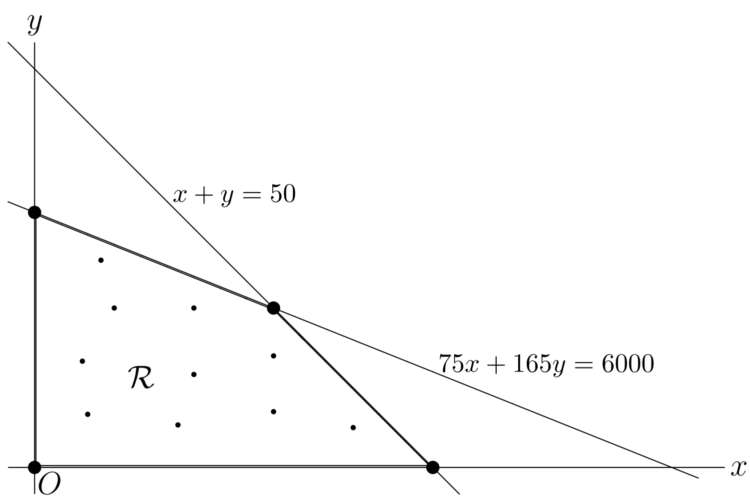
Theorem 4 *If a linear function $P(x,y)=ax+by+e$ achieves a maximum or a minimum at a point (p,q) of $\mathcal{R}$, then $\mathcal{R}$ lies completely in one of the two closed half-planes of L, where L is the line which is the level set of P passing through (p,q). In particular, a linear function $P(x,y)=ax+by+e$ cannot achieve a maximum or a minimum at an interior point of a region $\mathcal{R}$.*

In view of what we have already discussed, only two minor comments on this theorem are called for. First, although we have always assumed in the above arguments

[37]It may help for the understanding of this statement to recall that the definition of L^- depends only on the fact that L is a line and has nothing to do amy linear functions P or Q.

that $b > 0$ or $b < 0$, the case of $b = 0$ in $P(x, y) = ax + by + e$ (which is not excluded in Theorem 3) can be easily taken care of by noting that resulting level set L passing through a maximum point (p, q) would be vertical. Then we simply replace the erstwhile L^+ and L^- by the "positive" and "negative" half-planes of L, respectively, in all the arguments. Second, the reason for the last statement of Theorem 4 is that if (p, q) is an interior point of a region $\mathcal{R}$, then any line L passing through (p, q) would have the property that both of its open half-planes contain points of $\mathcal{R}$.

For regions which are intersections of closed half-planes such as the one arising from the Manufacturing Problem, Theorem 4 has a noteworthy refinement. What is important for our purpose about intersections of closed half-planes is that their boundary consists of straight line segments and that the boundary is part of the region.[38] The points of intersection of these boundary line segments are called **corners**. The corners of the region arising from the Manufacturing Problem are indicated with a dot in the picture below.



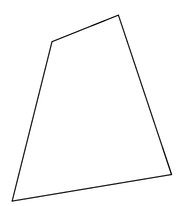
The refinement in question is:

[38]If the region is the intersection of open half-planes, the boundary of the region would not be part of the region. For example, consider $\{x > 0\} \cap \{y > 0\}$. This region is the first quadrant but without the two (positive) coordinate axes, so that this region does not contain its boundary.

Theorem 5 *Let a region $\mathcal{R}$ be an intersection of closed half-planes. If a linear function $P(x, y) = ax + by + e$ achieves its maximum or minimum in $\mathcal{R}$, then it does so at a corner.*

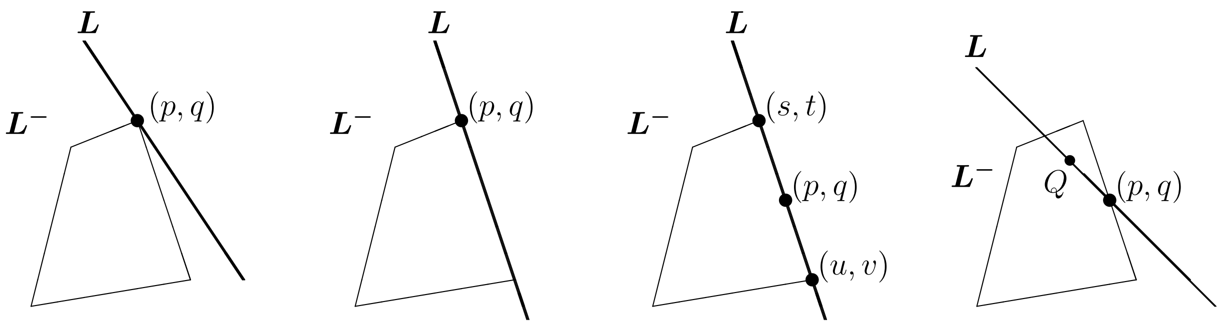
Please note what the theorem does *not* say. It does not say that P *only* achieves its maximum (resp., minimum) at a corner; it merely asserts that if P does achieve a maximum (resp., minimum) in $\mathcal{R}$ at some point, then it already achieves the maximum (resp., minimum) at one of the corners.

Proof We will deal with the case of a maximum; the case of a minimum is entirely similar. Consider a typical such region:



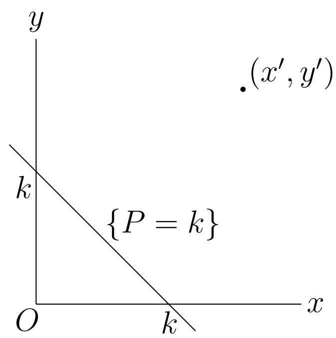
We will prove the theorem for the case $b > 0$. The case of $b < 0$ is entirely similar and will be left as an exercise. The case of $a = 0$ is simple and can be left to the reader.

Suppose P achieves its maximum at a point (p, q) of $\mathcal{R}$. Let L be the level set $\{P = k\}$ passing through (p, q). By Theorem 3, $\mathcal{R}$ must *lie in the closed half-plane $L^- \cup L$.* Because the boundary of $\mathcal{R}$ consists of line segments, there can only be four posibilities for this to happen: either (p, q) is already a corner and L contains no point of $\mathcal{R}$ except this corner, or (p, q) is a corner but L also contains an edge of the boundary, or (p, q) is not a corner but lies inside an edge of the boundary so that either L contains the edge in which (p, q) sits, or L does not contain this edge, as shown.



If (p, q) is already a corner, as in the first two pictures from the left, there would be nothing to prove. If (p, q) is not a corner, as in the two cases on the right, we want to eliminate the case on the extreme right, i.e., the case where L does not contain the edge in which (p, q) sits. In this case, L would contain an interior point Q of $\mathcal{R}$, as the picture shows. Since L is a level set of P, P assigns the same value to Q and (p, q), so that P attains a maximum in the region $\mathcal{R}$ at Q. Because Q is an interior point of $\mathcal{R}$, this is a contradiction to Theorem 4. Therefore, the scenario on the extreme right does not occur and L must contain the whole edge to which (p, q) belongs, as in the second picture from the right. In that event, the fact that $L = \{P = k\}$ means P assigns to the corners (s, t) and (u, v) of the edge containing (p, q) (and contained in L) the same value k, and therefore P also achieves its maximum in $\mathcal{R}$ at the corner (s, t) (or for that matter, (u, v)). This proves Theorem 5.

One has to pay attention to the careful wording of Theorem 5: it says that *if* P achieves a maximum in $\mathcal{R}$, then it does so at a corner, but it does not say that in any region $\mathcal{R}$ that is the intersection of closed half-planes, a linear function will always achieve a maximum in $\mathcal{R}$ at a corner. Indeed, this assertion is not even true. Consider for example the linear function $P(x, y) = x + y$ in the region $\mathcal{R}_0$ which is the graph of the pair of inequalities $x \geq 0$ and $y \geq 0$, i.e., the first quadrant including the positive coordinate axes, as shown:



Clearly the value P assigns to a point (x', y') increases without bound as (x', y') moves up in the direction of the upper right corner, and so P cannot achieve a maxiumum in $\mathcal{R}_0$. However, if $\mathcal{R}$ is a *finite* region (in the sense of being contained inside some circle) which is the intersection of closed half-planes, then any linear function will always achieve a maximum in $\mathcal{R}$. While this fact is fairly believable, its proof requires some advanced mathematics.

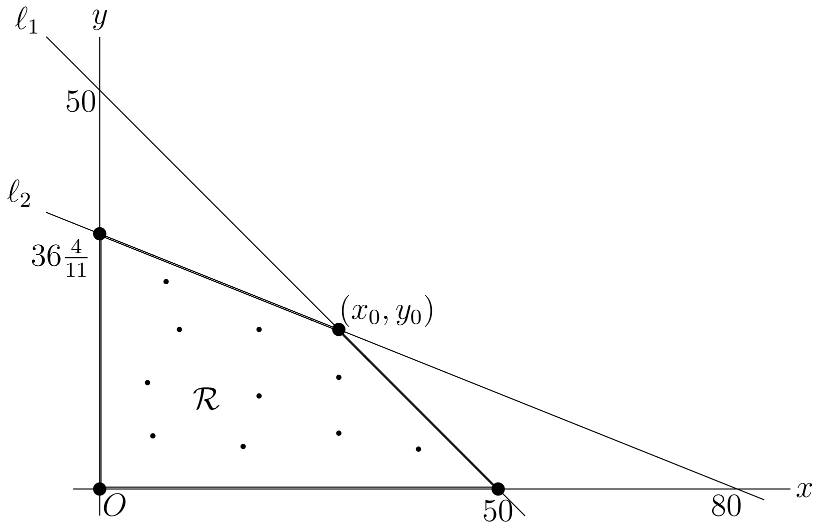
Solution of the Manufacturing Problem

The solution of the Manufacturing Problem is now relatively simple.

We are looking at the profit function $P(x, y) = 125x + 185y$ on the graph $\mathcal{R}$ of the following weak inequalities:

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x + y \leq 50 \\ 75x + 165y \leq 6000 \end{cases}$$

We have already seen a picture of $\mathcal{R}$:



$\mathcal{R}$ is a quadrilateral together with its interior bounded by the positive x- and y-axes, the line ℓ_1 defined by $x+y=50$ and the line ℓ_2 defined by $75x+165y=6000$. The four corners of $\mathcal{R}$ are

$$(0,0), \quad (50,0), \quad (x_0,y_0), \quad (0,36\frac{4}{11})$$

where every point is there for the obvious reason except for (x_0,y_0), which is the point of intersection of ℓ_1 and ℓ_2 and is therefore the solution (by Section 6) of the simultaneous system:

$$\begin{cases} x + y = 50 \\ 75x + 165y = 6000 \end{cases}$$

Solving this system in the standard way (but note that a simplification can be achieved by reducing the second equation to $x + 2.2\,y = 80$, we get $x_0 = y_0 = 25$. So $(x_0, y_0) = (25, 25)$.

Now the profit in Manufacturing Problem must have a maximum; for example, try out all possible profits with n A games and $50-n$ B games for $n = 0, 1, 2, \ldots, 50$ and the maximum is among these. Theorem 5 therefore tells us where to look for the maximum point: check the profit at each of the four corners above. The profits at $(0,0)$, $(50,0)$, $(25,25)$, $(0,36\frac{4}{11})$ are, respectively, 0, 6250, 7750, and $6725\frac{3}{11}$. Thus the maximum point of the profit function is $(25,25)$, and the solution is: *manufacture exactly 25 A games and 25 B games to make the maximum profit of \$7750.*

It remains to round off the preceding discussion by tying up a loose end. It is obvious that we could have left out any consideration of the corner $(0, 0)$ in search of the *maximum* possible profit. But the corner $(0, 36\frac{4}{11})$ had to be accounted for even if it makes no sense to talk about $36\frac{4}{11}$ B games. This is because there is always the possibility that if $(0, 36\frac{4}{11})$ is a maximum point, then one of the nearby points (*inside the feasibility region*) with whole number coordinates, i.e., $(0, 36)$ and $(1, 35)$, may be a maximum point among those points with whole number coordinates. Thus, it could be that 36 B games or 1 A game and 35 B games produces the maximum profit. In our case, this did not happen because even the profit at $(0, 36\frac{4}{11})$ is not large enough. In other situations, one would need to check such a possibility. (See Problem 10 in the following Exercises.)

To give some intuitive content to this way of approaching the problem, consider the solution of 25 each of A and B games. Why did we not check to see if the \$6000 manufacturing budget is enough to cover the production of 25 A games and 25 B games? This is because $(25, 25)$ is in the feasibility region of the problem, so that in particular it is in the half-plane of $75x + 165y \leq 6000$. Therefore, with $x = 25$ and $y = 25$, $(75 \times 25) + (165 \times 25) \leq 6000$, which is precisely the statement about the manufacturing cost of these 50 games being at most \$6000.

EXERCISES

1. (a) Graph the inequality $\frac{2}{3}x - (2 + 7x) \geq (6 + x) - (1 - \frac{1}{2}x)$ on the number line. (b) Graph the inequality $\frac{2}{5} - \frac{1}{2}x \geq \frac{1}{5}x + \frac{1}{6}$ on the number line.

2. Graph the following inequalities in the plane:

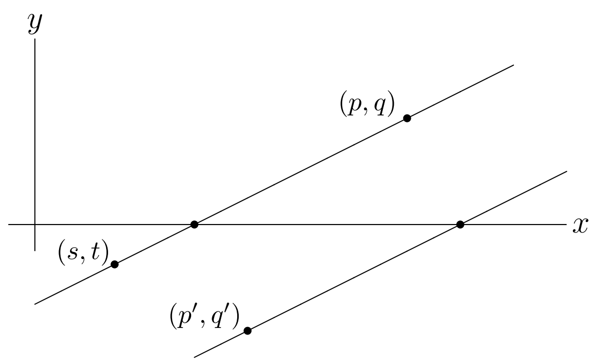
$$\begin{cases} \frac{5}{2}x + \frac{3}{4}y & \leq & 2 \\ -2x + 3y & \leq & 12 \\ \frac{1}{3}x - y & \leq & 5 \end{cases}$$

3. In the proof of Theorem 1: (a) Complete the proof when L is a vertical line, i.e., show that the L^+ and L^- so defined satisfy properties (ii) and (iii). (b) Write out a complete proof of the fact that L^- satisfies property (ii).

4. Give a proof of the half of Theorem 2 concerning $b > 0$.

5. Give a proof of the case of minimum in Theorem 4.

6. Given a linear function of two variables $P(x, y) = ax + by + e$ with $a > 0$ and $b \neq 0$. Suppose we have two lines both with slope $-\frac{a}{b}$ as shown:



(a) Compare the values that P assigns to (s, t) and (p, q). (b) Compare the values that P assigns to (p, q) and (p', q').

7. Let the linear function of two variables ℓ be defined by $\ell(x, y) = 5x - y + 7$. Sketch $\{\ell = 1\}$, $\{\ell = 10\}$, $\{\ell < 1\}$, and $\{\ell < 10\}$. What is the relationship between $\{\ell < 1\}$, and $\{\ell < 10\}$?

8. Let $\mathcal{S}$ be the region defined by the inequalities

$$\begin{cases} 3x - 3y \leq 5 \\ 2x + y \leq -4 \end{cases}$$

Does the linear function $P(x, y) = \frac{2}{3}x + y$ achieve a maximum in $\mathcal{S}$? Does it achieve a minimum in $\mathcal{S}$? Explain.

9. Let $\mathcal{R}$ be the graph of the following inequalities:

$$\begin{cases} \frac{1}{2}x & + & \frac{4}{5}y & \geq & 4 \\ -2x & + & 5y & \leq & 10 \\ \frac{1}{3}x & - & \frac{1}{4}y & \leq & 2 \end{cases}$$

Where would the linear function $2x + \frac{1}{3}y - 4$ attain its maximum in $\mathcal{R}$? Its minimum? What are the maximum and minimum values of the function?

10. Find the point (x_0, y_0) at which the linear function $10x + 3y$ achieves a maximum in the graph of the inequalities:

$$\begin{cases} x \geq 0 \\ x \leq 10 \\ y \geq 5 \\ 0.5x + y \leq 20 \end{cases}$$

Where does $10x + 3y$ achieve a minimum in this graph?

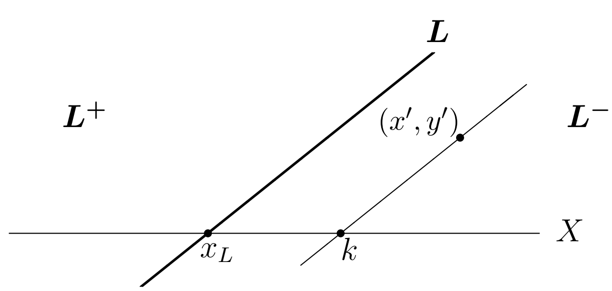
11. The nutritional values of a basic unit of two food items are tabulated below:

	calorie	vitamin C (i.u.)	protein (mg)
A	156	50	30
B	116	75	80

A mountain climber wants to bring enough of both items for her trip so that she would get at least 2600 calories, 1500 i.u. of vitamin C, and 1250 mgs of protein.

Suppose each unit of Item A costs \$2.80 and each unit of Item B costs \$5. How many units of each should she buy so that the total cost is minimum and her nutritional requirements are met? Use a scientific calculator.

12. Let a line L be defined by $ax + by = c$, where a, b, c are constants and $a \neq 0$ (thus L is not horizontal). Let its x-intercept be x_L. Let $\mathcal{A}$ (respectively, $\mathcal{B}$) be the set of all points (x', y') with the following property: (x', y') does not lie on L, and if the line passing through (x', y') and parallel to L has x-intercept k, then $k > x_L$ (respectively $k < x_L$). Prove that $\mathcal{A}$ and $\mathcal{B}$ are the half-planes of L.



10 Exponents

So far we have dealt with *linear* equations, *linear* inequalities, and *linear* functions of two variables. Linear objects are important because they are the basic building blocks of mathematics, but life is often not linear. A good example is Kepler's famous Third Law governing the motion of an object around the sun: the square of the *period*[39] divided by the cube of the so-called *semi-major axis* of the elliptic orbit[40] is a fixed constant no matter what the object may be (e.g., any planet, any meteor, any asteroid). In symbols, this means there is a number C so that, if T is the period and D is the semi-major axis of an object revolving around the sun,

$$\frac{T^2}{D^3} = C$$

Thus if the object is far from the sun compared with the earth (e.g., Pluto), then it would take much more than a year for that object to complete a revolution around the sun. By multiplying both sides with D^3, we can rewrite this equality as

$$T^2 - CD^3 = 0$$

You can see that this is *not* a linear equation in T and D. What this means is that, to progress further into mathematics, we would have to deal with powers of numbers, such as T^3 and even $x^{-3/5}$, as well as the absolute value of a number, i.e., $|x|$. These are the most basic nonlinear quantities.

Here are the subsections in this section.

Whole-number exponents

[39] The time it takes the object to complete a revolution around the sun.

[40] The maximum distance of the object from the sun. In some school mathematics and physics textbooks, this law is stated using "mean distance" in place of "major axis", and that is an error.

Let us start from the beginning. Unless stated to the contrary, let us agree that for the remainder of this section, α, β, γ will always stand for positive numbers, and m, n, k, ℓ will stand for positive *integers*. Recall that, by definition, $\alpha^2 = \alpha\alpha$, $\alpha^3 = \alpha\alpha\alpha$, $\alpha^4 = \alpha\alpha\alpha\alpha$, ... and in general,

$$\alpha^n = \alpha\alpha\cdots\alpha \qquad (n \text{ times})$$

The positive integer n is called the **exponent** or **power** of α^n. One also speaks of α^n as **raising $\boldsymbol{\alpha}$ to the $\boldsymbol{n}$-th power**. Here are the most basic facts concerning exponents:

(9.1) $\boldsymbol{\alpha^m \alpha^n = \alpha^{m+n}}$

(9.2) $\boldsymbol{(\alpha^m)^n = \alpha^{mn}}$

(9.3) $\boldsymbol{(\alpha\beta)^m = \alpha^m \beta^m}$

where m, n are positive integers. These three facts are, simultaneously, trivial to prove and "fun" to use due to their simplicity. For example, (9.1) says that, *in some vague sense*, exponents are additive under multiplication. As to the triviality of their proofs, there is no doubt of that. For example,

$$\begin{aligned}(\alpha^3)^5 &= (\alpha\alpha\alpha)^5 \\ &= (\alpha\alpha\alpha)(\alpha\alpha\alpha)(\alpha\alpha\alpha)(\alpha\alpha\alpha)(\alpha\alpha\alpha) \\ &= \alpha^{5\times 3} = \alpha^{3\times 5}\end{aligned}$$

The general proof of (9.2) is almost identical, and the same is true of the proofs of the other two identities.

Having established the desirability of (9.1)–(9.3), we may ask why something so good should be restricted to positive integer values of m and n. What should α^0 mean? What about fractions in the exponents? In fact, what about rational numbers? We now approach these three questions systematically.

First, what could 5^0 mean? While we do not know what meaning to give 5^0, we know what we want out of it: (9.1)–(9.3) should still hold even when m, or n is allowed to be 0. Therefore we shall perform a bold experiment by throwing out all the usual rules in mathematics. As we have emphasized, mathematics only deals with concepts with precise definitions. Right now *we do not know what* 5^0 *means but, for once, will pretend that we do.* We further assume that this object 5^0 that we know nothing about satisfies (9.1) even when n is 0. The boldness of this experiment therefore lies in the fact that *we are assuming that we know the very thing we are trying to find out.* This doesn't matter to us because we are merely trying to make the right guess about what 5^0 ought to be, and we are not claiming to be proving anything. So with this understood, the version of (9.1) with $n = 0$ gives:

$$5^1 \cdot 5^0 = 5^{1+0}$$

In other words, we should have $5 \cdot 5^0 = 5$. Multiplying both sides by $\frac{1}{5}$ gives $\frac{1}{5} \cdot 5 \cdot 5^0 = \frac{1}{5} \cdot 5$, so that $5^0 = 1$.

To recapitulate: if (9.1)–(9.3) are to remain meaningful and continue to be valid even when one or more of the exponents is allowed to be 0, then it is *necessary* to define $5^0 = 1$. This heuristic argument is the same if 5 is replaced by any positive number α: if the equality $\alpha^1 \alpha^0 = \alpha^{1+0}$ makes sense and is correct, then $\alpha\alpha^0 = \alpha$, so that necessarily $\alpha^0 = 1$.

On the basis of this heuristic argument, we define **the 0-th power of a positive numbex** α to be

$$\boldsymbol{\alpha^0 \;=\; 1}$$

We pause to make a point. The presentation of the definition of α^0 in some books gives the impression that one can *prove* $\alpha^0 = 1$. The "proof", such as it is, is exactly the argument we gave above. It is therefore necessary to underscore once more the fact that *the preceding heuristic argument is not a proof of why* $\alpha^0 = 1$. What it does is to give us confidence that we have probably made the correct definition to ensure the validity of (9.1)–(9.3) even when one or both of m, n is 0.

Fractional exponents: the definitions

Fractional exponent is next. We start with the simplest case: what could $\alpha^{\frac{1}{2}}$ mean? We will be guided by (9.1)–(9.3) again, and (9.2) suggests that

$$(\alpha^{\frac{1}{2}})^2 = \alpha^{\frac{1}{2}\,2} = \alpha$$

If we write γ for $\alpha^{\frac{1}{2}}$, this says γ should be a number so that $\gamma^2 = 2$. You recognize this as saying γ is the square root of 2.

A good mathematics education sometimes has the beneficial effect of making you stop and think about things that you may have taken for granted all along, and gain new understanding in the process. A case in point for most of us is our first serious encounter with the number π; we learn what it actually is and how to directly estimate it. The "square root of 2", something most of us have been familiar with since perhaps primary school, may yet be another such case. How do we know that *there is a number whose square is exactly 2*? One can rattle off $1.4142135\ldots$ as that number, but one may also be aware that the decimal expansion of the square root of 2 is non-repeating, so that no matter how many decimal digits one writes down, it will just be an approximation. For example, $1.142135^2 = 1.99999091405925$. So what gives us the confidence that there is a "square root of 2"?

This is where mathematical knowledge is helpful by providing the answer we need. There is a theorem, proved in advanced courses, that not only square roots, but any so-called n-th roots exist and are unique. Precisely, let n be a positive integer. Then given a *positive* number α, a *positive* number γ is said to be a **positive $\boldsymbol{n}$-root of $\boldsymbol{\alpha}$** if $\gamma^n = \alpha$. (Recall that $\gamma^n = \gamma\gamma\cdots\gamma$ (n times), by definition.)

> Note the emphasis throughout on the positivity of α and γ. This is because if $\alpha = -2$, then there is no number on the number line whose square is a negative number. (Do you know why?) Moreover, in case $\alpha > 0$, e.g., $\alpha = 4$, there will at least two numbers whose square is 2, namely, 2 and -2. This is why we have to specify the positivity of γ in the preceding paragraph.

Then the theorem that resolves all doubts in this context is the following.

Theorem *Given a positive number α and a positive integer n, there is one and only one positive number γ so that $\gamma^n = \alpha$.*

It is to be remarked that the uniqueness part of the theorem, which says that there is at most one such γ, is actually not too difficult to prove, but we will postpone this proof so as not to interrupt our discussion. Henceforth, we shall refer to the γ in the theorem as *the* positive n-th root of α and, if there is no fear of confusion, more simply as **the $\boldsymbol{n}$-root of $\boldsymbol{\alpha}$**. The standard notation for the positive n-th root of α is $\boldsymbol{\sqrt[n]{\alpha}}$. Note that the case $n = 2$ is distinguished and the notation for the positive square root is $\boldsymbol{\sqrt{\alpha}}$ rather than the more elaborate $\sqrt[2]{\alpha}$. Please remember that $\sqrt[n]{\alpha}$ is always positive, **by convention**. Therefore $\sqrt{4} = 2$, and *never* -2. In any case, one message of the theorem is that there *is* such a thing as *the positive square root of* 2.

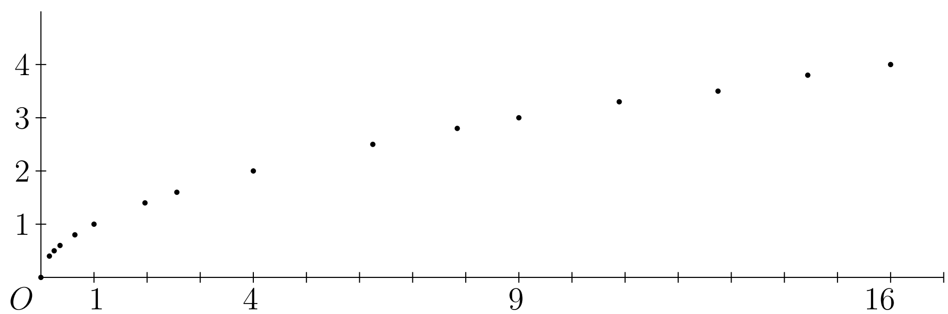
The third root of α is traditionally called its **cube root**.

Example Graph the function given by $f(x) = \sqrt{x}$.

We note that this is not a function from all numbers to all numbers, but

$$f: \{\text{all real numbers } \geq 0\} \to \{\text{all real numbers}\}$$

The following sequence of points on the graph of f are self-explanatory:



This sequence of points exhibit two different patterns: when $0 < x < 1$, $x < \sqrt{x}$, but when $1 < x$, $x > \sqrt{x}$. (This will be an exercise.) Note also that if $0 < a < b$, then $\sqrt{a} < \sqrt{b}$ (also an exercise). This fact then tells us that there is no need to see more points on the graph because as we go towards the right end of the positive x-axis, the graph simply rises in the same way as it does here for the values of $1 \leq x \leq 16$.

We are now in a position to resume our discussion of rational exponents. We will continue to use (9.1)–(9.3) as a guide for the correct definition of rational exponents. Thus, if the analogue of (9.2) is valid in general, we must have $(\alpha^{\frac{1}{2}})^2 = \alpha^{\frac{1}{2}\cdot 2} = \alpha^1 = \alpha$ for any positive α. By Theorem 1, $\alpha^{\frac{1}{2}}$ must be the positive square root of α.

The same heuristic argument would yield the fact that, for any positive integer n, $(\alpha^{\frac{1}{n}})^n = \alpha$, and therefore $\alpha^{\frac{1}{n}}$ has to be the positive n-th root of α. Thus for any positive integer n, we feel sufficiently confident to **define the $\boldsymbol{\frac{1}{n}}$-th power of a positive number $\boldsymbol{\alpha}$** to be

$$\boldsymbol{\alpha^{\frac{1}{n}} \;=\; \textbf{the positive } n\textbf{-th root } \sqrt[n]{\alpha} \textbf{ of } \alpha}$$

The next question is: what should $2^{\frac{3}{2}}$ mean? Again, assuming that the analogue of (9.2) for rational exponents is valid, we must have

$$2^{\frac{3}{2}} = 2^{\frac{1}{2}\cdot 3} = (2^{\frac{1}{2}})^3 = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$$

i.e., the cube of $2^{\frac{1}{2}}$, which we have just defined. This suggests that, in general, for any positive integers m and n, and for any positive number α, we *define*

$$\boldsymbol{\alpha^{\frac{m}{n}} \;=\; \alpha^{\frac{1}{n}}\alpha^{\frac{1}{n}}\cdots\alpha^{\frac{1}{n}} \quad (\, m \text{ times})}$$

In other words,

$$\boldsymbol{\alpha^{\frac{m}{n}} \;=\; (\alpha^{\frac{1}{n}})^m}$$

For example, $4^{\frac{5}{3}}$ means the cube root of 4 raised to the fifth power, i.e.,

$$4^{\frac{5}{3}} = 4^{\frac{1}{3}} \cdot 4^{\frac{1}{3}} \cdot 4^{\frac{1}{3}} \cdot 4^{\frac{1}{3}} \cdot 4^{\frac{1}{3}}$$

We are almost through with the definitions! There is only one more to go: negative exponents. What should $4^{-\frac{5}{3}}$ mean? We appeal to the analogue of (8.1) for rational exponents to obtain:

$$4^{-\frac{5}{3}} \cdot 4^{\frac{5}{3}} \;=\; 4^{-\frac{5}{3}+\frac{5}{3}} = 4^0 = 1$$

Thus we get

$$4^{-\frac{5}{3}} \cdot 4^{\frac{5}{3}} = 1$$

Multiplying both sides by

$$\frac{1}{4^{\frac{5}{3}}}$$

we get

$$4^{-\frac{5}{3}} = \frac{1}{4^{\frac{5}{3}}}$$

There is nothing special about either the number 4 or the exponent $-\frac{5}{3}$, so if we insist that (8.1) be valid for rational exponents, we should **define** for any positive number α and for any fraction (i.e., positive rational number) A,

$$\boldsymbol{\alpha^{-A} \;=\; \frac{1}{\alpha^{A}}}$$

We have now finished defining α^r for any positive number α and for any rational number r. But our work has barely begun! We do not want to get a general definition of α^r just to satisfy our idle curiosity; our goal is rather to generalize (9.1)–(9.3) to rational exponents. With this in mind, we now claim that for any positive number α and β, and for any *rational* numbers r and s, the following are valid:

(9.4) $\boldsymbol{\alpha^r \alpha^s = \alpha^{r+s}}$

(9.5) $\boldsymbol{(\alpha^r)^s = \alpha^{rs}}$

(9.6) $\boldsymbol{(\alpha\beta)^r = \alpha^r \beta^r}$

These are the **laws of exponents**. In advanced courses, it is shown how to define α^r for any number r, rational or irrational, and prove these laws in one fell swoop for all numbers r and s using sophisticated reasoning. Our more modest task here is to at least get a partial understanding of these laws for rational exponents.

Fractional exponents: a few proofs

The complete proofs of (9.4)–(9.6) are, without a doubt, unsuitable for school mathematics, especially grade eight. These proofs are as unpleasant as the proofs of (9.1)–(9.3) are trivial. Nevertheless, students should at least have a glimpse of why (9.4)–(9.6) are true, because they will be using these laws often and it is altogether a bad idea if they are total strangers to any basic tools they use. On the other hand, all prospective teachers should at least go through the complete proof of at least one of these laws, because if they will be telling their students that these proof are too difficult, then they should at least know whereof they speak.

Let us consider the problem of how to give students a glimpse of the reasoning behind the laws of exponents. We suggest first of all to check a few simple cases. For example: we check

$$64^{\frac{1}{2}} \cdot 64^{\frac{2}{3}} = 64^{\frac{7}{6}}$$

Note that $\frac{7}{6} = \frac{1}{2} + \frac{2}{3}$. Now $64 = 8^2$, so $64^{\frac{1}{2}} = 8$. Also $64 = 4^3$, so $64^{\frac{2}{3}} = (64^{\frac{1}{3}})^2 = 4^2 = 16$. Thus

$$64^{\frac{1}{2}} \cdot 64^{\frac{2}{3}} = 8 \cdot 16 = 128$$

On the other hand, $64 = 2^6$, so also

$$64^{\frac{7}{6}} = (64^{\frac{1}{6}})^7 = 2^7 = 128$$

and we are done.

We also suggest giving a proof of a special case of each of (9.4)–(9.6), such as the following:

$$\begin{cases} 5^{-\frac{1}{2}} \cdot 5^{-\frac{1}{4}} = 5^{-\frac{3}{4}} \\ \\ (5^2)^{\frac{1}{3}} = 5^{\frac{2}{3}} \\ \\ (\alpha\beta)^{\frac{1}{n}} = \alpha^{\frac{1}{n}}\beta^{\frac{1}{n}} \end{cases} \tag{11}$$

when the last equality is for all positive numbers α and β and for all positive integer n.

The proofs of all three items in (11) make use of the following useful lemma. (In the tradition of Euclid, a *lemma* is a theorem that is deemed to be of a lower status, either because it is not as central or not as comprehensive as others already designated as theorems. It is sometimes a very subjective judgment whether something should be a lemma or a theorem.)

Lemma 1 *If two positive numbers α and β satisfy $\alpha < \beta$, then for any positive integer n, $\alpha^n < \beta^n$.*

Proof Suppose $\alpha < \beta$, then (because $\alpha > 0$), multiplying both sides by α gives

$$\alpha^2 < \alpha\beta$$

Now we multiply both sides of $\alpha < \beta$ by β to get

$$\alpha\beta < \beta^2$$

Combining the preceding two displayed inequalities, we get $\alpha^2 < \beta^2$. Next we will prove that $\alpha^3 < \beta^3$. Multiplying both sides of $\alpha^2 < \beta^2$ by α, we get

$$\alpha^3 < \alpha\beta^2$$

Now we make use of the just-proven inequality $\alpha\beta < \beta^2$: multiply both sides by β to get

$$\alpha\beta^2 < \beta^3$$

Combining the preceding two displayed inequalities, we get $\alpha^3 < \beta^3$. Continuing one more step, we want to show next that $\alpha^4 < \beta^4$. Multiplying both sides of $\alpha^3 < \beta^3$ by α, we get

$$\alpha^4 < \alpha\beta^3$$

Now multiply both sides of the just-proven inequality $\alpha\beta^2 < \beta^3$ by β and we get

$$\alpha\beta^3 < \beta^4$$

Combining the last two displayed inequalities, we get $\alpha^4 > \beta^4$. In this way, we will eventually get to $\alpha^n < \beta^n$ no matter what n is. This completes the proof.

The following two corollaries (immediately drawn conclusions) of Lemma 1 are of independent interest.

Corollary 1 For two positive numbers α and β, if $\alpha^n < \beta^n$ for some positive integer n, then $\alpha < \beta$.

This is of course the converse of Lemma 1, so it is a curious fact that it is also a corollary of Lemma 1. Before explaining why, let us note first of all what it does not say: it does not say that if any two numbers α and β satisfy $\alpha^n < \beta^n$ for some positive integer n, then $\alpha < \beta$. For example, $3^2 < (-4)^2$, but $3 > (-4)$. So the truth of Corollary 1 depends critically on the positivity of both α and β.

As to the deduction of Corollary 1 from Lemma 1, we make use of what is called the **trichotomy law** among numbers, which states that for any two numbers a and b, one and only one of the three possibilities holds: either $a = b$, or $a < b$, or $a > b$. If you recall that numbers are points on the number line, then this assertion becomes obvious. The point is however that such an obvious fact can also be an effective tool in problem-solving. For the case at hand, we want to prove, under the assumptions $\alpha > 0$, $\beta > 0$, and $\alpha^n < \beta^n$, that $\alpha < \beta$. By the trichotomy law, it is suficient to show that neither $\alpha = \beta$ nor $\alpha > \beta$ is a possibility. Let us first rule out $\alpha = \beta$: in this case, clearly $\alpha^n = \beta^n$ for any positive integer n, thereby contradicting the hypothesis that $\alpha^n < \beta^n$. If $\alpha > \beta$, then Lemma 1 implies $\alpha^n > \beta^n$ for any positive integer n, again contradicting the hypothesis. Therefore, only $\alpha < \beta$ is possible.

The next corollary is the statement that the positive n-th root of a positive number is unique.

Corollary 2 If two positive numbers α and β satisfy $\alpha^n = \beta^n$ for some positive integer n, then $\alpha = \beta$.

We use the trichotomy law again to eliminate the possibility of either $\alpha < \beta$ or $\alpha > \beta$. Lemma 1 says that if either is true, then $\alpha^n \neq \beta^n$, which contradicts the hypothesis. Thus $\alpha = \beta$, as desired.

We are now ready for the proof of (11). To prove the first item of (11), it suffices to prove (according to Corollary 2) that

$$(5^{-\frac{1}{2}} \cdot 5^{-\frac{1}{4}})^4 = (5^{-\frac{3}{4}})^4$$

Now,

$$\begin{aligned}
(5^{-\frac{1}{2}} \cdot 5^{-\frac{1}{4}})^4 &= \left(\frac{1}{5^{\frac{1}{2}}} \cdot \frac{1}{5^{\frac{1}{4}}}\right)^4 \\
&= \left(\frac{1}{5^{\frac{1}{2}}} \cdot \frac{1}{5^{\frac{1}{4}}}\right) \cdots \left(\frac{1}{5^{\frac{1}{2}}} \cdot \frac{1}{5^{\frac{1}{4}}}\right) \qquad \text{(4 times)} \\
&= \left(\frac{1}{5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}}\right)\left(\frac{1}{5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}}\right)\left(\frac{1}{5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}}}\right)
\end{aligned}$$

But by definition of $5^{\frac{1}{2}}$, we know that $5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5$. Similarly, $5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}} = 5$. Thus,

$$(5^{-\frac{1}{2}} \cdot 5^{-\frac{1}{4}})^4 = \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{5^3}$$

On the other hand,

$$\begin{aligned}(5^{-\frac{3}{4}})^4 &= \frac{1}{5^{\frac{3}{4}}} \cdot \frac{1}{5^{\frac{3}{4}}} \cdot \frac{1}{5^{\frac{3}{4}}} \cdot \frac{1}{5^{\frac{3}{4}}} \\ &= \frac{1}{5^{\frac{3}{4}} \cdot 5^{\frac{3}{4}} \cdot 5^{\frac{3}{4}} \cdot 5^{\frac{3}{4}}}\end{aligned}$$

But by definition, $5^{\frac{3}{4}} = 5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}}$. So,

$$\begin{aligned}5^{\frac{3}{4}} \cdot 5^{\frac{3}{4}} \cdot 5^{\frac{3}{4}} \cdot 5^{\frac{3}{4}} &= 5^{\frac{1}{4}} \cdot 5^{\frac{1}{4}} \cdot \cdots \cdot 5^{\frac{1}{4}} \qquad \text{(12 times)} \\ &= \underbrace{(5^{\frac{1}{4}} \cdots 5^{\frac{1}{4}})}_{4}\underbrace{(5^{\frac{1}{4}} \cdots 5^{\frac{1}{4}})}_{4}\underbrace{(5^{\frac{1}{4}} \cdots 5^{\frac{1}{4}})}_{4} \\ &= 5 \cdot 5 \cdot 5 = 5^3\end{aligned}$$

Therefore also

$$(5^{-\frac{3}{4}})^4 = \frac{1}{5^3}$$

and the first item of (11) is proved.

We next prove the second item of (11), namely, $(5^2)^{\frac{1}{3}} = 5^{\frac{2}{3}}$. By Corollary 2, it suffices to prove

$$((5^2)^{\frac{1}{3}})^3 = (5^{\frac{2}{3}})^3$$

(It is tempting to immediately convert 5^2 to 25 in the preceding equality, but it will soon be apparent that this conversion will not simplify the proof and may even obscure the main line of the reasoning.) By the definition of the cube root of 5^2, we have

$$((5^2)^{\frac{1}{3}})^3 = 5^2$$

On the other hand, $5^{\frac{2}{3}} = 5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}}$, by the definition of $5^{\frac{2}{3}}$. Therefore,

$$\begin{aligned}(5^{\frac{2}{3}})^3 &= (5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}})(5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}})(5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}}) \\ &= (5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}})(5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}} \cdot 5^{\frac{1}{3}}) \\ &= 5 \cdot 5 = 5^2 = ((5^2)^{\frac{1}{3}})^3\end{aligned}$$

as desired.

Finally, we prove the last item of (11). In the usual notation of an n-th root, this says $\sqrt[n]{\alpha\beta} = \sqrt[n]{\alpha}\,\sqrt[n]{\beta}$, which is a very useful fact; the case of square root ($n = 2$) is used frequently in middle school and high school, but perhaps with no explanations. We use Corollary 2 once more and see that it suffices to prove

$$(\sqrt[n]{\alpha\beta})^n = (\sqrt[n]{\alpha}\,\sqrt[n]{\beta})^n$$

But

$$(\sqrt[n]{\alpha\beta})^n = \alpha\beta$$

by the definition of the n-th root, while

$$\begin{aligned}
(\sqrt[n]{\alpha}\,\sqrt[n]{\beta})^n &= (\sqrt[n]{\alpha}\,\sqrt[n]{\beta})\cdots(\sqrt[n]{\alpha}\,\sqrt[n]{\beta}) \qquad (n \text{ times})\\
&= \underbrace{(\sqrt[n]{\alpha}\cdots\sqrt[n]{\alpha})}_{n}\underbrace{(\sqrt[n]{\beta}\cdots\sqrt[n]{\beta})}_{n}\\
&= \alpha\beta = (\sqrt[n]{\alpha\beta})^n
\end{aligned}$$

We have proved (11) completely.

We will next give a proof of (9.4), but will not pursue (9.5) and (9.6) any further. However, the proof of (9.4) below requires a special case of (9.5), which is our next lemma.

Lemma 2 *For any positive number α and for any positive integers m, n, ℓ, $(\alpha^{\frac{m}{n}})^{\ell} = \alpha^{\frac{m\ell}{n}}$.*

Proof By definition, $\alpha^{\frac{m}{n}} = (\alpha^{\frac{1}{n}})^m$. Therefore,

$$(\alpha^{\frac{m}{n}})^{\ell} = ((\alpha^{\frac{1}{n}})^m)^{\ell}$$

But if we let $\gamma = \alpha^{\frac{1}{n}}$, then (8.2) implies that $(\gamma^m)^{\ell} = \gamma^{m\ell}$. Thus

$$((\alpha^{\frac{1}{n}})^m)^{\ell} = (\alpha^{\frac{1}{n}})^{m\ell}$$

However, by the definition of $\alpha^{\frac{m\ell}{n}}$, it is equal to $(\alpha^{\frac{1}{n}})^{m\ell}$ as well. Hence Lemma 2 is proved.

We now begin the proof of (9.4): $\alpha^r \alpha^s = \alpha^{rs}$ for all rational numbers r and s. We first deal with the case where $r > 0$ and $s > 0$. Let $r = \frac{m}{n}$ and $s = \frac{k}{\ell}$, where m, n, k, ℓ are positive integers. Then we have to prove:

$$\alpha^{\frac{m}{n}}\, \alpha^{\frac{k}{\ell}} = \alpha^{\frac{m}{n}+\frac{k}{\ell}}$$

Now $\frac{m}{n} + \frac{k}{\ell} = \frac{m\ell+nk}{n\ell}$, so what we need to prove is the following:

$$\alpha^{\frac{m}{n}}\, \alpha^{\frac{k}{\ell}} = \alpha^{\frac{m\ell+nk}{n\ell}} \tag{12}$$

(Incidentally, we need the explicit formula $\frac{m}{n} + \frac{k}{\ell} = \frac{m\ell+nk}{n\ell}$ here, and the use of this formula in this setting may be as good a reason as any as to why one should not follow the common method of defining fraction addition in terms of the LCM of the denominators n and ℓ.) By Corollary 2, it suffices to prove

$$(\alpha^{\frac{m}{n}}\, \alpha^{\frac{k}{\ell}})^{n\ell} = (\alpha^{\frac{m\ell+nk}{n\ell}})^{n\ell}$$

Now

$$\begin{aligned}
(\alpha^{\frac{m}{n}} \alpha^{\frac{k}{\ell}})^{n\ell} &= \underbrace{(\alpha^{\frac{m}{n}} \alpha^{\frac{k}{\ell}}) \cdots (\alpha^{\frac{m}{n}} \alpha^{\frac{k}{\ell}})}_{n\ell} \\
&= \underbrace{\alpha^{\frac{m}{n}} \cdots \alpha^{\frac{m}{n}}}_{n\ell}\, \underbrace{\alpha^{\frac{k}{\ell}} \cdots \alpha^{\frac{k}{\ell}}}_{n\ell} \\
&= (\alpha^{\frac{m}{n}})^{n\ell}\, (\alpha^{\frac{k}{\ell}})^{n\ell} \\
&= \alpha^{\frac{mn\ell}{n}}\, \alpha^{\frac{kn\ell}{\ell}} \qquad \text{(by Lemma 2)} \\
&= \alpha^{m\ell} \alpha^{kn} \\
&= \alpha^{m\ell+kn} \qquad \text{(by (8.1))}
\end{aligned}$$

But also,

$$(\alpha^{\frac{m\ell+nk}{n\ell}})^{n\ell} = \alpha^{\frac{(m\ell+nk)n\ell}{n\ell}} = \alpha^{m\ell+nk}$$

where the first equality again makes use of Lemma 2. We have therefore completed the proof of (9.4) in case $r > 0$ and $s > 0$.

To finish the proof of (9.4), we must also prove the remaining cases of $r < 0$ and $s > 0$, $r > 0$ and $s < 0$, and $r < 0$ and $s < 0$. In the usual notation, we need to prove:

(a) $\alpha^{-\frac{m}{n}}\,\alpha^{\frac{k}{\ell}} = \alpha^{-\frac{m}{n}+\frac{k}{\ell}}$

(b) $\alpha^{\frac{m}{n}}\,\alpha^{-\frac{k}{\ell}} = \alpha^{\frac{m}{n}-\frac{k}{\ell}}$

(c) $\alpha^{-\frac{m}{n}}\,\alpha^{-\frac{k}{\ell}} = \alpha^{-\frac{m}{n}-\frac{k}{\ell}}$

Case (a) and case (b) are similar, so we will prove case (a) only. There are two possibilities: either $-\frac{m}{n}+\frac{k}{\ell} > 0$ or $-\frac{m}{n}+\frac{k}{\ell} < 0$. If the former, then by (12),

$$\alpha^{\frac{m}{n}}\,\alpha^{-\frac{m}{n}+\frac{k}{\ell}} = \alpha = \alpha^{\frac{k}{\ell}}$$

Multiply both sides by

$$\frac{1}{\alpha^{\frac{m}{n}}}$$

we get:

$$\alpha^{-\frac{m}{n}+\frac{k}{\ell}} = \frac{\alpha^{\frac{k}{\ell}}}{\alpha^{\frac{m}{n}}} = \alpha^{\frac{k}{\ell}}\,\alpha^{-\frac{m}{n}}$$

which is (a). Now if $-\frac{m}{n}+\frac{k}{\ell} < 0$, then $\frac{m}{n}-\frac{k}{\ell} > 0$. So by (12) again,

$$\alpha^{\frac{m}{n}-\frac{k}{\ell}}\,\alpha^{\frac{k}{\ell}} = \alpha^{(\frac{m}{n}-\frac{k}{\ell})+\frac{k}{\ell}} = \alpha^{\frac{m}{n}}$$

In other words,

$$\alpha^{\frac{m}{n}-\frac{k}{\ell}}\,\alpha^{\frac{k}{\ell}} = \alpha^{\frac{m}{n}}$$

We now multiply both sides of this equality by

$$\frac{1}{\alpha^{\frac{m}{n}}\,\alpha^{\frac{m}{n}-\frac{k}{\ell}}}$$

we obtain

$$\alpha^{-\frac{m}{n}}\,\alpha^{\frac{k}{\ell}} = \alpha^{-(\frac{m}{n}-\frac{k}{\ell})}$$

which is exactly (a). The proof of case (c) is simpler. By (12), we know

$$\alpha^{\frac{m}{n}}\,\alpha^{\frac{k}{\ell}} = \alpha^{\frac{m}{n}+\frac{k}{\ell}}$$

By the cross-multiplication algorithm, this is equivalent to

$$\frac{1}{\alpha^{\frac{m}{n}}\,\alpha^{\frac{k}{\ell}}} = \frac{1}{\alpha^{\frac{m}{n}+\frac{k}{\ell}}}$$

which is of course the same as

$$\alpha^{-\frac{m}{n}}\,\alpha^{-\frac{k}{\ell}} = \alpha^{-\frac{m}{n}-\frac{k}{\ell}}$$

The proof of (9.4) is complete.

Remark We have chosen to give a complete proof of (9.4) because until one goes through such a proof, one doesn't know what it means that the proofs of (9.4)–(9.6) are unpleasant. Now in mathematics, when things get unpleasant, 90% of the time it is because they are not done "the right way". In this case, the right way is to prove (9.4)–(9.6) for *all* number r, s all at once without restricting oneself to only rational values of r and s. Such a proof is achieved by appealing to basic properties of the exponential and logarithmic functions and using differentiation. This will be done in advanced courses. For our task at hand, it remains to emphasize that you should feel free to use (9.4)–(9.6) for all values of r and s by appealing to FASM and worry about their proofs later.

We will put (9.4)–(9.6) to use in the next section.

Two applications

Before we leave the discussion of exponents, we would like to expand on the meaning of an *expression* as defined in Section 1. Recall that an expression (or number expression) is simply a collection of numbers x, y, etc. connected by the four arithmetic operations. Now that exponents are available, we can add to the meaning of an **expression** by defining it to mean a collection of numbers x, y, etc. which are connected by the four arithmetic operations and the use of rational exponents. In this context, an expression is also called an **algebraic expression**. Thus, the following is an algebraic expression:

$$x^{-3} + \{(yz)^2 + 5\}^{\frac{3}{4}} - (\frac{xy}{z})^5$$

It is usually not made clear to students that we need fractional exponents because we want to define *exponential functions* such as $H(x) = 2^x$. Having a definition of fractional exponents of 2 tells us how to define $H(x)$ for all rational values of x, and then FASM does the rest. Before giving a graph of H, let us note that the laws of exponents give a main reason why we should be interested in exponential functions:

this is because they enjoy the following special properties: for all numbers x and y,

$$\begin{aligned} H(x)H(y) &= g(x+y) \\ H(x)^y &= H(xy) \end{aligned}$$

We now plot enough points to give an idea of the graph of $H(x) = 2^x$.

Observe that $(0,1)$ is on the graph of H. We should obviously get the points of the graph above $x = \pm 1,\ \pm 2,\ \pm 3,\ \pm 4$. Note that

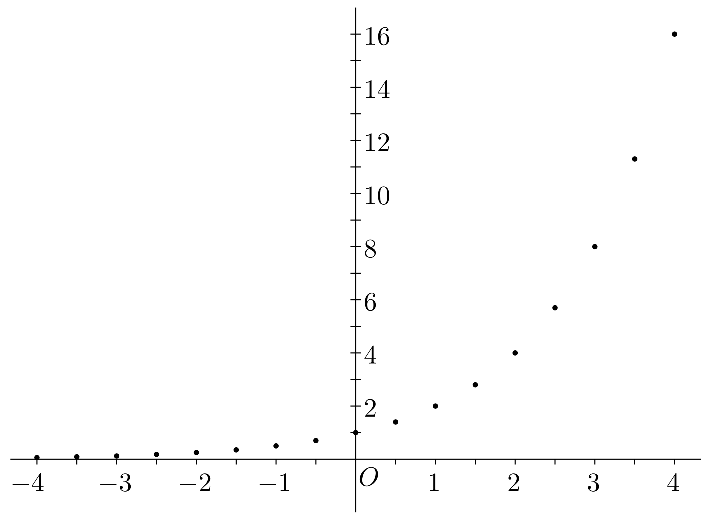
$$H(-1) = \tfrac{1}{2} = 0.5, \qquad H(-2) = \tfrac{1}{2^2} = 0.25,$$

$$H(-3) = \tfrac{1}{2^3} = 0.125, \qquad H(-4) = \tfrac{1}{2^4} = 0.0625.$$

We should get a few more points, e.g.,

$$\begin{aligned} H(\frac{1}{2}) &= \sqrt{2} \sim 1.4 \\ H(-\frac{1}{2}) &= \frac{1}{\sqrt{2}} \sim 0.7 \\ H(\frac{3}{2}) &= (\sqrt{2})^3 \sim 2.8 \\ H(-\frac{3}{2}) &= \frac{1}{(\sqrt{2})^3} \sim 0.35 \\ H(\frac{5}{2}) &= (\sqrt{2})^5 \sim 5.7 \\ H(-\frac{5}{2}) &= \frac{1}{(\sqrt{2})^5} \sim 0.18 \\ H(\frac{7}{2}) &= (\sqrt{2})^7 \sim 11.3 \\ H(-\frac{7}{2}) &= \frac{1}{(\sqrt{2})^7} \sim 0.09 \end{aligned}$$

where the symbol "$\sim$" means "approximately equal to". Here is the picture of these points:



Notice that the points rise steadily to the right, and this is because for all rational numbers r and s so that $r < s$, we have $2^r < 2^s$. The proof of a more general fact is left as an exercise.

We do not have the time here to pursue the study of this function H, or the functions α^x for a positive α. It is worthwhile to mention nevertheless that these functions are important in mathematics and the sciences.

EXERCISES

1. (a) Explain why there is no number on the number line whose square is a negative number. (b) Verify the equality $729^{\frac{1}{2}} \cdot 729^{\frac{1}{3}} = 729^{\frac{5}{6}}$ by direct computation. Do the same to $117649^{\frac{1}{2}} \cdot 117649^{\frac{1}{6}} = 117649^{\frac{2}{3}}$

2. (a) Prove that for all rational numbers r, s and t, $(\alpha^r \beta^s)^t = \alpha^{rt}\beta^{st}$. (b) Prove using only the definition of $\alpha^{\frac{m}{n}}$, but without using (9.5), that $(\alpha^{\frac{m}{n}})^{\frac{1}{m}} = \alpha^{\frac{1}{n}}$.

3. Prove that for any *rational number* r (i.e., r may not be positive), and for any positive number α,

$$\alpha^{-r} = \frac{1}{\alpha^r}$$

4. (a) Prove that when $0 < x < 1$, $x < \sqrt{x}$, and when $1 < x$, $x > \sqrt{x}$. (b)

Prove that if $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

5. *Plot enough points* in the graph of 4^x *without the use of a calculator* to get an accurate picture of its graph.

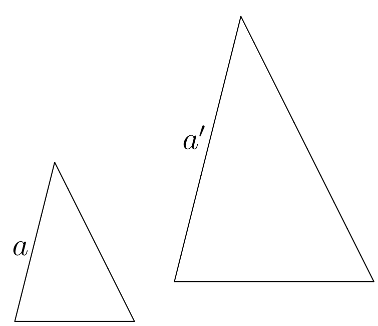
6. *Plot enough points* in the graph of each of the following functions to get an accurate picture of the graph: (*i*) $(\frac{1}{3})^x$, (*ii*) 3^x, (*iii*) 5^x. Compare the graphs of (*i*) – (*iii*) by putting all three in the same coordinate grid if necessary: what are the similarities, and what are the differences? (Use a scientific calculator.)

7. Prove that if $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$, for any positive interger n. (The fact that this is true for $n = 2$ is used in the Example.)

8. Prove that if $a > 1$, then for all rational numbers r and s so that $r < s$, $a^r < a^s$, while if $0 < b < 1$, $b^r > b^s$.

9. Prove that for any number x, $|x| = \sqrt{x^2}$.

10. Given two similar triangles as shown:



If the ratio of the area of the smaller triangle to the area of the bigger triangle is s, what is the ratio $\frac{a}{a'}$ in terms of s?

11. Recall that **an annual interest rate of x percent** means that an account of P dollars earns at the end of one full year an amount of

$$\left(\frac{x}{100}\right) P$$

dollars. Derive a formula which gives the amount of money in an account at the end of n years if the account has an initial deposit of P dollars and an annual interest rate of x percent.

12. Graph on the number line each of the following: (i) $|x| - 14 > -8$, (ii) $|x| - 4 < \frac{1}{3}$, (iii) $9 - |3x - 1| < 4$, (iv) $|2x + \frac{3}{5}| \geq \frac{1}{5}$, (v) $|6x + 1| + 2\frac{1}{4} < 5$.

11 Quadratic Functions and Their Graphs

If a function f from all numbers to all numbers is given by $f(x) = ax^2 + bx + c$ for some constants a, b, and c, we say f is a **quadratic polynomial function**, or more simply, **a quadratic function**. We have already come across the simplest quadratic function $f_0(x) = x^2$ in §7. The main theme of this section is the study of quadratic functions and their graphs. We shall see that, as in the case of linear functions, an understanding of such a function is based on an understanding of its graph. At the end of this section, we will prove that the graphs of all quadratic functions are similar (in the sense of the composition of a dilation followed by a congruence) to the graph of f_0, and each is a *parabola*.

Here are the subsections.

The main theorem on quadratic functions

We begin with two general observations. Given a quadratic function $f(x) = ax^2 + bx + c$ as above, we first observe that:

(**A**) $|f(x)|$ becomes arbitrarily large if $|x|$ gets sufficiently large.

(**B**) If $a > 0$, then $f(x) > 0$ for $|x|$ large. If $a < 0$, then $f(x) < 0$ for $|x|$ large.

In a suggestive notation, we will paraphrase the combined statements of (**A**) and (**B**) by writing:

$$\begin{aligned}\lim_{|x|\to\infty} f(x) &= \infty \qquad \text{for } a > 0 \\ \lim_{|x|\to\infty} f(x) &= -\infty \qquad \text{for } a < 0\end{aligned}$$

> Notice how we have to use absolute value to express both (**A**) and (**B**) succinctly. If the concept of absolute value were not at our disposal, we would have to write something like the following, for example, for (**A**): "$f(x) > R$ for a given large positive number R, or $f(x) < -R$ for a large positive number R if the distance of x from 0 is sufficiently large"

The reason for (**A**) is as follows. Write

$$f(x) = \left(a + \frac{b}{x} + \frac{c}{x^2}\right) x^2$$

Now if $|x|$ is sufficiently large, then $\frac{1}{|x|}$ would be sufficiently small, and therefore so are $|\frac{b}{x}|$ and $|\frac{c}{x^2}|$. Consequently, when $|x|$ is sufficiently large, $(a + \frac{b}{x} + \frac{c}{x^2})$ would be very close to a. If we use $\sim$ to indicate "approximately equal to", then we have $f(x) \sim ax^2$ when $|x|$ is sufficiently large. Since x^2 is always positive, $f(x) \sim a|x^2|$ when $|x|$ is sufficiently large. This proves (**B**), and (**A**) also follows from the same expression for $f(x)$ when $|x|$ is sufficiently large.

It remains to find out how $f(x)$ behaves when x is not very large. We will use a symmetry property of the graph of f to cut down our effort by half. Recall that a set S in the plane is said to be **symmetric with respect to a line** L if for every point Q in S, the reflection R across L maps Q to a point $R(Q)$ which *also lies in* S. If we use the terminology of Section 8, S being symmetric with respect to L means the part of S in the half-plane L^+ is congruent to the part of S in the other half-plane L^-. If there is such a symmetry, then the study of S itself reduces to a study of one of the two halves, $S \cap L^+$ or $S \cap L^-$. This explains the interest in such a symmetry. To describe this symmetry more precisely, we introduce a concept already used informally in Section 8. Given a function f from all numbers to all numbers (e.g., a quadratic function). We say f **achieves a minimum** at a point β if $f(x) \geq f(\beta)$ for any number x. Similarly, we say f **achieves a maximum** at a point β if $f(x) \leq f(\beta)$ for any number x. We also say that x_0 is a **zero of the function** f is $f(x_0) = 0$. The following is the main theorem about quadratic functions.

Theorem 1 *For any quadratic function $f(x) = ax^2 + bx + c$, there is a unique number p so that, if $a > 0$, then p is the number at which f achieves a minimum, and if $a < 0$, then p is the number at which f achieves a maximum. Furthermore, f can have at most two zeros, and the graph of f is symmetric with respect to the vertical line defined by $x = p$.*

Proof We will break up the proof into two parts. The *first part* assumes that the quadratic function is in the form $f(x) = a(x-p)^2 + q$ for some numbers p and q. Let us examine what this means. If we write

$$a(x-p)^2 + q = ax^2 - 2apx + (ap^2 + q),$$

then to say that f can be written as $f(x) = a(x-p)^2 + q$ is to say that the coefficients b and c in $f(x) = ax^2 + bx + c$ can be expressed in terms of two numbers p and q as

$$b = -2ap \qquad \text{and} \qquad c = ap^2 + q$$

In the *second part*, we will show that in fact every quadratic function can be written in the form $f(x) = a(x-p)^2 + q$.

As usual, we give a detailed proof of this theorem because the ideas of the proof contain important information about quadratic functions in general.

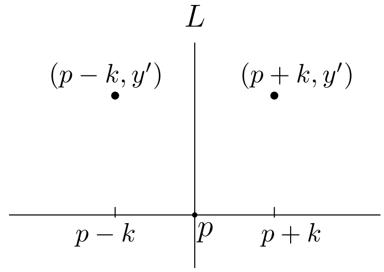
So suppose $f(x) = a(x-p)^2 + q$. We observe that for any number k,

$$f(p+k) = f(p-k) = ak^2 + q$$

This is easy:

$$f(p-k) = a((p-k)-p)^2 + q = ak^2 + q = a((p+k)-p)^2 + q = f(p+k)$$

We now use the fact that $f(p+k) = f(p-k)$ to show that the graph of f is symmetric with respect to the vertical line L defined by $x = p$. Indeed, the reflection R with respect to L maps a point $(p+k, y')$ to the point $(p-k, y')$, where k and y' are both arbitrary. The following picture assumes that k is positive.



Now if a point $(p-k, f(p-k))$ is on the graph of f, then under the reflection R, this point is moved to $(p+k, f(p-k))$, which is equal to $(p+k, f(p+k))$ because we have seen that $f(p-k) = f(p+k)$. Since $(p+k, f(p+k))$ is a point on the graph of f (the set of *all* the points of the form $(x, f(x))$, by definition), we have proved our claim of symmetry with respect to L.

To study the graph of f, this symmetry says it suffices to study $(x, f(x))$ where $x \geq p$; the rest (i.e., those $(x, f(x))$ where $x \leq p$) can be obtained by reflection. Now if $x \geq p$, we can write $x = p + k$, where $k \geq 0$, and from the above discussion, we have $f(p+k) = ak^2 + q$. Thus

> if $a > 0$, then $f(p+k)$ increases with k so that $f(p+k)$ is smallest when $k = 0$, that is, f attains its minimum only at $x = p$, and
>
> if $a < 0$, then $f(p+k)$ decreases with k so that $f(p+k)$ is biggest when $k = 0$, that is, f attains its maximum only at $x = p$.

The preceding assertions is about the behavior of $f(x)$ for $x \geq p$, but by the symmetry of the graph of f with respect to L (i.e., the line $x = p$), they remain true for all x. Moreover, for $a > 0$, if $q > 0$, then $f(p+k) = ak^2 + q > 0$ and there would be no zero for f when $x \geq p$. By symmetry with respect to L, f has no zero in this case. If $q = 0$, then $f(p) = 0$, and since p is the only minimum of f, there is exactly one zero for f. Finally, if $q < 0$, then $f(p) = q < 0$. But from (A) and (B) above, we know that $f(p+k)$ becomes large positive when k is large. Because $f(p+k)$ increases with k, we see that $f(p+k_0) = 0$ for some $k_0 > 0$. Thus f has one zero at $p + k_0$ among the numbers $p + k$ for $k \geq 0$. By symmetry with respect to L, $f(p - k_0)$ is also zero, so that f has another zero among the numbers $p - k$ for $k > 0$. Altogether, f has exactly two zeros in this case.

In a similar manner, we can show that if $a < 0$, then f has no zero, one zero, and two zeros, according to whether $q < 0$, $q = 0$, and $q > 0$, respectively. So the proof of Theorem 1 is complete in case $f(x) = a(x-p)^2 + q$.

Activity Plot points on the graph of $f(x) = 4(x-2)^2 + \frac{1}{3}$ which are symmetric with respect to the line $x = 2$. Does the graph cross the x-axis?

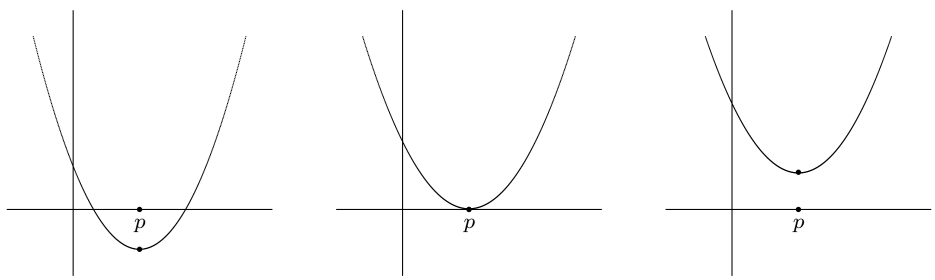
Remark Before continuing with the proof of Theorem 1, we digress to interpret the information from the preceding discussion in terms of the shape and the location of the graph of $f(x) = a(x-p)^2 + q$ relative to the coordinate axes. As mentioned above, because every quadratic function will be shown presently to be expressible in the form of $a(x-p)^2 + q$, what we are going to say is valid for all quadratic functions.

With L as the line $x = p$ and $f(x) = a(x-p)^2 + q$, first assume $a > 0$. Then we have seen that $f(p+k)$ with $k > 0$ gets bigger as k gets bigger. In terms of the graph of f, which consists of all the points $(p+k, ak^2+q)$ where k is arbitrary, *the graph rises as we go to the right*. Because of the symmetry with repsect to L, this means *the graph of f also rises as we go to the left*. Moreover, if $q > 0$, then $f(p+k) = ak^2 + q > q > 0$ for every k. Thus the whole graph of f would be above the x-axis and it would not intersect the x-axis. If on the other hand $q < 0$, then $f(p) = q < 0$ whereas $\lim_{|x|\to\infty} f(x) = \infty$. Therefore the graph of f goes from below the x-axis at p to above the x-axis when $|x|$ is very large. Thus the graph of f crosses the x-axis at two points exactly, once in L^+ because the graph rises as we go to the right, and therefore also once in L^- because of the symmetry with respect to L. Finally, if $q = 0$, then $f(p) = 0$, and the graph would touch the x-axis at exactly $(p, 0)$.

Our interest in the intersection of the graph of f with the x-axis naturally coincides with our interest in locating the zeros of the polynomial function $f(x) = ax^2 + bx + c$. (A zero of the polynomial $ax^2 + bx + c$ is also called a **root** of the polynomial equation $ax^2 + bx + c = 0$.) If the graph of f intersects the x-axis at $(x_0, 0)$, then $(x_0, 0)$ being on the graph means $0 = f(x_0)$, so that x_0 is a zero of f. Conversely, if $f(x_0) = 0$, then the point $(x_0, 0)$ (which is of course $(x_0, f(x_0))$) is on both the graph of f and the x-axis and is therefore a point of the intersection of the graph with the x-axis. Thus

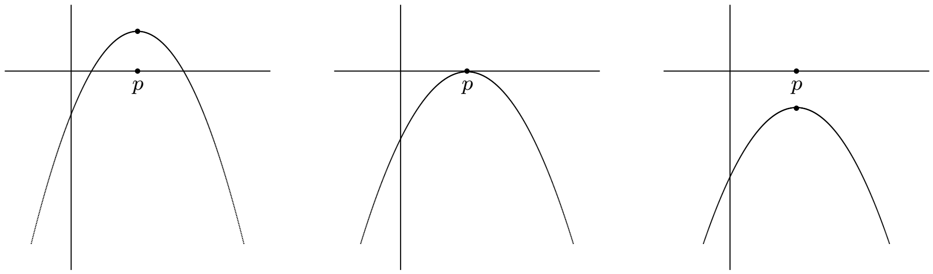
> *locating the points of intersection of the graph of f with the x-axis is equivalent to locating the zeros of f.*

We will discuss the zeros of a quadratic function a bit more in the next section and we will see that these zeros provide critical information in applications. What we have established is that, in case $a > 0$ in $f(x) = a(x-p)^2 + q$, f may have one zero, two zeros, or no zero, depending on whether $q = 0$, $q < 0$, and $q > 0$, respectively. Typically, these three cases lead to the following kinds of graphs, with the case of $q < 0$ on the left, $q = 0$ in the middle, and $q > 0$ on the right:



In the case of $a > 0$, we call $f(p)$ (which equals q) the **minimum value** of f, and the point $(p, f(p))$ is then called the **vertex** of the parabola. The vertex is the lowest point of the parabola in this case. Precisely, the x-coordinate of the vertex is the point at which f achieves its minimum, and the y-coordinate is the minumum value of f.

Still with $f(x) = a(x-p)^2 + q$, now assume $a < 0$. The same discussion can be carried through, but now f achieves a maximum at p and, because $f(p+k) = ak^2 + q$ and $a < 0$ so that f gets more and more negative as k goes to the right of the x-axis, and *the graph of f goes down as we go to the right of the x-axis.* By symmetry with respect to L, *the graph of f also goes down as we go to the left of the x-axis.* The point $(p, f(p))$ is still called the **vertex** of the parabola; it is now the highest point on the parabola when $a < 0$. Furthermore, if $q < 0$, the whole graph would be below the x-axis and f would have no zero. If $q > 0$, then because $f(p) = q$, the graph of f will now have points both above and below the x-axis and will therefore intersect the axis at precisely two points (the reason for "two" is the same as before). If $q = 0$, then the graph touches the x-axis at exactly $(p, 0)$. The three possibilites are given in the following pictures:



It is common to refer to the graph of a quadratic function in the case of $a > 0$ as a **up parabola**, and in the case of $a < 0$ as a **down parabola**. We will justify this terminology presently.

We can now return to the **proof of the second part of Theorem 1**.[41]

Given $f(x) = ax^2 + bx + c$, we want to find numbers p and q so that $f(x) = a(x-p)^2 + q$. We first do it algebraically. One should first work out some concrete examples such as $f(x) = 2x^2 + 6x - 7$ or $f(x) = 3x^2 - 4x + 1$ to get some feeling for the problem before embarking on the general case. Be that as it may, here is the general argument. Write

$$f(x) = a(x^2 + \frac{b}{a}x) + c$$

and compare it with what we want, which is

$$f(x) = a(x-p)^2 + q = a(x^2 - 2px + p^2) + q$$

For these two expressions to be equal for *all* x, i.e.,

$$a(x^2 + \frac{b}{a}x) + c = a(x^2 - 2px + p^2) + q$$

for all x, we must have, at least heuristically, the equality of all the corresponding coefficients.[42] In particular, the two coefficients of x must be equal. Thus $\frac{b}{a} = -2p$, which would suggest that we let $p = -\frac{b}{2a}$. We try that and get:

$$a(x-p)^2 + q = a(x + \frac{b}{2a})^2 + q = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + q = ax^2 + bx + (\frac{b^2}{4a} + q)$$

[41] You will see that the identities $(x+y)^2 = x^2 + 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$ of Section 1 play a crucial role in the subsequent argument.

[42] This fact can be proven in general, but for quadratic functions, the proof is not too difficult. See the Exercises in the next section.

If we hope to make this last expression equal to $= ax^2 + bx + c$, we have to set $\frac{b^2}{4a} + q = c$, or what is the same thing, set $q = c - \frac{b^2}{4a} = \frac{4ac-b^2}{4a}$. With this heuristic reasoning in the background, we now get to work.

Formally, let

$$p = -\frac{b}{2a} \quad \text{and} \quad q = \frac{4ac-b^2}{4a}$$

Then:

$$\begin{aligned} a(x-p)^2 + q &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a} \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \frac{4ac-b^2}{4a} \\ &= ax^2 + bx + \frac{b^2}{4a} + \frac{4ac-b^2}{4a} \\ &= ax^2 + bx + c \end{aligned}$$

It follows that with these values for p and q, we now know that for the given $f(x) = ax^2 + bx + c$,

$$f(x) = a(x-p)^2 + q$$

Then the proof of the first part of Theorem 1 now takes over to completely prove the theorem.

There are two comments on this proof that are of equal importance, and we go through both carefully. With $f(x) = ax^2 + bx + c$ understood, the first one is that, from the proof of the first part of Theorem 1, we know that p is the point at which f achieves its maximum or minimum, depending on whether $a < 0$ or $a > 0$, and the maximum or minimum value of f at p is q. But we have the explicit values of p and q given above in terms of the coefficients a, b, and c of f. We can therefore summarize these conclusions in the following important theorem about quadratic functions.

Theorem 2 *The quadratic function* $f(x) = ax^2 + bx + c$

(i) achieves its maximum at $-\frac{b}{2a}$ *if* $a < 0$, *and the maximum value* $f(-\frac{b}{2a})$ *is* $\frac{4ac-b^2}{4a}$,

(ii) achieves its minimum at $-\frac{b}{2a}$ *if* $a > 0$, *and the minimum value* $f(-\frac{b}{2a})$ *is* $\frac{4ac-b^2}{4a}$.

Furthermore, the quadratic function can be rewritten as

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

It goes without saying that the vertex of the graph of f has coordinates $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$.

A second comment is that, looking back, we can see that the most critical step in the proof of the second part of Theorem 1, one that allows us to reduce the most general quadratic function to the manageable special case of $a(x-p)^2 + q$, is the identity that

$$x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

Indeed, if we knew this identity, then we would immediately recognize that ax^2+bx+c could be put in the form of $a(x-p)^2 + q$ with $p = -\frac{b}{2a}$ and $q = c - \frac{b^2}{4a}$, because it is a simple computation to see that

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right\} + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right). \end{aligned}$$

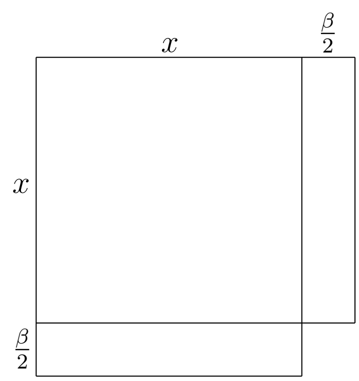
Such an identity has wide applicability in mathematics and should be stated more generally, as follows: for any number β, we have

$$x^2 + \beta x = \left(x + \frac{\beta}{2}\right)^2 - \left(\frac{\beta}{2}\right)^2 \tag{13}$$

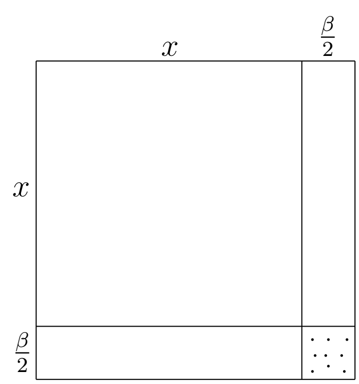
In this form, identity (13) is known under the name of **completing the square**. Its validity is not in doubt if you start with the right side and simply expand it and simplify to arrive at the left side. What is more interesting is how, starting with the *left* side $x^2 + \beta x$, one would be led into thinking that it can be expressed as the difference of two squares, and what the name "completing the square" means. Both questions are answered simultaneously if we retrace the steps of the Babylonians who made extensive use of such an identity some 38 centuries ago (around 1800 B.C.).[43] Writing $x^2 + \beta x$ as

$$x^2 + 2\left(\frac{\beta}{2} \cdot x\right)$$

and remembering that β and x are numbers, we see that this expression is exactly the area of the following figure consisting of a square with one side of length x and two rectangles each with sides of length $\frac{\beta}{2}$ and x.



Looking at this picture, it would be natural to *complete it to a bigger square with a side of length* $x + \frac{\beta}{2}$ by adding the small dotted square at the lower right corner:



[43]No, the Babylonians did not state completing the square as identity (13), but if they had the symbolic notation that we have today, they might have.

Now the dotted square has a side of length $\frac{\beta}{2}$, so its area is $(\frac{\beta}{2})^2$. With this dotted square added to the original figure, the total area is now the area of the big square with a side of length $x + \frac{\beta}{2}$. Thus:

$$(x^2 + \beta x) + \left(\frac{\beta}{2}\right)^2 = \left(x + \frac{\beta}{2}\right)^2$$

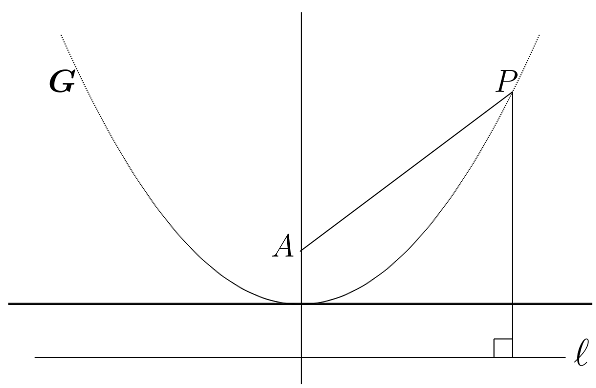
Transposing $(\frac{\beta}{2})^2$ to the right, we get immediately identity (13).

Two properties of the graph

We conclude this section by fulfilling the promise made at the beginning of the section about parabolas. We are going to prove the following two theorems.

Theorem 3 *The graph of any quadratic function is similar to the graph G_0 of the quadratic function given by $f_0(x) = x^2$ for all x.*

Before stating the next Theorem, we define a **parabola** G to be the set of all points equidistant from a fixed point A and a fixed line ℓ. The point A is called the **focus** and the line is called the **directrix** of the parabola, as shown.



Theorem 4 *The graph of every quadratic function is a parabola.*

A few comments before the proofs. It is quite clear that these theorems are not part of the standard algebra curriculum, whether it be Algebra I or Algebra II. So why spend time on them here? There are three reasons. The first is that Theorem 3 is a surprising results, and even mentioning something like this in an algebra class could be inspiring or intriguing to students. A second reason is that the proof of Theorem 3 shows why it is essential that we know a *precise* definition of similarity. In this case, we have to prove that the graphs of any two quadratic functions, which are not rectilinear figures, are similar. There are no line segments to measure and no angles to compare; instead we are forced to use the definition of similarity in terms of dilation. The final reason is that Theorem 4 is usually stated in textbooks *without* any definition of what a parabola is. Such intellectual sloppiness is simply not acceptable, and we wish to set the record straight.

Proof of Theorem 3 We break up the proof into two parts:

Part 1: Let G_k be the graph of the function $f_k(x) = kx^2$ where $k > 0$, then the dilation $\mathcal{D}$ defined by $\mathcal{D}(x, y) = (\frac{1}{k}x, \frac{1}{k}y)$ maps G_0 to G_k.

Part 2: Let G be the graph of a quadratic function $f(x) = ax^2 + bx + c$. Then there is a congruence that maps a G_k for some k to G.

Because a similarity is by definition a dilation followed by a congruence, Part 1 and 2 together prove that G_0 is similar to the graph of any quadratic function.

Part 1 is proved as follows. By definition, G_k consists of all points of the form (x, kx^2), where x is arbitrary, while G_0 consists of all points of the form (x, x^2) for all x. We want to show that $\mathcal{D}(G_0) = G_k$. For an arbitrary (x, x^2) of G_0, we have

$$\mathcal{D}(x, x^2) = (\frac{1}{k}, \frac{1}{k}x^2) \;=\; ((\frac{1}{k}x), k(\frac{1}{k}x)^2)\,,$$

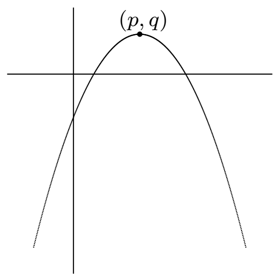
which shows that $\mathcal{D}$ maps G_0 into G_k. On the other hand, if (x, kx^2) is an arbitrary point of G_k, then we have a point $(kx, (kx)^2)$ of G_0 so that

$$\mathcal{D}(kx, (kx)^2) \;=\; (x, kx^2)$$

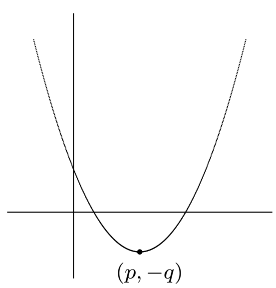
so that $\mathcal{D} : G_0 \to G_k$ is surjective. Thus $\mathcal{D}(G_0) = G_k$. This proves Part I.

For Part 2, let the graph G of a quadratic function be given. If the vertex of G is (p, q), then by Theorem 2, G is the graph of the quadratic function $f(x) = a(x - p)^2 + q$. We are going to assume the worst case scenario so that both p and q

are nonzero, and that $a < 0$. In other words, G is a down parabola. We will produce a reflection and a translation that carry G to G_k for some k. The composition of the inverse translation and the reflection then furnishes the desired congruence from G_k to G.



Let R be the reflection with respect to the x-axis. Then R maps a point (x, y) to $(x, -y)$. Now a point on G, which is the graph of $f = a(x-p)^2 + q$, is of the form $(x, f(x)) = (x, a(x-p)^2 + q)$. Therefore each point $(x, a(x-p)^2 + q)$ of G is mapped by R to $(x, (-a)(x-p)^2 - q)$. Observe that the collection of all these point $\{ (x, (-a)(x-p)^2 - q) \}$ is exactly the graph of the function g given by $g(x) = (-a)(x-p)^2 - q$. Because $a < 0$, $(-a) > 0$. Write γ for $(-a)$ for simplicity, and we see that the reflected parabola $R(G)$ is the graph of g given by $g(x) = \gamma(x-p)^2 - q$, where $\gamma > 0$ so that $R(G)$ is an up parabola, as shown:



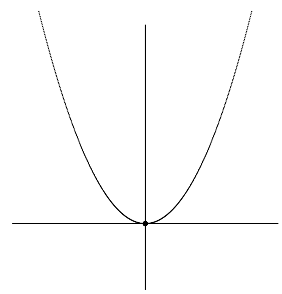
Notice that the vertex of $R(G)$ is $(p, -q)$.

Next we will use a translation to move $(p, -q)$ to $(0, 0)$. The translation T is described algebraically by

$$T(x, y) = (x - p, y + q)$$

for all (x, y). Observe that $T(p, -q) = (0, 0)$. The points $(x, \gamma(x-p)^2 - q)$ of the reflected parabola $R(G)$ are now translated to $(x - p, \gamma(x-p)^2 - q + q) =$

$(x-p, \gamma(x-p)^2)$. So the translated parabola $T(R(G))$ consists of all points of the form $(x-p, \gamma(x-p)^2)$ where x is arbitrary. This means that $T(R(G))$ is G_γ, the graph of the function $f_\gamma(x) = \gamma x^2$.



In case $a > 0$ in $f(x) = a(x-p)^2 + q$, then in the preceding argument, we simply dispense with the use of the reflection R and directly apply the translation T to this G. Then $T(G)$ is G_a, the graph of the function $f_a(x) = ax^2$. This completes the proof.

Proof of Theorem 4 Let G be the graph of a quadratic function f. We know that $f(x) = a(x-p)^2 + q$ for some numbers p and q (see proof of the second part of Theorem 1). To cut a long story short, we simply let A be the point $A = (p, \frac{1}{4a} + q)$ and ℓ be the horizontal line $y = -\frac{1}{4a} + q$. Then for a point $P = (x, a(x-p)^2 + q)$ on G, the *square* of its distance from A is

$$\begin{aligned}(x-p)^2 + (\frac{1}{4a} + q - a(x-p)^2 - q)^2 &= (x-p)^2 + (\frac{1}{4a} - a(x-p)^2)^2 \\ &= (\frac{1}{4a})^2 + \frac{1}{2}(x-p)^2 + (a(x-p)^2)^2 \\ &= (\frac{1}{4a} + a(x-p)^2)^2\end{aligned}$$

But the *square* of the distance from $P = (x, a(x-p)^2 + q)$ on G to ℓ is

$$(a(x-p)^2 + q - (-\frac{1}{4a} + q))^2 = (\frac{1}{4a} + a(x-p)^2)^2$$

Hence the two distances are equal. This shows that every point of G is equidistant from A and ℓ. If a point Q is equidistant from A and ℓ, then same computation shows that Q must be of the form $(x, a(x-p)^2 + q)$, which is a point on G. Thus G is the parabola with focus A and directrix ℓ. The proof of Theorem 4 is complete.

EXERCISES

1. Write each of the following quadratic functions in the form $a(x-p)^2+q$ for suitably chosen numbers a, p and q by completing the square: (*i*) $f(x)=x^2-8x+7$. (*ii*) $g(x)=-2x^2+6x-21$. (*iii*) $h(x)=3x^2+4x+6$. (*iv*) $f(x)=-\frac{2}{3}x^2+x-1$. (*v*) $g(x)=5x^2-\frac{2}{5}+2$. (*vi*) $g(x)=\sqrt{2}x^2-\sqrt{6}x+5$. (*vii*) $h(x)=-3x^2+\sqrt{5}x-1$.

2. Find the maximum or the minimum of each of the following quadratic functions: (*i*) $f(x)=2x^2+3x+4$. (*ii*) $g(x)=\frac{3}{4}x^2-2x+\frac{8}{5}$. (*iii*) $h(x)=-6x^2+x+\frac{5}{3}$. (*iv*) $h(x)=3x^2-2x+\frac{8}{3}$.

3. Determine whether any of the following quadratic functions has zeros by using only what has been done in this section (in particular, *don't* use the quadratic formula!): (*i*) $f(x)=-3x^2+4x-\frac{8}{3}$. (*ii*) $g(x)=\frac{3}{2}x^2-x-8$. (*iii*) $h(x)=-x^2+\frac{4}{5}x-2$. (*iv*) $f(x)=-\frac{3}{10}x^2+7x-35$.

4. Graph each of the functions in the preceding problems; indicate whether it is an up parabola or a down parabola, estimate the zeros of the function if there is any, and locate the vertex.

5. (a) Let G be the graph of $g(x)=x^2$. Let G' be the set obtained by changing each point (x,y) of G to $(x+5,y)$. Then G' is the graph of which function? (b) G as above, let G' be the set obtained by changing each point (x,y) of G to $(x,y-2)$. Then G' is the graph of which function? (c) G as above, let G' be the set obtained by changing each point (x,y) of G to $(x-3,y+2)$. Then G' is the graph of which function? (d) Let G be the graph of the function $h(x)=x^3$. If G' is the set obtained by changing each point (x,y) of G to $(x+1,y+2)$. Then G' is the graph of which function?

6. Write down the explicit transformation of the plane that realizes the similarity of the parabola $f(x)=x^2+2x$ with the standard parabola $f_0(x)=x^2$. Do the

same with the parabola $g(x) = -4x^2 + 12x - 9$.

7. In the text, it is asserted that the translation T which goes in the direction from a point (p, q) to the origin is the transformation of the plane given by $T(x, y) = (x - p, y - q)$. Explain why this is true.

8. Among all rectangles with a perimeter of s feet, which has the greatest area?

9. A hifi store sells only 35 CD players per month when the price is marked up to make a pofit of \$50 per player. Suppose for each \$2 decrease in the price, the store can sell 5 more players. What should the price be per player in order to maximize total monthly profit? What will the total profit be per month?

12 The Quadratic Formula and Applications (outlined)

Consider the problem:

> *A rectangle has a perimeter 180 linear units and an area of 1800 area units. What are its dimensions?*

If the length of one side is x linear units, then we have an equation in x, namely, $x(90 - x) = 180$. This is known as a **quadratic equation** in the variable x, and this problem illustrates how such equations arise naturally without the intervention of a quadratic function. We want to **solve this equation**, i.e., determine all the numbers x_0 so that $x_0(90 - x_0) = 180$, or, $-x_0^2 + 90x_0 - 180 = 0$. Such an x_0 is also called a **root** or a **zero** of the polynomial equation $-x^2 + 90x - 180 = 0$. From the perspective of the last section, solving this equation is the same as locating the zeros of the quadratic function $f(x) = -x^2 + 90x - 180$. From Theorem 1 of Section 11, we know that there are at most two zeros of f, and therefore the quadratic equation has at most two roots. The proof of Theorem 1 in facts gives an algorithm (completing the square) for locating these zeros if they exist. Thus an understanding of the quadratic function itself gives a comprehensive understanding of the nature of its zeros. Our study of the quadratic function therefore covers more ground than the solving of quadratic equations.

There is value, nevertheless, in studying a quadratic equation by itself without reference to the associated quadratic function in the school classroom, the primary reason being that "solving an equation" is easier to students than "understanding the concept of a function". For this reason, we now approach the quadratic polynomial $ax^2 + bx + c$ purely algebraically, without considering its graph, and show again that it has at most two zeros. Thus, given

$$ax^2 + bx + c = 0,$$

we start from the beginning and assume nothing about quadratic functions. *Suppose there is a root* s. By completing the square, we get

$$\left(s + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Because the right side is equal to the square of the number $s + \frac{b}{2a}$, *necessarily* $b^2 - 4ac \geq 0$. Hence we can take the square root of $b^2 - 4ac \geq 0$ to get

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This shows that *if there is a root* s, then s must be one of the numbers on the right side.

Now we must actually verify directly that both

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are roots of $ax^2 + bx + c = 0$. We do so by a direct computation (and making use of the identity $(x - y)(x + y) = x^2 - y^2$ in the process):

$$a\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)^2 + b\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) + c =$$

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)\left(a \cdot \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) + b\right) + c =$$

$$\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)\left(\left(\frac{-b \pm \sqrt{b^2-4ac}}{2}\right) + \frac{2b}{2}\right) + c =$$

$$\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)\left(\frac{b \pm \sqrt{b^2-4ac}}{2}\right) + c =$$

$$\frac{1}{4a}\left(-b \pm \sqrt{b^2-4ac}\right)\left(b \pm \sqrt{b^2-4ac}\right) + c =$$

$$\frac{1}{4a}\left(\pm\sqrt{b^2-4ac}\; - \;b\right)\left(\pm\sqrt{b^2-4ac}\; + \;b\right) + c =$$

$$\frac{1}{4a}\left((b^2-4ac) \; - \; b^2\right) + c \;=\; 0$$

We have therefore proved the **quadratic formula (QF)**:

Theorem 1 *If $b^2-4ac \geq 0$, then the equation $ax^2+bx+c=0$ has the following two roots (which coincide if $b^2-4ac=0$):*

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Conversely, if $ax^2+bx+c=0$ has a root, then $b^2-4ac \geq 0$.

Theorem 1 gives a direct proof that a quadratic equation has 0, 1, or 2 roots.

We continue with the study of the algebraic properties of the polynomial ax^2+bx+c. The following is the quadratic-polynomial analog of the division-with-remainder among whole numbers.

Theorem 2 *Given ax^2+bx+c and a number r, there is a linear polynomial $(ax+\beta)$ so that*

$$ax^2+bx+c = (ax+\beta)(x-r)+\gamma$$

for some constant γ and for all x.

The linear polynomial is called the **quotient** of the division, and γ the **remainder**. If this terminology seems strange, we review the division-with-remainder for whole numbers. When we say "59 divided by 7 has quotient 8 and remainder 3", we are making a statement about *multiplication*, namely, among all whole number multiples of 7, the one that is *closest to* 59 *but not exceeding it* is the 8th multiple (which is of course 56), and it misses 59 by 3 ($3 = 59 - (8 \times 7)$). The *quotient* is therefore this multiple, and the *remainder* is determined by the quotient once the quotient is clearly defined. Analogously, we look for a "multiple" of $(x - r)$ that is "closest" to $ax^2 + bx + c$, but in this case, "multiple" would have to mean "multiplication by another polynomial". Since the degrees add when polynomials are multiplied, and since $ax^2 + bx + c$ has degree 2, this "multiple" has degree either 1 or 0. But how close is "close"? We would obviously welcome a multiple of $(x - r)$ that matches $ax^2 + bx + c$ coefficient-by-coefficient for as many coefficients as possible. *It turns out that if we let* $\beta = ar + b$, *then* $(ax + \beta)(x - r)$ *would match* $ax^2 + bx + c$ *up to the coefficients of* x^2 *and* x, i.e., a and b. So the *remainder*

$$ax^2 + bx + c - (ax + \beta)(x - r)$$

in this case is just a number (or a **constant**, which is the common terminology). This is what γ is all about.

Theorem 2 can appear to be deceptively simple (or perhaps plain opaque), but through its many substantial consequences, such as those listed below, one gains a better understanding of it.

(a) A number r is a root of $ax^2 + bx + c = 0$ if and only if $(x - r)$ *divides* $ax^2 + bx + c$. (We say $(x - r)$ **divides** a polynomial $p(x)$ if $p(x)$ factors as $p(x) = q(x)(x - r)$ for some polynomial $q(x)$. Thus to say $(x - r)$ divides $ax^2 + bx + c$ means precisely that $\gamma = 0$ in Theorem 2).

(b) If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, then for all x,

$$ax^2 + bx + c = a(x - r_1)(x - r_2)$$

(This identity may seem *too* obvious to you because some may even regard this as the *definition* of the roots r_1 and r_2. But it is not obvious. What it says is that, *if* r_1 *and* r_2 *are any two numbers which satisfy* $a(r_1)^2 + br_1 + c = 0$ *and* $a(r_2)^2 + br_2 + c = 0$, *then the equality*

$ax^2 + bx + c = a(x - r_1)(x - r_2)$ *is valid for all* x. Such a striking statement undoubtedly demands an explanation. One way is to get explicit expressions of r_1 and r_2 in terms of a, b, and c by appealing to QF, and then explcitly compute $a(x - r_1)(x - r_2)$ to show that it is equal to $a(r_1)^2 + br_1 + c$. It is also possible to obtain a more conceptual, but also more sophisticated proof of this fact by a judicious use of (a).)

(c) Let r_1 and r_2 be the roots of $x^2 + bx + c = 0$, then

$$b = -(r_1 + r_2) \quad \text{and} \quad c = r_1 r_2$$

(This needs (b) and Problem 2 in the following Exercises.)

(d) If a quadratic polynomial $ax^2 + bx + c$ can be factored into a product of linear polynomials, then the factorization is $a(x - r_1)(x - r_2)$, where r_1 and r_2 are the roots of $ax^2 + bx + c = 0$. (In short, the QF provides a factorization for all quadratic polynomials.)

Activity Factor $10x^2 - 13x - 30$.

Here is a typical example of how all this knowledge about quadratic functions and quadratic equations is put to use in solving word problems.

Example If an object is thrown from a height of h meters from the ground with an initial velocity of v_0 m/sec, then its distance $f(t)$ above the ground t seconds after it is thrown (in meters) is

$$f(t) = -4.9t^2 + v_0 t + h$$

(This follows from Newton's second law.) Now if $h = 20$ meters and $v_0 = 2$ m/sec, what is the highest point of the object above the ground, when does it get there, and when does it hit the ground?

The highest point above the ground is the maximum of the quadratic function $f(t) = -4.9t^2 + 2t + 20$, which is $20\frac{10}{49}$ meters. The object hits the ground after t_0

seconds if $f(t_0) = 0$. Thus solving $-4.9t^2 + 2t + 20 = 0$, we get $t_0 = 2.2$ seconds, approximately.

EXERCISES

1. (Everybody must do this problem.) Starting with $ax^2 + bx + c = 0$, give a self-contained and coherent derivation of the quadratic formula.

2. Suppose we have two quadratic functions $f(x) = ax^2 + bx + c$ and $g(x) = a'x^2 + b'x + c'$, and suppose $f(x) = g(x)$ for *all* x. Prove that $a = a'$, $b = b'$, and $c = c'$.

3. Solve each of the following quadratic equations: (*i*) $6x^2 - 13x = 5$. (*ii*) $s^2 = 6s + 3$. (*iii*) $-3x^2 + 4\sqrt{3}\,x - 4 = 0$. (*iv*) $\frac{1}{6}x^2 + x - \frac{1}{3} = 0$. (*v*) $x^2 + \frac{1}{4}x = \frac{1}{4}$. (*vi*) $-t^2 - \sqrt{13}t + 3 = 0$. (*vii*) $-x^2 - \sqrt{13}x = 3$. (*viii*) $\sqrt{180}x^2 + 7x = \sqrt{5}$. (*ix*) $x^2 - 3\sqrt{2}x + 4 = 0$. (*x*) $3s^2 - \frac{4}{\sqrt{3}}s + \frac{1}{5} = 0$.

4. Use mental math to decide whether each of the following quadratic functions has two distinct zeros, only one zero, or no zero: (*i*) $f(x) = 215x^2 - 87x + 21$. (*ii*) $f(s) = 5s^2 + \frac{22}{3}s + 7$. (*iii*) $g(x) = -83x^2 + 5.2x - \frac{9}{76}$. (*iv*) $h(s) = \frac{1}{2}s^2 - \sqrt{\frac{15}{7}}\,s + 1.5$. (*v*) $h(x) = 3.2x^2 - 9.5x + 22$.

5. In the quadratic function $f(x) = 3x^2 - ux + 2$, u is a number. For what value of u would f have two zeros? One zero? No zero?

6. In the quadratic function $g(x) = 3x^2 + x + 2u$, u is a number. For what value of u would g have two zeros? One zero? No zero?

7. Factor the following polynomials: (*i*) $30x^2 + 13x - 36$. (*ii*) $5x^2 - x - 7$. (*iii*) $105x^2 + 766x + 72$. (*iv*) $4x^2 - \frac{11}{6}x - 3$.

8. Find a quadratic polynomial with the indicated pair of zeros: (*i*) 2 and -5. (*ii*) $-\frac{3}{5}$ and 4. (*iii*) $\frac{3}{4}$ and $\frac{7}{3}$. (*iv*) $2 + \sqrt{5}$ and $2 - \sqrt{5}$. (*v*) $\sqrt{6}$ and 5. (*vi*) $\sqrt{2}$ and $\sqrt{3}$. (*vii*) $\frac{2}{3} + \sqrt{5}$ and $\frac{2}{3} - \sqrt{5}$. (*viii*) $1 - \frac{\sqrt{10}}{3}$ and $1 + \frac{\sqrt{10}}{3}$.

9. Find two numbers whose difference is 7, and the difference of their cubes is 721.

10. A merchant has a cask full of wine. He draws out 6 gallons and fills the cask with water. Again he draws out 6 gallons, and fills the cask with water. There are now 25 gallons of pure wine in the cask. How many gallons does the cask hold?

11. Two workmen can do a piece of work (think of painting a house) together in 6 days. In how many days can each do it alone if it takes one of them 5 days longer than the other? (Assume both work at a constant rate.)

12. A train makes a run of 120 miles. A second train starts one hour later and, traveling at 6 mph faster, reaches the end of the same run 20 minutes later than the first train. Find the time of the run of each train. (Assume that both trains make the run at a constant rate.)

13. A tank can be filled by the larger of two faucets in 5 hours less time than by the smaller one. It is filled by them both together in 6 hours. If the water flows from the faucets at a constant rate, how many hours will it take to fill the tank by each faucet separately?

14. A line passing through the points $(t, 2)$ and $(3, t)$ has slope $2t$. What is t?