

MATRIX FORMULAE AND SKEIN RELATIONS FOR CLUSTER ALGEBRAS FROM SURFACES

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ABSTRACT. This paper concerns cluster algebras with principal coefficients $\mathcal{A}_\bullet(S, M)$ associated to bordered surfaces (S, M) , and is a companion to a concurrent work of the authors with Schiffler [MSW2]. Given any (generalized) arc or loop in the surface – with or without self-intersections – we associate an element of (the fraction field of) $\mathcal{A}_\bullet(S, M)$, using products of elements of $PSL_2(\mathbb{R})$. We give a direct proof that our matrix formulas for arcs and loops agree with the combinatorial formulas for arcs and loops in terms of matchings, which were given in [MSW, MSW2]. Finally, we use our matrix formulas to prove *skein relations* for the cluster algebra elements associated to arcs and loops. Our matrix formulas and skein relations generalize prior work of Fock and Goncharov [FG1, FG2, FG3], who worked in the coefficient-free case. The results of this paper will be used in [MSW2] in order to show that certain collections of arcs and loops comprise a vector-space basis for $\mathcal{A}_\bullet(S, M)$.

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1. INTRODUCTION

Since their introduction by Fomin and Zelevinsky [FZ1], cluster algebras have been related to diverse areas of mathematics such as total positivity, quiver representations, tropical geometry, Lie theory, Poisson geometry, and Teichmüller theory. There is an important class of cluster algebras arising from *bordered surfaces with marked points*, introduced by Fomin, Shapiro, and Thurston in [FST] (which in turn generalized work of Fock and Goncharov [FG1, FG2] and Gekhtman, Shapiro, and Vainshtein [GSV05]), and further developed in [FT]. Such cluster algebras are interesting for several reasons: they comprise “most” of the mutation-finite skew-symmetric cluster algebras [FeShTu], and also they can be thought of as coordinate rings for the *decorated Teichmüller space* of (S, M) . More specifically, the cluster variable associated to an arc in (S, M) corresponds to the *Penner coordinate* [Pen] or *exponentiated lambda length* of that arc.

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Because of the interpretation in terms of decorated Teichmüller space, one can express the Laurent expansion of a cluster variable associated to an arc in terms of a product of matrices in $PSL_2(\mathbb{R})$; this was explained by Fock and Goncharov in [FG1, FG3] in the coefficient-free case. On the other hand, in our previous work with Schiffler [MSW], we gave formulas for the Laurent expansion of every cluster variable in terms of perfect matchings of certain planar graphs. We worked in the generality of principal coefficients, and as a consequence proved the positivity conjecture for cluster algebras from surfaces whose coefficient system is of geometric type.

Besides the positivity conjecture, a main open problem about cluster algebras is to construct (vector-space) bases which have “good” positivity properties. It is expected (and in many cases proved) that the cluster monomials are linearly independent [CK, FZ4, DWZ], and should be a part of such a basis, but in general, one needs some extra elements to complete the cluster monomials to a basis. In concurrent work [MSW2], we construct bases for the cluster algebras $\mathcal{A}_\bullet(S, M)$ from surfaces, with principal coefficients with respect to a seed T . In order to construct these bases and prove that they span the cluster algebra, we need to associate cluster algebra elements X_γ not only to arcs but also to generalized arcs and closed loops γ (with self-intersections allowed). Our formulas in [MSW2] for such elements also involve matchings of certain graphs, but in the case of closed loops, these graphs lie on a Möbius strip or an annulus.

In this paper we associate cluster algebra elements χ_γ to generalized arcs and closed loops γ using products of matrices in $PSL_2(\mathbb{R})$. We prove that this definition using matrices agrees with the definition using matchings, that is, that $\chi_\gamma = X_\gamma$ for any generalized arc or closed loop γ . In order to prove this, we prove a general combinatorial result which explains how to enumerate matchings of a *snake* or *band* graph using products of 2×2 matrices. We then prove *skein relations*, which allow us to multiply elements χ_α and χ_β , where α and β are generalized arcs or closed loops. Topologically, these skein relations resolve crossings in the corresponding arcs or loops.

Note that our matrix formulas and skein relations generalize prior work of Fock and Goncharov [FG1, FG2, FG3], who worked in the coefficient-free case. However, it is crucial for us to work in the context of principal coefficients, because our proofs in [MSW2] use the notion of **g-vectors**, which are defined in the case of principal coefficients. Matrix formulas have also appeared in related literature, including [ADSS, ARS, BW, BW2, Pen2, Pen3].

This paper is organized as follows. We give background on cluster algebras from surfaces in Section 2. In Sections 3 and 4 we give our combinatorial formulas from [MSW2] and our matrix formulas for the cluster algebra elements associated to generalized arcs and closed loops. In Section 5 we prove that these two formulas coincide. Finally, in Section 6 we prove the skein relations.

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2. CLUSTER ALGEBRAS ARISING FROM SURFACES

We assume that the reader is familiar with the notion of a cluster algebra and the terminology of [FZ4], including principal coefficients and F -polynomials. We will begin by providing background on cluster algebras from surfaces.

Building on work of Fock and Goncharov [FG1, FG2], and of Gekhtman, Shapiro and Vainshtein [GSV05], Fomin, Shapiro and Thurston [FST] associated a cluster algebra to any *bordered surface with marked points*. In this section we will recall that construction, as well as further results of Fomin and Thurston [FT].

Definition 2.1 (*Bordered surface with marked points*). Let S be a connected oriented 2-dimensional Riemann surface with (possibly empty) boundary. Fix a nonempty set M of *marked points* in the closure of S with at least one marked point on each boundary component. The pair (S, M) is called a *bordered surface with marked points*. Marked points in the interior of S are called *punctures*.

For technical reasons, we require that (S, M) is not a sphere with one, two or three punctures; a monogon with zero or one puncture; or a bigon or triangle without punctures.

2.1. Ideal triangulations and tagged triangulations.

Definition 2.2 (*Ordinary arcs*). An *arc* γ in (S, M) is a curve in S , considered up to isotopy, such that: the endpoints of γ are in M ; γ does not cross itself, except that its endpoints may coincide; except for the endpoints, γ is disjoint from M and from the boundary of S ; and γ does not cut out an unpunctured monogon or an unpunctured bigon.

Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are *boundary segments*. Note that boundary segments are not ordinary arcs.

Definition 2.3 (*Crossing numbers and compatibility of ordinary arcs*). For any two arcs γ, γ' in S , let $e(\gamma, \gamma')$ be the minimal number of crossings of arcs α and α' , where α and α' range over all arcs isotopic to γ and γ' , respectively. We say that arcs γ and γ' are *compatible* if $e(\gamma, \gamma') = 0$.

Definition 2.4 (*Ideal triangulations*). An *ideal triangulation* is a maximal collection of pairwise compatible arcs (together with all boundary segments). The arcs of a triangulation cut the surface into *ideal triangles*.

There are two types of ideal triangles: triangles that have three distinct sides and triangles that have only two. The latter are called *self-folded* triangles. Note that a self-folded triangle consists of an arc ℓ whose endpoints coincide, together with an arc r to an enclosed puncture which we dub a *radius*. Following the notation of [GSV11], we will refer to an arc ℓ cutting out a once-punctured monogon as a *noose*.

Definition 2.5 (*Ordinary flips*). Ideal triangulations are connected to each other by sequences of *flips*. Each flip replaces a single arc γ in a triangulation T by a (unique) arc $\gamma' \neq \gamma$ that, together with the remaining arcs in T , forms a new ideal triangulation.

In a cluster algebra associated to an unpunctured surface, the cluster variables correspond to arcs, the clusters to triangulations, and the mutations to flips. However, in a cluster algebra associated to a surface with punctures, one needs to generalize the notion of arc and triangulation in order to get a combinatorial framework that encodes the whole cluster complex. In [FST], the authors introduced *tagged arcs* and *tagged triangulations*, and showed that they are in bijection with cluster variables and clusters.

Definition 2.6 (*Tagged arcs*). A *tagged arc* is obtained by taking an arc that is not a noose and marking (“tagging”) each of its ends in one of two ways, *plain* or *notched*, so that the following conditions are satisfied:

- an endpoint lying on the boundary of S must be tagged plain

- if the endpoints of an arc coincide, then they must be tagged in the same way.

Definition 2.7 (*Representing ordinary arcs by tagged arcs*). One can represent an ordinary arc β by a tagged arc $\iota(\beta)$ as follows. If β is not a noose, then $\iota(\beta)$ is simply β with both ends tagged plain. Otherwise, β is a noose based at point a , which contains the puncture b inside it. Let α be the unique arc connecting a and b and compatible with β . Then $\iota(\beta)$ is obtained by tagging α plain at a and notched at b .

Definition 2.8 (*Compatibility of tagged arcs*). Tagged arcs α and β are called *compatible* if and only if the following properties hold:

- the arcs α^0 and β^0 obtained from α and β by forgetting the taggings are compatible;
- if $\alpha^0 = \beta^0$ then at least one end of α must be tagged in the same way as the corresponding end of β ;
- $\alpha^0 \neq \beta^0$ but they share an endpoint a , then the ends of α and β connecting to a must be tagged in the same way.

Definition 2.9 (*Tagged triangulations*). A maximal (by inclusion) collection of pairwise compatible tagged arcs is called a *tagged triangulation*.

2.2. From surfaces to cluster algebras. One can associate an exchange matrix and hence a cluster algebra to any bordered surface (S, M) [FST].

Definition 2.10 (*Signed adjacency matrix of an ideal triangulation*). Choose any ideal triangulation T , and let $\tau_1, \tau_2, \dots, \tau_n$ be the n arcs of T . For any triangle Δ in T which is not self-folded, we define a matrix $B^\Delta = (b_{ij}^\Delta)_{1 \leq i \leq n, 1 \leq j \leq n}$ as follows.

- $b_{ij}^\Delta = 1$ and $b_{ji}^\Delta = -1$ in the following cases:
 - (a) τ_i and τ_j are sides of Δ with τ_j following τ_i in the clockwise order;
 - (b) τ_j is a radius in a self-folded triangle enclosed by a noose τ_ℓ , and τ_i and τ_ℓ are sides of Δ with τ_ℓ following τ_i in the clockwise order;
 - (c) τ_i is a radius in a self-folded triangle enclosed by a noose τ_ℓ , and τ_ℓ and τ_j are sides of Δ with τ_j following τ_ℓ in the clockwise order;
- $b_{ij}^\Delta = 0$ otherwise.

Then define the matrix $B_T = (b_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ by $b_{ij} = \sum_{\Delta} b_{ij}^\Delta$, where the sum is taken over all triangles in T that are not self-folded.

Note that B_T is skew-symmetric and each entry b_{ij} is either $0, \pm 1$, or ± 2 , since every arc τ is in at most two triangles.

Theorem 2.11. [FST, Theorem 7.11] and [FT, Theorem 5.1] *Fix a bordered surface (S, M) and let \mathcal{A} be the cluster algebra associated to the signed adjacency matrix of a tagged triangulation (see [FST, Definition 9.18]). Then the (unlabeled) seeds Σ_T of \mathcal{A} are in bijection with tagged triangulations T of (S, M) , and the cluster variables are in bijection with the tagged arcs of (S, M) (so we can denote each by x_γ , where γ is a tagged arc). Moreover, each seed in \mathcal{A} is uniquely determined by its cluster. Furthermore, if a tagged triangulation T' is obtained from another tagged triangulation T by flipping a tagged arc $\gamma \in T$ and obtaining γ' , then $\Sigma_{T'}$ is obtained from Σ_T by the seed mutation replacing x_γ by $x_{\gamma'}$.*

Remark 2.12. By a slight abuse of notation, if γ is an ordinary arc which is not a noose (so that the tagged arc $\iota(\gamma)$ is obtained from γ by tagging both ends plain), we will often write x_γ instead of $x_{\iota(\gamma)}$.

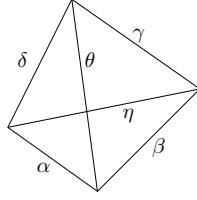


FIGURE 1. Quadrilateral illustrating Proposition 2.14

Remark 2.13. In this paper we will typically fix a triangulation $T = (\tau_1, \dots, \tau_n)$ of (S, M) . The initial cluster variables correspond to the arcs τ_i , and we will denote them by either x_{τ_i} or x_i . Similarly, we will denote the initial coefficient variables by either y_{τ_i} or y_i .

Given a surface (S, M) with a puncture p and a tagged arc γ , we let both $\gamma^{(p)}$ and γ^p denote the arc obtained from γ by changing its notching at p . (So if γ is not incident to p , $\gamma^{(p)} = \gamma$.) If p and q are two punctures, we let $\gamma^{(pq)}$ denote the arc obtained from γ by changing its notching at both p and q . Given a tagged triangulation T of S , we let T^p denote the tagged triangulation obtained from T by replacing each $\gamma \in T$ by $\gamma^{(p)}$.

Besides labeling cluster variables of $\mathcal{A}(B_T)$ by x_τ , where τ is a tagged arc of (S, M) , we will also make the following conventions:

- If ℓ is an unnotched noose with endpoints at q cutting out a once-punctured monogon containing puncture p and radius r , then we set $x_\ell = x_r x_{r^{(p)}}$.
- If β is a boundary segment, we set $x_\beta = 1$.

The exchange relation corresponding to a flip in an ideal triangulation is called a *generalized Ptolemy relation*. It can be described as follows.

Proposition 2.14. [FT] *Let $\alpha, \beta, \gamma, \delta$ be arcs (including nooses) or boundary segments of (S, M) which cut out a quadrilateral; we assume that the sides of the quadrilateral, listed in cyclic order, are $\alpha, \beta, \gamma, \delta$. Let η and θ be the two diagonals of this quadrilateral; see Figure 1. Then*

$$(2.1) \quad x_\eta x_\theta = Y x_\alpha x_\gamma + Y' x_\beta x_\delta$$

for some coefficients Y and Y' .

Proof. This follows from the interpretation of cluster variables as *lambda lengths* and the Ptolemy relations for lambda lengths [FT, Theorem 7.5 and Proposition 6.5]. \square

Note that some sides of the quadrilateral in Proposition 2.14 may be glued to each other, changing the appearance of the relation. There are also generalized Ptolemy relations for tagged triangulations, see [FT, Definition 7.4].

2.3. Principal coefficients. In this paper we work with cluster algebras $\mathcal{A} = \mathcal{A}_\bullet(B_T)$ with *principal coefficients* with respect to the seed Σ_T . Concretely, these are defined by using a $2n \times n$ extended exchange matrix whose top $n \times n$ part is B_T , and whose bottom $n \times n$ part is the identity matrix. See [FZ4] for more details. When B_T comes from a triangulation of a bordered surface (S, M) , one can compute the coefficients using Thurston's theory of measured laminations; see [FT] and also [FG3]. Concretely, one can compute principal coefficients with respect to the seed Σ_T (where T is a tagged triangulation) using the *shear coordinates* with respect to the n *elementary laminations* associated to the n tagged arcs of T [FT, Definition 16.2].

For a cluster algebra \mathcal{A} with exchange matrix B_T and an arbitrary semifield \mathbb{P} of coefficients, Laurent expansions of cluster variables can be computed from the formula in $\mathcal{A}_\bullet(B, T)$ by the following theorem.

Theorem 2.15. [FZ4, Theorem 3.7] *Let \mathcal{A} be a cluster algebra over an arbitrary semifield \mathbb{P} and contained in the ambient field \mathcal{F} , with a seed at an initial vertex t_0 given by*

$$((x_1, \dots, x_n), (y_1^*, \dots, y_n^*), B^0).$$

Then the cluster variables in \mathcal{A} can be expressed as follows:

$$(2.2) \quad x_{\ell; t} = \frac{X_{\ell; t}^{B^0; t_0} |_{\mathcal{F}}(x_1, \dots, x_n; y_1^*, \dots, y_n^*)}{F_{\ell; t}^{B^0; t_0} |_{\mathbb{P}}(y_1^*, \dots, y_n^*)}.$$

An important class of semifields \mathbb{P} are the *tropical semifields*.

Definition 2.16 (*Tropical semifield*). Let $\text{Trop}(u_1, \dots, u_m)$ be an abelian group (written multiplicatively) freely generated by the u_j . We define \oplus in $\text{Trop}(u_1, \dots, u_m)$ by

$$(2.3) \quad \prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)},$$

and call $(\text{Trop}(u_1, \dots, u_m), \oplus, \cdot)$ a *tropical semifield*.

A cluster algebra is of *geometric type* whenever \mathbb{P} is such a semifield. Notice that in this case, the denominator of equation (2.2) is a monomial.

3. MATCHING FORMULAS FOR GENERALIZED ARCS AND CLOSED LOOPS

In this section we fix a bordered surface (S, M) , an ideal triangulation $T = (\tau_1, \dots, \tau_n)$, and the cluster algebra $\mathcal{A} = \mathcal{A}_\bullet(B_T)$ with principal coefficients with respect to the seed Σ_T . We will explain how to associate an element X_γ^T of (the fraction field of) \mathcal{A} to each generalized arc or closed loop γ in (S, M) . Our definition of the cluster algebra element associated to a closed loop comes from joint work of the authors together with Ralf Schiffler [MSW2].

Each element will be defined as a sum over matchings of a graph. When γ is an ordinary arc, X_γ^T recovers the cluster expansion formula for the cluster variable associated to γ , with respect to Σ_T [MSW].

Definition 3.1 (*Generalized arcs*). A *generalized arc* in (S, M) is a curve γ in S such that: the endpoints of γ are in M ; except for the endpoints, γ is disjoint from M and the boundary of S ; γ does not cut out an unpunctured monogon or an unpunctured bigon. Note that we allow a generalized arc to cross itself a finite number of times. We consider generalized arcs up to isotopy (of immersed arcs). In particular, an isotopy cannot remove a contractible kink from a generalized arc.

Definition 3.2 (*Closed loops*). A closed loop in (S, M) is a closed curve γ in S which is disjoint from M and the boundary of S . We allow a closed loop to have a finite number of self-intersections. As in Definition 3.1, we consider closed loops up to isotopy.

Definition 3.3. A closed loop in (S, M) is called *essential* if:

- it is not contractible nor contractible onto a single puncture;
- it does not have self-intersections.

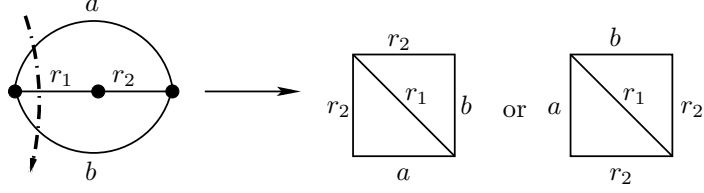


FIGURE 2. Possible tiles corresponding to crossing radius of a bigon

3.1. Tiles. Let γ be a generalized arc in (S, M) which is not in T . Choose an orientation on γ , let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. We denote by $s = p_0, p_1, p_2, \dots, p_{d+1} = t$ the points of intersection of γ and T in order. Let τ_{i_j} be the arc of T containing p_j , and let Δ_{j-1} and Δ_j be the two ideal triangles in T on either side of τ_{i_j} .

To each p_j we associate a *tile* G_j , an edge-labeled triangulated quadrilateral (see the right-hand-side of Figure 2), which is defined to be the union of two edge-labeled triangles Δ_1^j and Δ_2^j glued at an edge labeled τ_{i_j} . The triangles Δ_1^j and Δ_2^j are determined by Δ_{j-1} and Δ_j as follows.

If neither Δ_{j-1} nor Δ_j is self-folded, then they each have three distinct sides (though possibly fewer than three vertices), and we define Δ_1^j and Δ_2^j to be the ordinary triangles with edges labeled as in Δ_{j-1} and Δ_j . We glue Δ_1^j and Δ_2^j at the edge labeled τ_{i_j} , so that the orientations of Δ_1^j and Δ_2^j both either agree or disagree with those of Δ_{j-1} and Δ_j ; this gives two possible planar embeddings of a graph G_j which we call an *ordinary tile*.

If one of Δ_{j-1} or Δ_j is self-folded, then in fact T must have a local configuration of a bigon (with sides a and b) containing a radius r incident to a puncture p inscribed inside a loop ℓ , see Figure 3. If γ has no self-intersection as it passes through the self-folded triangle, then γ must either

- (1) intersect the loop ℓ and terminate at puncture p , or
- (2) intersect the loop ℓ , radius r and then ℓ again.

In case (1), we associate to p_j (the intersection point with ℓ) an *ordinary tile* G_j consisting of a triangle with sides $\{a, b, \ell\}$ which is glued along diagonal ℓ to a triangle with sides $\{\ell, r, r\}$. As before there are two possible planar embeddings of G_j .

In case (2), we associate to the triple of intersection points p_{j-1}, p_j, p_{j+1} a union of tiles $G_{j-1} \cup G_j \cup G_{j+1}$, which we call a *triple tile*, based on whether γ enters and exits through different sides of the bigon or through the same side. These graphs are defined by the first three examples in Figure 3 (each possibility is denoted in boldface within a concatenation of five tiles). Note that in each case there are two possible planar embeddings of the triple tile. We call the tiles G_{j-1} and G_{j+1} within the triple tile *ordinary tiles*.

On the other hand, if γ has a self-intersection inside the self-folded triangle, then the local configuration in the associated graph is as in the last two examples of Figure 3.

Definition 3.4 (*Relative orientation*). Given a planar embedding \tilde{G}_j of an ordinary tile G_j , we define the *relative orientation* $\text{rel}(\tilde{G}_j, T)$ of \tilde{G}_j with respect to T to be ± 1 , based on whether its triangles agree or disagree in orientation with those of T .

Note that in Figure 3, the lowest tile in each of the five graphs in the middle (respectively, rightmost) column has relative orientation $+1$ (respectively, -1), as indicated by the signs in the figures. Also note that by construction, the planar embedding of a triple tile $\tilde{G}_{j-1} \cup \tilde{G}_j \cup \tilde{G}_{j+1}$ satisfies $\text{rel}(\tilde{G}_{j-1}, T) = \text{rel}(\tilde{G}_{j+1}, T)$.

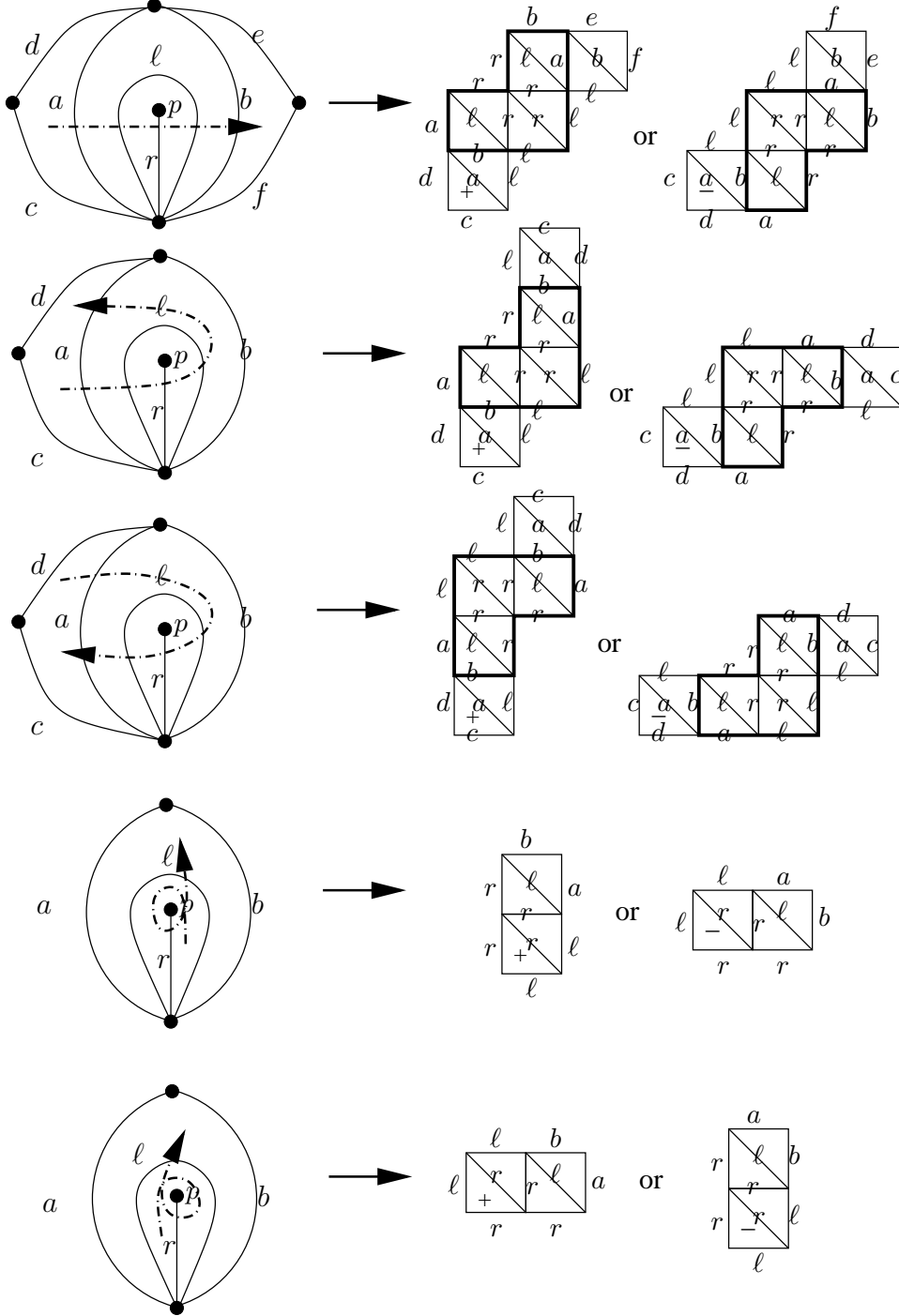


FIGURE 3. Possible triple tiles for crossing a self-folded triangle

Definition 3.5. Using the notation above, the arcs τ_{i_j} and $\tau_{i_{j+1}}$ form two edges of a triangle Δ_j in T . Define a_j to be the third arc in this triangle if Δ_j is not self-folded, and to be the radius in Δ_j otherwise.

3.2. The snake graph $G_{T,\gamma}$. We now recursively glue together the tiles G_1, \dots, G_d in order from 1 to d , subject to the following conditions.

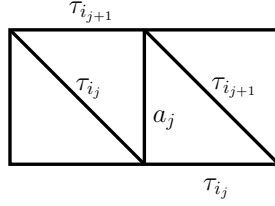


FIGURE 4. Gluing tiles \tilde{G}_j and \tilde{G}_{j+1} along the edge labeled a_j

- (1) Triple tiles must stay glued together as in Figure 3.
- (2) For two adjacent ordinary tiles, each of which may be an exterior tile of a triple tile, we glue G_{j+1} to \tilde{G}_j along the edges labeled a_j , choosing a planar embedding \tilde{G}_{j+1} for G_{j+1} so that $\text{rel}(\tilde{G}_{j+1}, T) \neq \text{rel}(\tilde{G}_j, T)$. See Figure 4.

After gluing together the d tiles, we obtain a graph (embedded in the plane), which we denote $\overline{G}_{T,\gamma}$. Let $G_{T,\gamma}$ be the graph obtained from $\overline{G}_{T,\gamma}$ by removing the diagonal in each tile. We call $G_{T,\gamma}$ the *snake graph* associated to γ with respect to T . Figure 3 gives examples of a dotted arc γ and the corresponding graph $\overline{G}_{T,\gamma}$. Each γ intersects T five times, so each $\overline{G}_{T,\gamma}$ has five tiles.

3.3. Definition of cluster algebra elements associated to generalized arcs. Recall that if τ is a boundary segment then $x_\tau = 1$, and if τ is a noose cutting out a once-punctured monogon with radius r and puncture p , then $x_\tau = x_r x_{r(p)}$.

Definition 3.6 (*Crossing Monomial*). If γ is an ordinary arc and $\tau_{i_1}, \tau_{i_2}, \dots, \tau_{i_d}$ is the sequence of arcs in T which γ crosses, we define the *crossing monomial* of γ with respect to T to be

$$\text{cross}(T, \gamma) = \prod_{j=1}^d x_{\tau_{i_j}}.$$

Definition 3.7 (*Perfect matchings and weights*). A *perfect matching* of a graph G is a subset P of the edges of G such that each vertex of G is incident to exactly one edge of P . If the edges of a perfect matching P of $G_{T,\gamma}$ are labeled $\tau_{j_1}, \dots, \tau_{j_r}$, then we define the *weight* $x(P)$ of P to be $x_{\tau_{j_1}} \dots x_{\tau_{j_r}}$.

Definition 3.8 (*Minimal and Maximal Matchings*). By induction on the number of tiles it is easy to see that $G_{T,\gamma}$ has precisely two perfect matchings which we call the *minimal matching* $P_- = P_-(G_{T,\gamma})$ and the *maximal matching* $P_+ = P_+(G_{T,\gamma})$, which contain only boundary edges. To distinguish them, if $\text{rel}(\tilde{G}_1, T) = 1$ (respectively, -1), we define e_1 and e_2 to be the two edges of $\overline{G}_{T,\gamma}$ which lie in the counter-clockwise (respectively, clockwise) direction from the diagonal of \tilde{G}_1 . Then P_- is defined as the unique matching which contains only boundary edges and does not contain edges e_1 or e_2 . P_+ is the other matching with only boundary edges.

For an arbitrary perfect matching P of $G_{T,\gamma}$, we let $P_- \ominus P$ denote the symmetric difference, defined as $P_- \ominus P = (P_- \cup P) \setminus (P_- \cap P)$.

Lemma 3.9. [MS, Theorem 5.1] *The set $P_- \ominus P$ is the set of boundary edges of a (possibly disconnected) subgraph G_P of $G_{T,\gamma}$, which is a union of cycles. These cycles enclose a set of tiles $\cup_{j \in J} G_{i_j}$, where J is a finite index set.*

We use this decomposition to define *height monomials* for perfect matchings. Note that the exponents in the height monomials defined below coincide with the definition of height

functions given in [Pr1] for perfect matchings of bipartite graphs, based on earlier work of [CL], [EKLP], and [Th90] for domino tilings.

Definition 3.10 (*Height Monomial and Specialized Height Monomial*). Let $T = \{\tau_1, \tau_2, \dots, \tau_n\}$ be an ideal triangulation of (S, M) and γ be an ordinary arc of (S, M) . By Lemma 3.9, for any perfect matching P of $G_{T, \gamma}$, $P \ominus P_-$ encloses the union of tiles $\cup_{j \in J} G_{i_j}$. We define the *height monomial* $h(P)$ of P by

$$h(P) = \prod_{k=1}^n y_{\tau_k}^{m_k},$$

where m_k is the number of tiles in $\cup_{j \in J} G_{i_j}$ whose diagonal is labeled τ_k .

We define the *specialized height monomial* $y(P)$ of P to be the specialization $\Phi(h(P))$, where Φ is defined below.

$$\Phi(y_{\tau_i}) = \begin{cases} y_{\tau_i} & \text{if } \tau_i \text{ is not a side of a self-folded triangle;} \\ \frac{y_r}{y_{r(p)}} & \text{if } \tau_i \text{ is a radius } r \text{ to puncture } p \text{ in a self-folded triangle;} \\ y_{r(p)} & \text{if } \tau_i \text{ is a noose in a self-folded triangle with radius } r \text{ to puncture } p. \end{cases}$$

Definition 3.11. For an arc $\tau \in T$ and a puncture p , let $e_p(\tau)$ denote the number of ends of τ incident to p (so if both ends of τ are at p , $e_p(\tau) = 2$). Additionally, if γ is a generalized arc or loop, then $e(\tau, \gamma)$ denotes the number of crossings between τ and γ .

Definition 3.12. Let (S, M) be a surface, $T = (\tau_1, \dots, \tau_n)$ an ideal triangulation, and $\mathcal{A} = \mathcal{A}_\bullet(B_T)$ be the cluster algebra with principal coefficients with respect to Σ_T . Let γ be a generalized arc and let $G_{T, \gamma}$ denote its snake graph. We will define a Laurent polynomial X_γ^T which lies in (the fraction field of) \mathcal{A} , as well as a Laurent polynomial F_γ^T obtained from X_γ^T by specialization.

- (1) If γ cuts out a contractible monogon, then X_γ^T is equal to zero.
- (2) If γ has a contractible kink, let $\bar{\gamma}$ denote the corresponding tagged arc with this kink removed, and define $X_\gamma^T = (-1)X_{\bar{\gamma}}^T$.
- (3) Otherwise, define

$$X_\gamma^T = \frac{1}{\text{cross}(T, \gamma)} \sum_P x(P)y(P),$$

where the sum is over all perfect matchings P of $G_{T, \gamma}$.

Define F_γ^T to be the Laurent polynomial obtained from X_γ^T by specializing all the x_{τ_i} to 1.

Theorem 3.13. [MSW, Thm 4.9] *Use the notation of Definition 3.12. When γ is an arc (with no self-intersections), X_γ^T is the Laurent expansion of the cluster variable $x_\gamma \in \mathcal{A}$, with respect to the seed Σ_T , and F_γ^T is its F-polynomial.*

Remark 3.14. In a few cases (see Lemma 3.22), the elements F_γ^T may not be polynomials, only Laurent polynomials. However, it's known that a cluster algebra with principal coefficients with respect to the seed $((x_1, \dots, x_n), (y_1, \dots, y_n), B)$ is contained in $\mathbb{Z}[x_1^\pm, \dots, x_n^\pm; y_1, \dots, y_n]$ [FZ4, Proposition 3.6]. Motivated by this fact and also Theorem 2.15, we see that we should define the cluster algebra element associated to a generalized arc as in Definition 3.15.

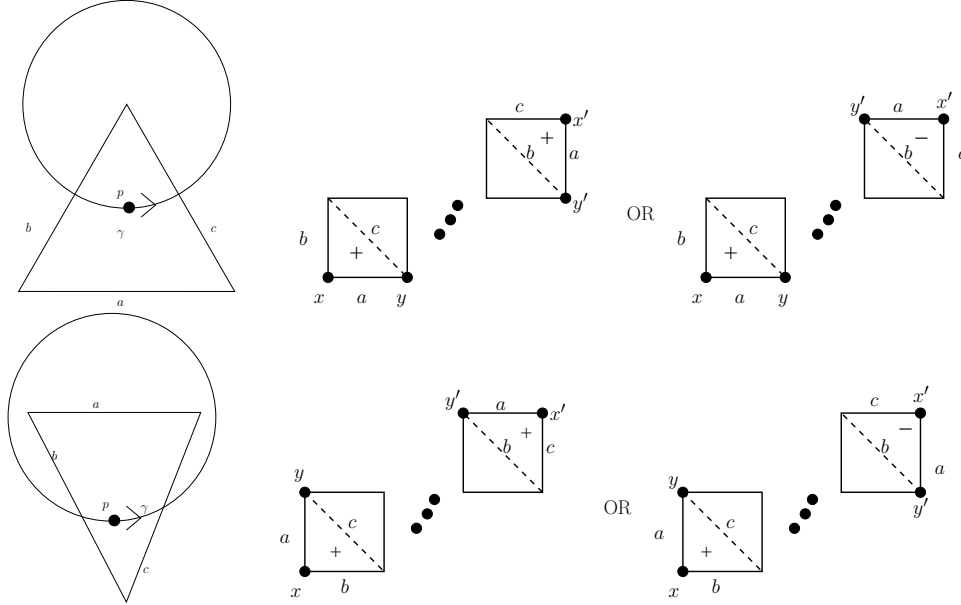


FIGURE 5. (Left): Triangle containing p along closed loop γ . (Right): Corresponding Band graph (with $x \sim x'$, $y \sim y'$) depending on whether γ crosses an odd or even number of arcs. The +’s and -’s denote relative orientation of each tile

Definition 3.15. Let \mathbb{P} be the tropical semifield $Trop(y_1, \dots, y_n)$. We define the cluster algebra element $x_\gamma^T \in \mathcal{A} = \mathcal{A}_\bullet(B_t)$ associated to a generalized arc by

$$(3.1) \quad x_\gamma^T = \frac{X_\gamma^T}{F_\gamma^T|_{\mathbb{P}(y_1, \dots, y_n)}}.$$

Remark 3.16. Most of the time (see Lemma 3.22) F_γ^T is a polynomial with constant term 1, and hence $x_\gamma^T = X_\gamma^T$.

3.4. Definition of the cluster algebra elements associated to closed loops. By using a variant of the above construction, for each closed loop γ we will associate an element of (the fraction field of) \mathcal{A} [MSW2].

Definition 3.17 (*Band Graph corresponding to a closed loop γ*). Let γ be a closed loop in (S, M) , which may or not have self-intersections, but is not contractible and has no contractible kinks. We pick a triangle Δ traversed by γ arbitrarily, let p be a point in the interior of this triangle which lies on γ , and let b and c be the two sides of triangle crossed by γ immediately before and following its travel through point p , respectively. Let a be the third side of Δ . Note that these definitions make sense even if Δ is self-folded. We let $\tilde{\gamma}$ denote the arc from p back to itself that exactly follows closed loop γ . See the left of Figure 5.

We start by building the snake graph $G_{T, \tilde{\gamma}}$ as defined above. In the first tile of $G_{T, \tilde{\gamma}}$, let x denote the vertex at the corner of the edge labeled a and the edge labeled b , and let y denote the vertex at the other end of the edge labeled a . Similarly, in the last tile of $G_{T, \tilde{\gamma}}$, let x' denote the vertex at the corner of the edge labeled a and the edge labeled c , and let y' denote the vertex at the other end of the edge labeled a . See the right of Figure 5.

From $G_{T, \tilde{\gamma}}$, we build $\tilde{G}_{T, \gamma}$, the band graph for the closed loop γ , by identifying the edges labeled a in the first and last tiles so that the vertices x and x' and the vertices y and

y' are glued together. We refer to the two vertices obtained by identification as x and y , and to the edge obtained by identification as the *cut edge*. The resulting graph lies on an annulus or a Möbius strip.

Definition 3.18 (*Good matchings on a band graph*). Let P be a perfect matching of a band graph \tilde{G} . We call P *good* if either x and y are matched to each other ($P(x) = y$ and $P(y) = x$) or if both edges $(x, P(x))$ and $(y, P(y))$ lie on one side of the cut edge.

Remark 3.19. Let \tilde{G} be a band graph obtained by identifying two edges of the snake graph G . The good matchings of \tilde{G} can be identified with a subset of the perfect matchings of G . Let \tilde{P} be a good matching of \tilde{G} . Thinking of \tilde{P} as a subset of edges of G , then by definition of good we can add to it either the edge (x, y) or the edge (x', y') to get a perfect matching P of G . In this case, we say that the perfect matching P of G *descends* to a good matching \tilde{P} of \tilde{G} . In particular, the minimal matching of G descends to a good matching of \tilde{G} , which we also call *minimal*. (To see this, just consider the cases of whether G has an odd or even number of tiles, and observe that the minimal matching of G always uses one of the edges (x, y) and (x', y') .)

Definition 3.20. Let (S, M) be a surface, $T = (\tau_1, \dots, \tau_n)$ an ideal triangulation, and $\mathcal{A} = \mathcal{A}_\bullet(B_T)$ be the cluster algebra with principal coefficients with respect to Σ_T . Let γ be a closed curve. We define a Laurent polynomial X_γ^T which lies in (the fraction field of) \mathcal{A} , as well as a Laurent polynomial F_γ^T obtained from X_γ^T by specialization.

- (1) If γ is a contractible loop, then let $X_\gamma^T = -2$.
- (2) If γ is a closed loop without self-intersections enclosing a single puncture p :
 - If T contains a self-folded triangle containing p , then let $X_\gamma^T = 1 + \frac{y_r}{y_r^{(p)}}$, where r is the radius incident to p .
 - Otherwise, let $X_\gamma^T = 1 + \prod_{\tau \in T} y_\tau^{e_p(\tau)}$, where $e_p(\tau)$ is given by Definition 3.11.¹
- (3) If γ has a contractible kink, let $\bar{\gamma}$ denote the corresponding closed curve with this kink removed, and define $X_\gamma^T = (-1)X_{\bar{\gamma}}^T$.
- (4) Otherwise, let

$$X_\gamma^T = \frac{1}{\text{cross}(T, \gamma)} \sum_P x(P)y(P),$$

where the sum is over all good matchings P of the band graph $\tilde{G}_{T, \gamma}$.

Define F_γ^T to be the Laurent polynomial obtained from X_γ^T by specializing all the x_{τ_i} to 1.

Remark 3.21. We also apply Definition 3.15 to define the cluster algebra element $x_\gamma^T \in \mathcal{A} = \mathcal{A}_\bullet(B_t)$ when γ is a closed loop.

Lemma 3.22. *If T has no self-folded triangles, then for any generalized arc or loop γ , F_γ^T is a polynomial with constant term 1. If T has self-folded triangles, then for any arc (without self-intersections) or essential loop γ , F_γ^T is a polynomial with constant term 1.*

Proof. By definition, $F_\gamma^T = \sum_P \Phi(h(P))$, where the sum is over all matchings (resp. good matchings) of the snake graph (resp. band graph) of γ . Clearly $\sum_P h(P)$ is a polynomial in the variables y_{τ_i} , with constant term 1. The only way that Proposition 3.22 might fail is if the specialization Φ produces a Laurent monomial which is not a monomial, or an extra constant term. However, whenever γ is an arc or an essential loop, each time γ goes through a noose, its local configuration must look like one of the first three diagrams in

¹We could instead define X_γ^T in terms of matchings in a band graph, as below, but this definition in terms of e_p is simpler to compute.

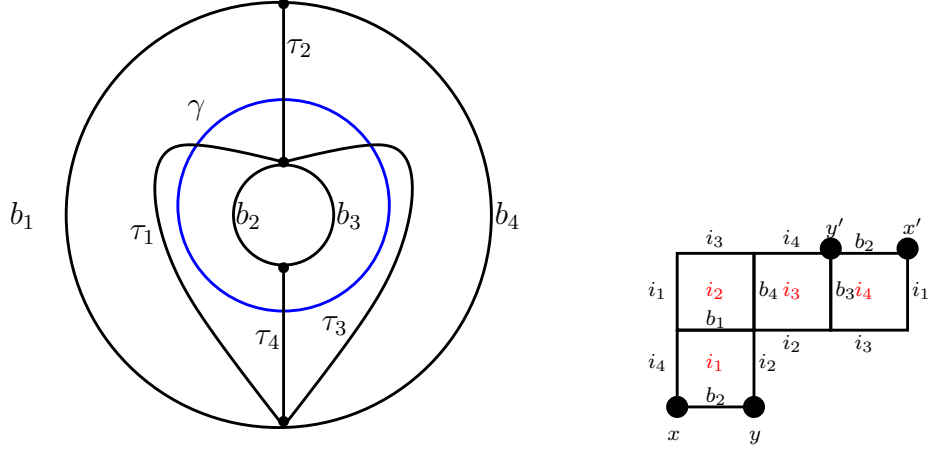


FIGURE 6. (Left): An ideal triangulation T and a closed loop γ . (Right) Corresponding band graph $\tilde{G}_{T,\gamma}$

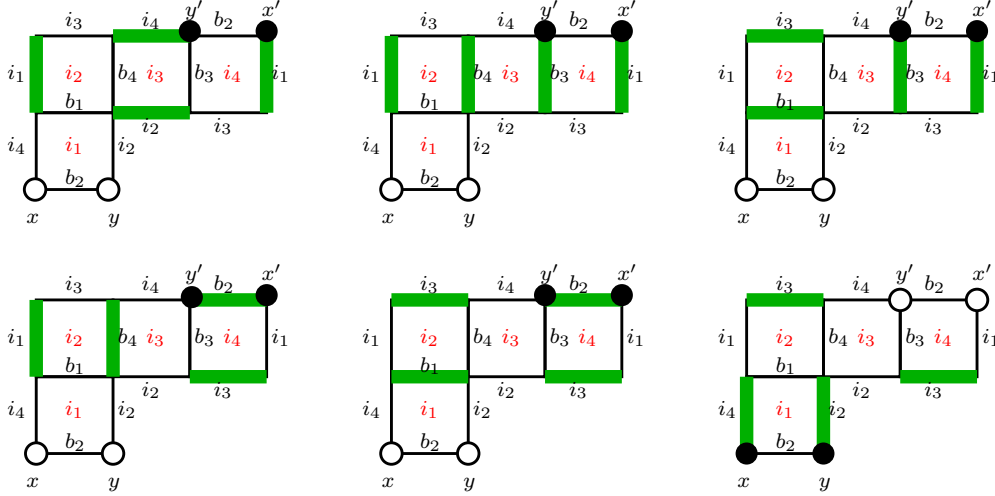


FIGURE 7. The good matchings in the band graph $\tilde{G}_{T,\gamma}$ of the above example

Figure 3. By inspection, it's impossible for the height monomial $h(P)$ of a matching P of the corresponding band graph to include a contribution from the tile labeled r *without* also including a contribution from one of the two adjacent tiles labeled ℓ . Therefore $\Phi(h(P))$ will never produce a denominator. Also, it's impossible for $\Phi(h(P))$ to produce an addition term equal to 1: such a term would need to have a factor of $\frac{y_r}{y_{r(p)}}$, but then there would be no way to cancel the numerator y_r . \square

We end this section with two examples illustrating the computation of X_γ^T .

Example 3.23 (Example of a Laurent expansion corresponding to a closed loop). Consider the ideal triangulation T and closed loop γ on the left of Figure 6. In this case, we obtain the band graph $\tilde{G}_{T,\gamma}$ appearing on the right of Figure 6. We thus compute $X_\gamma^T = \frac{x_1^2 x_2 x_4 + y_3 x_1^2 + (y_2 y_3 + y_3 y_4) x_1 x_3 + y_2 y_3 y_4 x_3^2 + y_1 y_2 y_3 y_4 x_2 x_3^2 x_4}{x_1 x_2 x_3 x_4}$ by specializing $b_1 = b_2 = b_3 = b_4 = 1$. In particular, $\tilde{G}_{T,\gamma}$ has six good matchings, as listed in Figure 7.

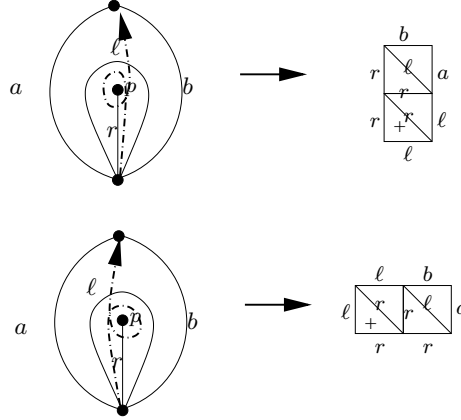


FIGURE 8. (Left): An ideal triangulation T containing a self-folded triangle and generalized arcs γ_1 and γ_2 . (Right): Corresponding snake graphs G_{T, γ_1} and G_{T, γ_2}

Example 3.24 (Examples of Laurent expansions for generalized arcs through a self-folded triangle). Consider the ideal triangulation T and the generalized arc γ_1 on the top (resp. γ_2 on the bottom) on the left of Figure 8. In this case, we obtain the snake graph G_{T, γ_1} (resp. G_{T, γ_2}) appearing on the right of Figure 8. We thus compute

$$X_{\gamma_1}^T = \frac{ax_r x_\ell + y_\ell b x_r x_\ell + y_r y_\ell b x_r x_\ell}{x_r x_\ell} = b(y_{\tau(p)} + y_\tau) + a$$

and

$$X_{\gamma_2}^T = \frac{ax_r x_\ell + y_r a x_r x_\ell + y_r y_\ell b x_r x_\ell}{x_r x_\ell} = a\left(1 + \frac{y_\tau}{y_{\tau(p)}}\right) + y_\tau b$$

by specializing $y_\ell = y_{\tau(p)}$, $y_r = \frac{y_\tau}{y_{\tau(p)}}$.

Notice that in the case of $X_{\gamma_2}^T$, we obtain a Laurent polynomial expansion with a $y_{\tau(p)}$ in the denominator. The corresponding algebraic quantity $F_{\gamma_2}^T = 1 + \frac{y_\tau}{y_{\tau(p)}} + y_\tau$ is in fact a *Laurent* polynomial and not a polynomial in this case. To obtain the associated cluster algebra element $x_{\gamma_2}^T$, we follow Definition 3.15, and divide $X_{\gamma_2}^T$ by the tropical evaluation of $F_{\gamma_2}^T$, which is $1/y_{\tau(p)}$. This gives us $x_{\gamma_2}^T = a(y_{\tau(p)} + y_\tau) + by_\tau y_{\tau(p)}$.

4. MATRIX PRODUCT FORMULAS FOR GENERALIZED ARCS AND CLOSED LOOPS

In this section we also fix a bordered surface (S, M) and ideal triangulation T of (S, M) . We then associate a Laurent polynomial to each arc, generalized arc, and closed loop in (S, M) ; this Laurent polynomial represents the corresponding element of the cluster algebra $\mathcal{A}_T(S, M)$ with principal coefficients with respect to the seed T . Our formula works by associating a product of matrices to each such arc (respectively, loop), and then computing its upper right entry (respectively, trace). Our formulas are closely related to those given in [FG1] and [FG3]. In particular, if we set our initial cluster variables equal to 1, we recover Fock and Goncharov's *X-coordinates*, and if we set our coefficient variables equal to 1, we recover their *A-coordinates*.

Before presenting our formulas, we need to define some *elementary steps*, and the matrices associated to them.

Definition 4.1. (The points $v_{m, \tau}$, $v_{m, \tau}^+$, $v_{m, \tau}^-$) For each marked point $m \in M$, draw a small circle h_m locally around m . If m is on the boundary of S , then we only consider the

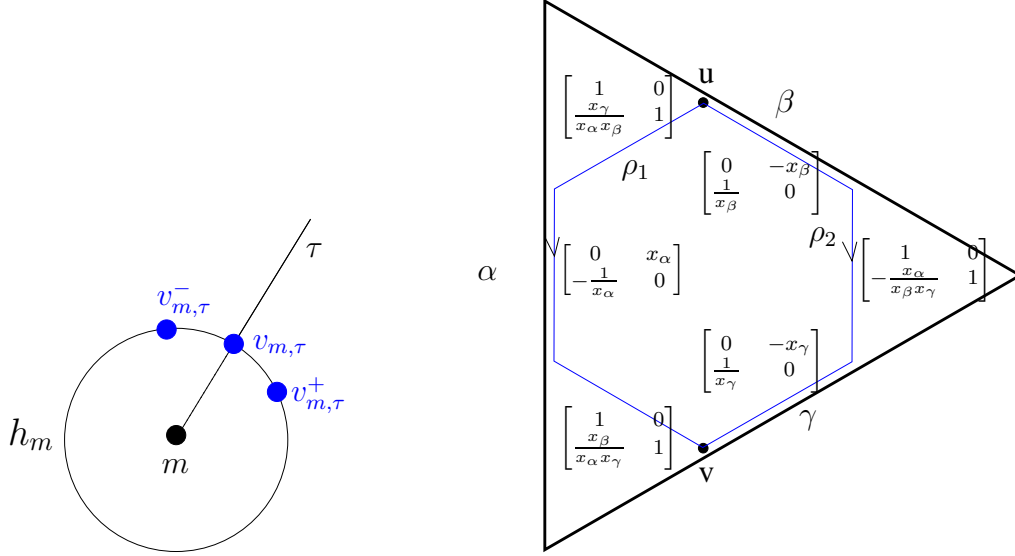


FIGURE 9. (Left): h_m and $v_{m,\tau}^\pm$. (Right): two M -paths from u to v

intersection $h_m \cap S$. The circles are chosen small enough so that $h_m \cap h_{m'} = \emptyset$ for each pair of distinct marked points m and m' . For each arc $\tau \in T$ and marked point $m \in M$ incident to τ , we let $v_{m,\tau}$ denote the intersection point $h_m \cap \tau$. We let $v_{m,\tau}^+$ (resp. $v_{m,\tau}^-$) denote a point on h_m which is very close to $v_{m,\tau}$ but in the clockwise (resp. counterclockwise) direction from $v_{m,\tau}$. See Figure 9.

Definition 4.2. (Elementary steps) Given (S, M) and T , we define three types of *elementary steps*, each of which connects two points of the form $v_{m,\tau}^\pm$ and $v_{m',\tau'}^\pm$. We also associate a 2×2 matrix $M(\rho)$ to each elementary step ρ .

- The first type of step is shown in Figure 10. We consider two arcs τ and τ' from T which are both incident to a marked point m and which form a triangle with third side $\sigma \in T$. Then our first type of step is a curve which travels either clockwise or counterclockwise around h_m between τ and τ' , without crossing them. The matrix corresponding to this step is $\begin{bmatrix} 1 & 0 \\ \pm \frac{x_\sigma}{x_\tau x_{\tau'}} & 1 \end{bmatrix}$, where we choose the positive (resp. negative) sign if the step is clockwise (resp. counterclockwise).
- The second type of step is shown in Figure 11. This step moves along a circle h_m connecting two points $v_{m,\tau}^+$ and $v_{m,\tau}^-$, so in particular it crosses the arc τ . If the step travels clockwise (resp. counterclockwise), we associate to it the matrix $\begin{bmatrix} 1 & 0 \\ 0 & y_\tau \end{bmatrix}$ (resp., $\begin{bmatrix} y_\tau & 0 \\ 0 & 1 \end{bmatrix}$).
- The third type of step is shown in Figure 12. Given two marked points m and m' connected by some $\tau \in T$, such a step follows a path parallel to τ , and connects $v_{m,\tau}^\pm$ and $v_{m',\tau}^\mp$. We associate the matrix $\begin{bmatrix} 0 & x_\tau \\ -\frac{1}{x_\tau} & 0 \end{bmatrix}$ to such a step if τ lies beneath it (when we orient the step from left to right), and we associate the inverse matrix $\begin{bmatrix} 0 & -x_\tau \\ \frac{1}{x_\tau} & 0 \end{bmatrix}$ to the step otherwise.

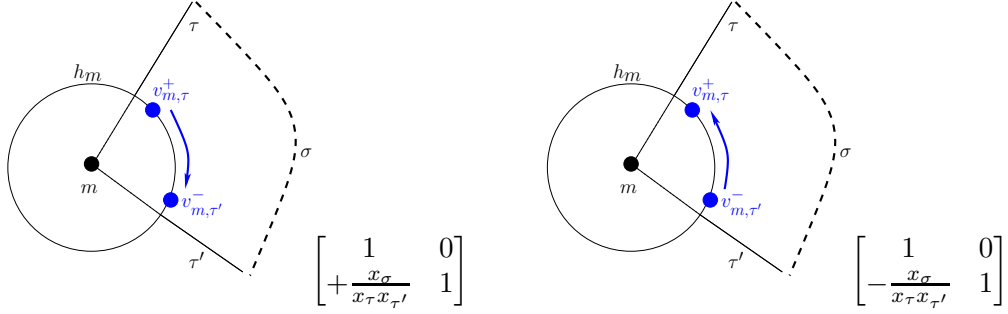


FIGURE 10. Elementary step of type 1

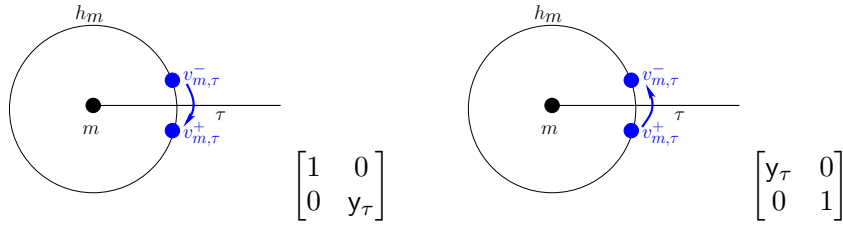


FIGURE 11. Elementary step of type 2

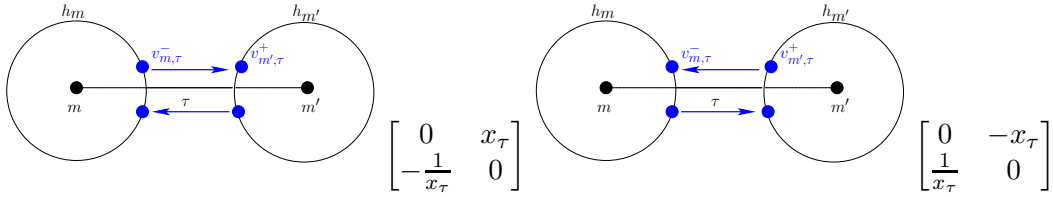


FIGURE 12. Elementary steps of type 3 in the positive and negative direction

Remark 4.3. Note that the two matrices associated to elementary steps of type 1 and elementary steps of type 3 are inverses in $SL_2(\mathbb{R})$. Also, the product of the two matrices associated to elementary steps of type 2 is y_τ times the identity.

We are now ready to associate a matrix to each generalized arc and closed loop.

Definition 4.4. (Matrix associated to an arc or loop) Given (S, M) , T , and a generalized arc γ in (S, M) from s to t , we choose a curve ρ_γ which has the following properties:

- It begins at a point P_s of the form $v_{s,\tau}^\pm$, where τ is an arc of T incident to s .
- It ends at a point P_t of the form $v_{t,\tau'}^\pm$.
- It is a concatenation of elementary steps, and is isotopic to the portion of γ between $h_s \cap \gamma$ and $h_t \cap \gamma$.
- The intersections of ρ_γ with T are in bijection with the intersections of γ with T .

Given a closed loop γ in (S, M) , we choose a curve ρ_γ which has the following properties:

- It starts and ends at a point $P_s = P_t$ which has the form $v_{m,\tau}^\pm$, where $\tau \in T$ is crossed by γ .
- It is a concatenation of elementary steps, and is isotopic to γ .
- The intersections of ρ_γ with T are in bijection with the intersections of γ with T .

In both cases, we refer to the curve ρ_γ as an M -path. If $\rho_\gamma = \rho_t \circ \cdots \circ \rho_2 \circ \rho_1$ is the decomposition of ρ_γ into elementary steps, then we define $M(\rho_\gamma) = M(\rho_t) \cdots M(\rho_2)M(\rho_1)$. By convention, the matrix associated to the empty path is the identity matrix.

Definition 4.5. (Reduced Matrix associated to an arc or loop) As noted in Remark 4.3, the matrices corresponding to elementary steps of type 2 are not in $SL_2(\mathbb{R})$. For some applications, it will be more useful to use the matrices

$$\begin{bmatrix} y_\tau^{-1/2} & 0 \\ 0 & y_\tau^{1/2} \end{bmatrix} \text{ and } \begin{bmatrix} y_\tau^{1/2} & 0 \\ 0 & y_\tau^{-1/2} \end{bmatrix}$$

instead for a step in the clockwise (resp. counterclockwise) direction. In other words, we divide by $\sqrt{y_\tau}$ for every elementary step of type 2 crossing the arc τ . If so, we will let $\overline{M}(\rho_\gamma) = \overline{M}(\rho_t) \cdots \overline{M}(\rho_2)\overline{M}(\rho_1)$ denote the corresponding matrix product in $SL_2(\mathbb{R})$.

Remark 4.6. Clearly $M(\rho_\gamma)$ and $\overline{M}(\rho_\gamma)$ depend on our choice of ρ_γ . However, it turns out that nevertheless, we can read off from them invariants which depends only on γ .

Definition 4.7. Given a 2×2 matrix $M = (m_{ij})$, let $\text{ur}(M)$ denote m_{12} . Let $\text{tr}(M)$ denote the trace of M .

Lemma 4.8. Fix (S, M) and T as usual. Let γ_1 and γ_2 be a generalized arc and a closed loop, respectively, with no contractible kinks. Then for any two M -paths ρ and ρ' associated to γ_1 , we have that

$$|\text{ur}(M(\rho))| = |\text{ur}(M(\rho'))|.$$

And for any two M -paths ρ and ρ' associated to γ_2 , we have that

$$|\text{tr}(M(\rho))| = |\text{tr}(M(\rho'))|.$$

The analogous results also hold for the reduced matrices corresponding to an M -path.

Lemma 4.8 allows us to make the following definitions.

Definition 4.9. Let γ be a generalized arc and γ' be a closed loop, and let ρ and ρ' denote arbitrary M -paths associated to γ and γ' , respectively. We associate three (related) algebraic quantities to γ and γ' :

- (1) $\hat{\chi}_{\gamma,T} = |\text{ur}(M(\rho))|$ and $\hat{\chi}_{\gamma',T} = |\text{tr}(M(\rho'))|$.
- (2) $\overline{\chi}_{\gamma,T} = |\text{ur}(\overline{M}(\rho))|$ and $\overline{\chi}_{\gamma',T} = |\text{tr}(\overline{M}(\rho'))|$.
- (3) $\chi_{\gamma,T} = \Phi(\hat{\chi}_{\gamma,T})$ and $\chi_{\gamma',T} = \Phi(\hat{\chi}_{\gamma',T})$.

When T has no self-folded triangles, the first and third definitions coincide. The third definition is the most fundamental, and we will show in Section 5 that $\chi_{\gamma,T} = X_{\gamma,T}$, where $X_{\gamma,T}$ is the sum over matchings which we defined in Section 3. The second definition will be used in our proofs of skein relations in Section 6.

Proof of Lemma 4.8. Consider an M -path ρ . First note that the two M -paths ρ_1 and ρ_2 in Figure 9 have the property that $M(\rho_1) = -M(\rho_2)$; in other words, they are equal as elements of PSL_2 . This means that a local adjustment of an M -path which replaces one segment ρ_1 by another segment ρ_2 will not affect the value of $|\text{ur}(M(\rho))|$ or $|\text{tr}(M(\rho))|$.

Next, note that $\begin{bmatrix} 0 & x_\tau \\ \frac{-1}{x_\tau} & 0 \end{bmatrix} \begin{bmatrix} y_\tau & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & y_\tau \end{bmatrix} \begin{bmatrix} 0 & x_\tau \\ \frac{-1}{x_\tau} & 0 \end{bmatrix}$. Dividing by $\sqrt{y_\tau}$ on both sides also preserves this identity. This implies that if at some point an M -path crosses an arc and then travels along that arc, then a local adjustment which switches the order of these two steps will not affect the final value of $|\text{ur}(M(\rho))|$ or $|\text{tr}(M(\rho))|$.

Finally, if ρ is the M -path for a generalized arc γ , note that we can change its starting point $P_s = v_{s,\tau}^\pm$ by choosing the other point $v_{s,\tau}^\pm$ which is obtained from P_s by traveling around the circle h_s without crossing any arcs of T . The result will still be an M -path ρ' for γ , and we will have that $M(\rho') = M(\rho)N$ where N is lower-triangular with 1's on the

diagonal. Therefore $|\text{ur}(M(\rho'))| = |\text{ur}(M(\rho))|$. Similarly, we can change the ending point P_t , and we will still have an M -path ρ' for γ such that $|\text{ur}(M(\rho'))| = |\text{ur}(M(\rho))|$.

Now the proof of Lemma 4.8 for generalized arcs follows from the fact that any two M -paths for γ can be obtained from each other by a combination of the above local adjustments. To complete the proof in the case that γ is a closed loop, note that choosing a different starting point $P_s = P_t$ for the M -path ρ amounts to circularly re-arranging the matrix product. The fact that this operation does not affect $|\text{tr}(M(\rho))|$ follows from the identity $\text{tr}(UV) = \text{tr}(VU)$. The analogous identities for $|\text{ur}(\overline{M}(\rho))|$ and $|\text{tr}(\overline{M}(\rho))|$ also hold. \square

Remark 4.10. We can actually strengthen Lemma 4.8 in the case of reduced matrices. Whenever ρ is an M -path (for some generalized arc or closed loop) from s to t and ρ' is another path from s to t that is isotopic to ρ (with possibly some extra intersections with T), we have that $\overline{M}(\rho) = \pm \overline{M}(\rho')$.

Our first main result is the following.

Theorem 4.11. *Let (S, M) be a bordered surface with an ideal triangulation T , and let $\iota(T) = \{\tau_1, \tau_2, \dots, \tau_n\}$ be the corresponding tagged triangulation. Let \mathcal{A} be the corresponding cluster algebra with principal coefficients with respect to $\Sigma_T = (\mathbf{x}_T, \mathbf{y}_T, B_T)$.*

- *Suppose γ is a generalized arc in S without contractible kinks (this may include a noose). Let $G_{T,\gamma}$ be the graph constructed in Section 3.2. Then*

$$\chi_{\gamma,T} = \frac{1}{\text{cross}(T, \gamma)} \sum_P x(P)y(P),$$

where the sum is over all perfect matchings P of $G_{T,\gamma}$. Combining this with Theorem 3.13, it follows that when γ is an arc, $\chi_{\gamma,T}$ is the Laurent expansion of x_γ with respect to Σ_T .

- *Suppose that γ is a closed loop which is not contractible, has no contractible kinks, and does not enclose a single puncture. Then*

$$\chi_{\gamma,T} = \frac{1}{\text{cross}(T, \gamma)} \sum_P x(P)y(P),$$

where the sum is over all good matchings P of the band graph $\tilde{G}_{T,\gamma}$.

Comparing our formulas to those of Fock and Goncharov [FG3], we observe the following.

Proposition 4.12. *Fix (S, M) and T . Let γ be a generalized arc or closed loop, and suppose that it crosses arcs $\tau_{i_1}, \dots, \tau_{i_d}$ in T . Then if we substitute $x_i = 1$ and $y_i = X_i$ into $\overline{\chi}_{\gamma,T}$, the resulting expression will give the associated X -coordinates for γ with respect to T . On the other hand, if we substitute $x_i = A_i$ and $y_i = 1$ into $\chi_{\gamma,T}$, the resulting expression will give the associated A -coordinate for γ with respect to T .*

Proof. In the case of A -coordinates, this result is immediate. Comparing our notation with that of Fock and Goncharov [FG3], their F -matrices realize elementary steps of type 1 and their D -matrices realize elementary steps of type 3. (Note that when we set $y_i = 1$, a matrix corresponding to an elementary step of type 2 is the identity matrix.)

To obtain their formula for X -coordinates we coarsen our vertex structure on (S, M) by using the subset V^- consisting only of the $v_{m,\tau}^-$'s. We thereby get a coarsened graph which contains a triangle (as opposed to a hexagon) for each triangle of the triangulation T and a single edge (as opposed to two edges) crossing each $\tau \in T$. See Figure 13.

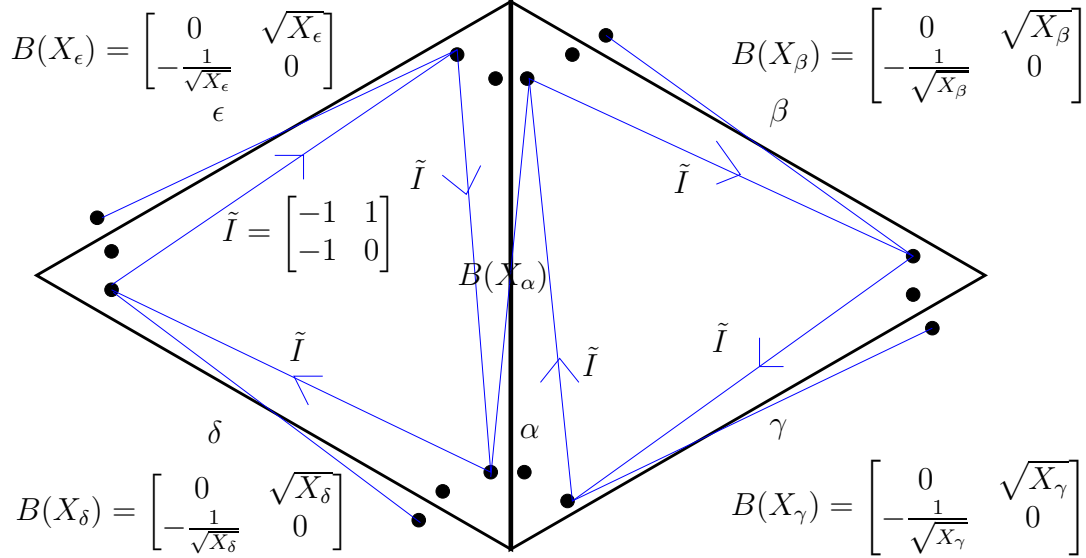


FIGURE 13. A quadrilateral inside triangulation T with steps between vertices of V^- highlighted

Any M -path from $P_s = v_{m_1, \tau_1}^-$ to $P_t = v_{m_2, \tau_2}^-$ can be decomposed into quasi-elementary steps, each of which goes between vertices of V^- . One possible quasi-elementary step combines a (counterclockwise) step of type 1 followed by a step of type 3. We obtain

$$\tilde{I} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

The other possible quasi-elementary step crosses an arc $\tau \in T$ and combines a step of 2 and a step of type 3 (in the positive direction). These steps correspond to the matrices

$$B(X_\tau) = \frac{1}{\sqrt{X_\tau}} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & X_\tau \end{bmatrix} = \frac{1}{\sqrt{X_\tau}} \cdot \begin{bmatrix} X_\tau & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{X_\tau} \\ -\frac{1}{\sqrt{X_\tau}} & 0 \end{bmatrix},$$

where we have divided by $\sqrt{X_\tau}$ as our M -path transverses $\tau \in T$, which is crossed by the arc γ .

Note that the matrix \tilde{I} differs slightly from the matrix $I = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ used by Fock and Goncharov. We will utilize the matrix equality

$$\begin{bmatrix} -A_1 & B_1 \\ C_1 & -D_1 \end{bmatrix} \begin{bmatrix} -A_2 & B_2 \\ C_2 & -D_2 \end{bmatrix} = - \begin{bmatrix} -(A_1 A_2 + B_1 C_2) & (A_1 B_2 + B_1 D_2) \\ (C_1 A_2 + D_1 C_2) & -(C_1 B_2 + D_1 D_2) \end{bmatrix},$$

which indicates how the matrix product changes if we negate the diagonal entries of the constituent matrices. As our formulas for $\bar{\chi}_{\gamma, T}|_{x_i=1, y_i=X_i}$ only involve matrix products consisting of \tilde{I} , \tilde{I}^2 , and the anti-diagonal matrices $B(X_\tau)$, we have by induction that the absolute values of the upper right entries and traces of these matrix products agree with the formulas for X -coordinates in Section 4.1 of [FG3]. \square

Theorem 4.11 immediately implies the following.

Corollary 4.13. *The quantity $\chi_{\gamma, T}$ is a Laurent polynomial with all coefficients positive.*


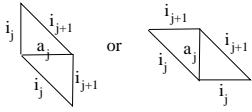
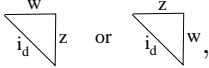
In the case that each $y_i = 1$, Corollary 4.13 was also proved by Fock and Goncharov in [FG1, Section 12.2].

5. THE MATCHING AND MATRIX-PRODUCT FORMULAS COINCIDE

In this section we will prove Theorem 4.11. We will start by giving two general combinatorial results in Section 5.1 about how one can enumerate matchings of (abstract) snake and band graphs using appropriate products of 2×2 matrices, and then apply these results in the case that the snake and band graphs come from arcs and loops in a surface.


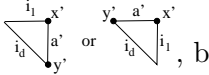
5.1. Matchings of abstract snake and band graphs.

Definition 5.1 (Abstract snake graph). An *abstract snake graph* with d tiles is formed by concatenating the following *puzzle pieces*:

- An *initial triangle* 
- $d - 1$ *parallelograms* H_1, \dots, H_{d-1} , where each H_j is either (a *north-pointing* or *east-pointing* parallelogram) 
- A *final triangle* , based on whether d is odd or even.

We then erase all diagonal edges (those with slope -1) from the figure.

Definition 5.2 (Abstract band graph). An *abstract band graph* with d tiles is formed by concatenating the following *puzzle pieces*:

- An *initial triangle* 
- $d - 1$ *parallelograms* H_1, \dots, H_{d-1} , where each H_j is as before.
- A *final triangle*  , based on whether d is odd or even.

We then identify the edges a and a' , the vertices x and x' , and the vertices y and y' . Finally, we erase all diagonal edges (those with slope -1) from the figure.

Just as in Definitions 3.7 and 3.18, we can consider the perfect matchings of an abstract snake graph and the good matchings of an abstract band graph. Additionally, we can use Definitions 3.7 and 3.10 to associate to each perfect matching P of an abstract snake graph and to each good matching P of an abstract band graph its *weight* and *height* monomials $x(P)$ and $h(P)$.

Definition 5.3. Let G be an abstract snake or band graph with d tiles. We will associate to G a matrix M_d . First we define some matrices m_1, \dots, m_{d-1} , where each m_i is either

$$\begin{bmatrix} 1 & 0 \\ \frac{x_{a_j}}{x_{i_j} x_{i_{j+1}}} & y_{i_j} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{x_{i_{j+1}}}{x_{i_j}} & x_{a_j} y_{i_j} \\ 0 & \frac{x_{i_j} y_{i_j}}{x_{i_{j+1}}} \end{bmatrix}, \text{ subject to the following conditions:}$$

- m_1 is of the first type if H_1 is a north-pointing parallelogram, and otherwise it is of the second type;
- for $i > 1$, m_i is of the first type if both H_{i-1} and H_i have the same shape, and otherwise, it is of the second type.

Finally, if $d = 1$, we set $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and otherwise, we set $M_d = m_{d-1} \dots m_1$.

A main result of this section is the following.

Theorem 5.4. *Suppose that G is an abstract snake graph with d tiles. Then its perfect matching enumerator is given by*

$$\sum_P x(P)h(P) = x_{i_1} \dots x_{i_d} \operatorname{ur} \left(\begin{bmatrix} \frac{x_w}{x_{i_d}} & x_z y_{i_d} \\ -\frac{1}{x_z} & 0 \end{bmatrix} M_d \begin{bmatrix} 0 & x_a \\ -\frac{1}{x_a} & x_{i_1} \end{bmatrix} \right),$$

where the sum is over all perfect matchings of G .

Now suppose that G is an abstract band graph with d tiles. Then its good matching enumerator is given by

$$\sum_P x(P)h(P) = x_{i_1} \dots x_{i_d} \operatorname{tr} \left(\begin{bmatrix} \frac{x_{i_1}}{x_{i_d}} & x_a y_{i_d} \\ 0 & \frac{y_{i_d} x_{i_d}}{x_{i_1}} \end{bmatrix} M_d \right),$$

where the sum is over all good matchings of G .

The main step towards proving Theorem 5.4 is the following.

Proposition 5.5. *Let G be an abstract snake graph with d tiles. Write $M_d = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}$.*

Then we have

$$\begin{aligned} A_d &= \frac{\sum_{P \in S_A} x(P)h(P)}{(x_{i_1} \dots x_{i_{d-1}})x_a x_w} & B_d &= \frac{\sum_{P \in S_B} x(P)h(P)}{(x_{i_2} \dots x_{i_{d-1}})x_b x_w} \\ C_d &= \frac{\sum_{P \in S_C} x(P)h(P)}{(x_{i_1} \dots x_{i_d})x_a x_z y_{i_d}} & D_d &= \frac{\sum_{P \in S_D} x(P)h(P)}{(x_{i_2} \dots x_{i_d})x_b x_z y_{i_d}}, \end{aligned}$$

where S_A , S_B , S_C , and S_D are the sets of perfect matchings of G which use the edges $\{a, w\}$; $\{b, w\}$; $\{a, z\}$; and $\{b, z\}$, respectively.

Proof. The proof of this proposition is straightforward, using induction on d , and considering what happens as one adds one more tile to a snake graph. When $d = 1$, the graph G consists of an initial triangle glued to a final triangle, set $S_A = \{\text{minimal matching of } G\}$, sets S_B , S_C are empty, and set $S_D = \{\text{maximal matching of } G\}$. Thus the base case, where M_1 equals the 2-by-2 identity matrix, holds.

For $d > 1$, we let G' denote the graph obtained by gluing together the initial triangle, parallelograms H_1, \dots, H_{d-2} , and the final triangle. For convenience, we label the final triangle in G' with w' and z' , and note that the orientation of this triangle depends on whether $(d - 1)$ is odd or even. By changing edge labels, we observe that the graph G' is isomorphic to the subgraph of G consisting of the first $(d - 1)$ tiles. In particular, we either replace the edge label w' with a_{d-1} and z' with i_d , or vice-versa. In the first case, we have bijections between the following pairs of perfect matchings:

- $S_A(G) \leftrightarrow S_A(G')$,
- $S_B(G) \leftrightarrow S_B(G')$,
- $S_C(G) \leftrightarrow S_A(G') \sqcup S_C(G')$, and
- $S_D(G) \leftrightarrow S_B(G') \sqcup S_D(G')$.

See Figures 14 and 15. In the second case, we have bijections between the following pairs of perfect matchings:

- $S_A(G) \leftrightarrow S_A(G') \sqcup S_C(G')$,
- $S_B(G) \leftrightarrow S_B(G') \sqcup S_D(G')$,
- $S_C(G) \leftrightarrow S_C(G')$, and
- $S_D(G) \leftrightarrow S_D(G')$.

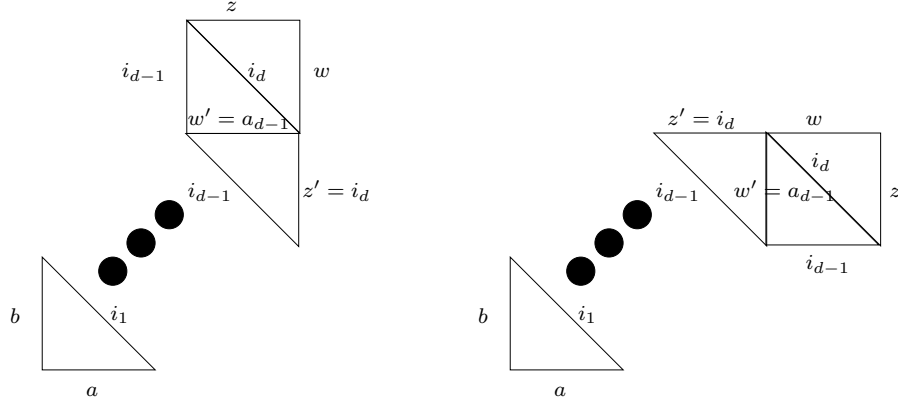


FIGURE 14. The graph G' as a subgraph of G in the case $w' = a_{d-1}$ and $z' = i_d$. (Left): d is even. (Right): d is odd

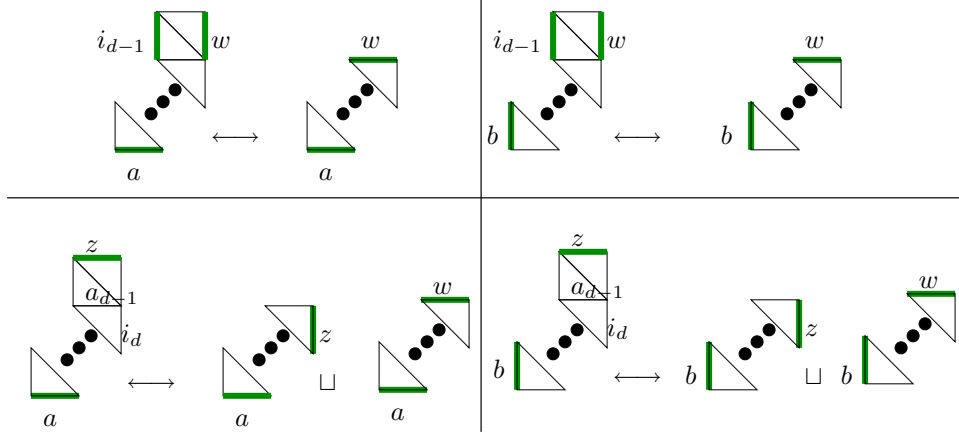


FIGURE 15. Decomposing matrix M_d in the first case and when d is even

Note that the set S_A contains the minimal matching of G , while S_D contains the maximal matching. Consequently, by altering the weights and heights accordingly, we obtain

$$(5.1) \quad \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{x_{a_{d-1}}}{x_{i_{d-1}}x_{i_d}} & y_{i_{d-1}} \end{bmatrix} \begin{bmatrix} A_{d-1} & B_{d-1} \\ C_{d-1} & D_{d-1} \end{bmatrix}$$

in the first case, and we obtain

$$(5.2) \quad \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} = \begin{bmatrix} \frac{x_{i_d}}{x_{i_{d-1}}} & x_{a_{d-1}}y_{i_{d-1}} \\ 0 & \frac{x_{i_{d-1}}y_{i_{d-1}}}{x_{i_d}} \end{bmatrix} \begin{bmatrix} A_{d-1} & B_{d-1} \\ C_{d-1} & D_{d-1} \end{bmatrix}$$

in the second case. Comparing these equations with the definition of matrix m_{d-1} , we see that the two cases agree with the two cases in Definition 5.3. \square

We have the following immediate corollary of Proposition 5.5.

Corollary 5.6. *Let G be an abstract snake graph with d tiles. Write $M_d = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}$.*

Then

$$(5.3) \quad \frac{\sum_P x(P)h(P)}{x_{i_1} \cdots x_{i_d}} = \frac{x_a x_w A_d}{x_{i_d}} + \frac{x_b x_w B_d}{x_{i_1} x_{i_d}} + y_{i_d} x_a x_z C_d + \frac{y_{i_d} x_b x_w D_d}{x_{i_1}}$$

$$(5.4) \quad = \text{ur} \left(\begin{bmatrix} \frac{x_w}{x_{i_d}} & x_z y_{i_d} \\ -\frac{1}{x_z} & 0 \end{bmatrix} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \begin{bmatrix} 0 & x_a \\ -\frac{1}{x_a} & \frac{x_b}{x_{i_1}} \end{bmatrix} \right),$$

where the sum at the left is over all perfect matchings of G .

Proposition 5.5 also implies the following.

Proposition 5.7. *Let G be an abstract snake graph with d tiles, but with a labeling obtained by substituting i_1 for w , a' for z , and i_d for b (the same labeling which is used for a band graph). Write $M_d = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}$. Then we have*

$$\begin{aligned} A_d &= \frac{\sum_{P \in S_A} x(P)h(P)}{(x_{i_1} \cdots x_{i_{d-1}})x_a x_{i_1}} & B_d &= \frac{\sum_{P \in S_B} x(P)h(P)}{(x_{i_2} \cdots x_{i_{d-1}})x_{i_d} x_{i_1}} \\ C_d &= \frac{\sum_{P \in S_C} x(P)h(P)}{(x_{i_1} \cdots x_{i_d})x_a x_{a'} y_{i_d}} & D_d &= \frac{\sum_{P \in S_D} x(P)h(P)}{(x_{i_2} \cdots x_{i_d})x_{i_d} x_{a'} y_{i_d}}, \end{aligned}$$

where S_A , S_B , S_C , and S_D are the sets of perfect matchings of G which respectively use the edges a and i_1 from the first and last triangle, i_d and i_1 from the first and last triangle, a and a' from the first and last triangle, and i_d and a' from the first and last triangle.

Corollary 5.8. *Let G be an abstract band graph with d tiles. Write $M_d = \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}$.*

Then

$$\begin{aligned} \frac{\sum_P x(P)h(P)}{x_{i_1} \cdots x_{i_d}} &= \frac{x_{i_1} A_d}{x_{i_d}} + y_{i_d} x_a C_d + \frac{y_{i_d} x_{i_d} D_d}{x_{i_1}} \\ &= \text{tr} \left(\begin{bmatrix} \frac{x_{i_1}}{x_{i_d}} & x_a y_{i_d} \\ 0 & \frac{y_{i_d} x_{i_d}}{x_{i_1}} \end{bmatrix} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \right), \end{aligned}$$

where the sum at the left is over all good matchings of G .

Note that Corollary 5.6 and Corollary 5.8 immediately imply the first and second parts of Theorem 5.4.

Proof. Consider the sets of matchings S_A , S_B , S_C and S_D which were defined in Proposition 5.7. Let G be the snake graph from Proposition 5.7, and let \tilde{G} denote the band graph obtained from G by identifying edge a and a' . Note that every perfect matching of G in S_A (respectively, S_C and S_D) descends to a good matching of \tilde{G} if we remove the edge a (respectively, a and a') from it. Moreover, every good matching of \tilde{G} can be obtained uniquely from one of the sets S_A , S_C , and S_D . (On the other hand, no matching P from S_B can give rise to a good matching of \tilde{G} .) This completes the proof. \square

5.2. The standard M -path. To facilitate the proof of Theorem 4.11, we will associate to each arc γ a standard M -path ρ_γ , and show that the matrix formula coming from that M -path has the same form as (5.4) from Corollary 5.6.

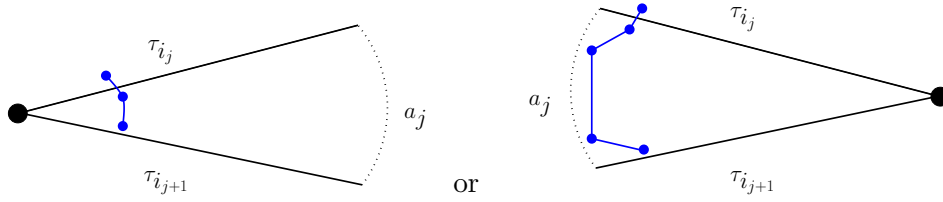


FIGURE 16. Possible transitions between τ_{i_j} and $\tau_{i_{j+1}}$ in the standard M -path.

Definition 5.9 (Standard M -path for an arc). Let γ be a generalized arc that goes from point P to point Q , crossing the arcs $\tau_{i_1}, \dots, \tau_{i_d}$ in order. Label the first triangle Δ_0 that γ crosses with sides a, b , and τ_{i_1} in clockwise order so that P is the intersection of the arcs a and b ; and label the last triangle Δ_d crossed with sides w, z , and τ_{i_d} in clockwise order, with Q being the intersection of the arcs w and z . See Figure 17.

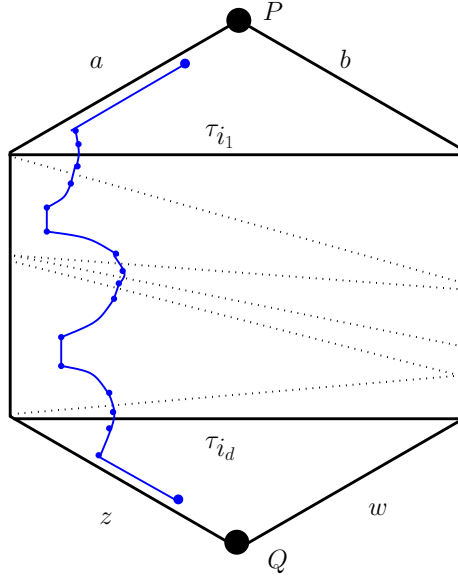
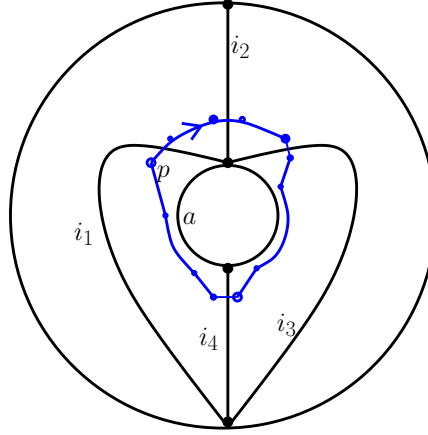
We define ρ_γ so that it starts and ends at points $v_{P,a}^\pm$ and $v_{Q,z}^\pm$, where the sign is chosen so that the starting and ending points lie inside the triangles Δ_0 and Δ_d . The path ρ_γ starts with an elementary step of type 3 along arc a , followed by a step of type 1 between arcs a and τ_{i_1} . The segment we have defined so far ends at a point of the form $v_{*,\tau_{i_1}}^\pm$ (and does not cross τ_{i_1}).

Subsequently, we define the sequence of elementary steps in ρ_γ based on whether the arc $\tau_{i_{j+1}}$ lies clockwise or counterclockwise from the arc τ_{i_j} in the unique triangle containing the corresponding segment of ρ_γ . If it is counterclockwise, we proceed with the definition of ρ_γ by adding an elementary step of type 2 which crosses τ_{i_j} and then a step of type 1 between τ_{i_j} and $\tau_{i_{j+1}}$. If the orientation is clockwise, we again begin with a step of type 2 which crosses τ_{i_j} , however, we then have a step of type 1 between τ_{i_j} and a_j . We then follow a step of type 3 in the positive direction along a_j , succeeded by a step of type 1 between a_j and $\tau_{i_{j+1}}$. In both of these cases, we do not cross or touch $\tau_{i_{j+1}}$. This progression keeps the path in the same relative position after each double or quadruple step, and as a consequence, we can iterate our construction. See Figure 16.

After $(d-1)$ transitions as in Figure 16, the path is at a point $v_{*,\tau_{i_d}}^\pm$, on the side closer to the arc labeled z , and is about to cross τ_{i_d} . We then add an elementary step of type 2 which crosses τ_{i_d} , a step of type 1, and then a step of type 3 which travels along z to the point $v_{Q,z}^\pm$. We call this particular M -path ρ_γ the *standard M -path associated to γ* .

Remark 5.10. We remark that in the above definition, if there are three arcs $\tau_{i_j}, \tau_{i_{j+1}},$ and $\tau_{i_{j+2}}$ such that $\tau_{i_{j+1}}$ is in the clockwise direction from τ_{i_j} and $\tau_{i_{j+2}}$ is in the clockwise direction from $\tau_{i_{j+1}}$, then the standard M -path ρ_γ will have some back-tracking: there will be two consecutive steps of type 3 which travel in opposite directions along $\tau_{i_{j+1}}$.

Definition 5.11 (Standard M -path for a closed loop). Let γ be a closed loop which crosses exactly d arcs of T (counted with multiplicity). Choose a triangle Δ in T such that two of its arcs are crossed by γ . Label those two arcs τ_{i_1} and τ_{i_d} , where τ_{i_1} is in the clockwise direction from τ_{i_d} . Label the third side of Δ by a . Let p be a point on γ which lies in Δ and has the form $v_{*,\tau_{i_1}}^\pm$. Finally, let $\tau_{i_1}, \dots, \tau_{i_d}$ denote the ordered sequence of arcs which are crossed by γ , when one travels from p away from Δ . We define the *standard M -path ρ_γ associated to γ* exactly as in Definition 5.9, starting and ending at the point p and travelling along elementary steps based on whether $\tau_{i_{j+1}}$ is counterclockwise or clockwise from τ_{i_j} . In this case, we need to consider indices modulo n : note that the last elementary steps of ρ_γ will be determined by the fact that the arc i_1 is in the clockwise direction from i_d . See Figure 18.


 FIGURE 17. The standard M -path ρ_γ of an arc γ .

 FIGURE 18. The standard M -path ρ_γ of a loop γ , which crosses $d = 4$ arcs of T

We now turn to the proof of Theorem 4.11.

Proof. First we consider the case that γ is a generalized arc. Consider its standard M -path ρ_γ , and recall Definition 4.4, which gives an algorithm for associating a product of matrices to a concatenation of elementary steps such as ρ_γ . Note that the first two steps of this path correspond to the matrix product

$$\begin{bmatrix} 1 & 0 \\ \frac{x_b}{x_a x_{i_1}} & 1 \end{bmatrix} \begin{bmatrix} 0 & x_a \\ -\frac{1}{x_a} & 0 \end{bmatrix} = \begin{bmatrix} 0 & x_a \\ -\frac{1}{x_a} & \frac{x_b}{x_{i_1}} \end{bmatrix},$$

and the last three steps of ρ_γ correspond to

$$\begin{bmatrix} 0 & x_z \\ -\frac{1}{x_z} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{x_w}{x_{i_d} x_z} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y_{i_d} \end{bmatrix} = \begin{bmatrix} \frac{x_w}{x_{i_d}} & x_z y_{i_d} \\ -\frac{1}{x_z} & 0 \end{bmatrix}.$$

In between, the matrix for the portion between τ_{i_j} and $\tau_{i_{j+1}}$ (for $1 \leq j \leq d-1$) corresponds respectively to

$$(5.5) \quad \begin{bmatrix} 1 & 0 \\ \frac{x_{a_j}}{x_{i_j}x_{i_{j+1}}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y_{i_j} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{x_{a_j}}{x_{i_j}x_{i_{j+1}}} & y_{i_j} \end{bmatrix} \text{ or}$$

$$(5.6) \quad \begin{bmatrix} 1 & 0 \\ \frac{x_{i_j}}{x_{a_j}x_{i_{j+1}}} & 1 \end{bmatrix} \begin{bmatrix} 0 & x_{a_j} \\ -\frac{1}{x_{a_j}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{x_{i_{j+1}}}{x_{a_j}x_{i_j}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & y_{i_j} \end{bmatrix} = \begin{bmatrix} \frac{x_{i_{j+1}}}{x_{i_j}} & x_{a_j}y_{i_j} \\ 0 & \frac{x_{i_j}y_{i_j}}{x_{i_{j+1}}} \end{bmatrix},$$

depending on whether $\tau_{i_{j+1}}$ lies counterclockwise or clockwise from τ_{i_j} . In both cases, we let a_j be the label of the third side in the triangle bounded by τ_{i_j} and $\tau_{i_{j+1}}$ which inscribes the appropriate part of the arc γ .

Applying Definition 4.4 to ρ_γ , we find that

$$(5.7) \quad \chi_{\gamma,T} = \text{ur} \left(\begin{bmatrix} \frac{x_w}{x_{i_d}} & x_z y_{i_d} \\ -\frac{1}{x_z} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & x_a \\ -\frac{1}{x_a} & \frac{x_b}{x_{i_1}} \end{bmatrix} \right),$$

where the middle matrix is obtained by multiplying together a sequence of matrices of the form

$$\begin{bmatrix} 1 & 0 \\ \frac{x_{a_j}}{x_{i_j}x_{i_{j+1}}} & y_{i_j} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{x_{i_{j+1}}}{x_{i_j}} & x_{a_j}y_{i_j} \\ 0 & \frac{x_{i_j}y_{i_j}}{x_{i_{j+1}}} \end{bmatrix},$$

for $1 \leq j \leq d-1$.

Note that this has precisely the same form as (5.4), and therefore by Corollary 5.6, $\chi_{\gamma,T}$ has an interpretation in terms of the perfect matchings of some abstract snake graph G . Moreover, if one compares Definitions 5.1 and 5.3 with the construction of snake graphs associated to arcs in Section 3.2, it is clear that the abstract snake graph G is precisely the snake graph $G_{T,\gamma}$ associated to the arc γ . This completes the proof of Theorem 4.11 when γ is a generalized arc.

Now we consider the case that γ is a closed loop. Consider its standard M -path ρ_γ . Note that as before, for j from 1 to $d-1$, the matrix for the portion of the path from τ_{i_j} to $\tau_{i_{j+1}}$ corresponds respectively to (5.5) or (5.6), depending on whether $\tau_{i_{j+1}}$ lies counterclockwise or clockwise from τ_{i_j} . Since τ_{i_1} is in the clockwise direction from τ_{i_d} (by construction), the

last few steps of ρ_γ correspond to $\begin{bmatrix} \frac{x_{i_1}}{x_{i_d}} & x_a y_{i_d} \\ 0 & \frac{y_{i_d} x_{i_d}}{x_{i_1}} \end{bmatrix}$.

Therefore from Definition 4.4 we have that

$$(5.8) \quad \chi_{\gamma,T} = \text{tr} \left(\begin{bmatrix} \frac{x_{i_1}}{x_{i_d}} & x_a y_{i_d} \\ 0 & \frac{y_{i_d} x_{i_d}}{x_{i_1}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right),$$

where the rightmost matrix is obtained by multiplying together a sequence of matrices of the form

$$\begin{bmatrix} 1 & 0 \\ \frac{x_{a_j}}{x_{i_j}x_{i_{j+1}}} & y_{i_j} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{x_{i_{j+1}}}{x_{i_j}} & x_{a_j}y_{i_j} \\ 0 & \frac{x_{i_j}y_{i_j}}{x_{i_{j+1}}} \end{bmatrix},$$

for $1 \leq j \leq d-1$.

Note that this has precisely the same form as the expression in Corollary 5.8, and therefore it follows that $\chi_{\gamma,T}$ has an interpretation in terms of the good matchings of some abstract band graph G . Moreover, if one compares Definitions 5.2 and 5.3 with the construction of band graphs associated to arcs in Definition 3.17, it is clear that the

abstract band graph G is precisely the band graph $\tilde{G}_{T,\gamma}$ associated to the arc γ . This completes the proof of Theorem 4.11. \square

Example 5.12. Consider the loop γ and ideal triangulation as in Example 3.23. Note that the standard M -path ρ_γ for this γ is illustrated in Figure 18, where the arcs of triangulation are labelled slightly differently.

Using the arc labels from Figure 6, we see in this case that $\hat{\chi}_{\gamma,T} = |\text{tr}(M(\rho_\gamma))|$, where

$$\begin{aligned} M(\rho_\gamma) &= \begin{bmatrix} \frac{x_1}{x_4} & b_2 y_4 \\ 0 & \frac{x_4 y_4}{x_1} \end{bmatrix} \begin{bmatrix} \frac{x_4}{x_3} & b_3 y_3 \\ 0 & \frac{x_3 y_3}{x_4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{b_4}{x_2 x_3} & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{b_1}{x_1 x_2} & y_1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{x_1^2 x_2 x_4 + y_3 x_1^2 b_1 b_4 + y_3 y_4 x_1 x_3 b_2 b_4 + y_2 y_3 x_1 x_3 b_1 b_3 + y_2 y_3 y_4 x_3^2 b_1 b_2}{x_1^2 x_2} & \frac{y_1 y_2 y_3 (x_1 b_3 + y_4 x_3 b_2)}{x_4} \\ \frac{x_1 x_2 x_3 x_4}{y_3 y_4 (x_1 b_4 + y_2 x_3 b_1)} & \frac{y_1 y_2 y_3 y_4 x_3}{x_1} \end{bmatrix}. \end{aligned}$$

Computing the trace of this matrix product, we see that X_γ^T and $\chi_{\gamma,T} = \hat{\chi}_{\gamma,T}|_{y_i=y_i, b_i=1}$ agree. The last equality follows from the absence of self-folded triangles in the triangulation T and the fact that the b_i 's label boundary segments.

5.3. Signs of $\hat{\chi}_{\gamma,T}$, $\bar{\chi}_{\gamma,T}$, and $\chi_{\gamma,T}$. Recall that the signs of three algebraic quantities defined in the last section, Definition 4.9, depended on the choice of the M -path ρ associated to the generalized arc or loop γ .

Lemma 5.13. *If we use the standard M -path, Definition 5.9, for the generalized arc or loop γ with no contractible kinks, then every coefficient of $\hat{\chi}_{\gamma,T}$, $\bar{\chi}_{\gamma,T}$, and $\chi_{\gamma,T}$ is positive.*

For the purposes of this lemma, we consider a contractible loop γ to contain a *contractible kink*. In particular, note that in this case that the signs of the three quantities are *negative*:

$$\hat{\chi}_{\gamma,T} = \bar{\chi}_{\gamma,T} = \chi_{\gamma,T} = \text{tr} \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) = -2 \text{ for such a loop.}$$

Proof. Given γ , which is a generalized arc or loop which has no contractible kinks, let ρ_γ denote a corresponding standard M -path. By inspection, the matrices $M(\rho_\gamma)$ and $\bar{M}(\rho_\gamma)$ can be decomposed into a product of matrices where almost all entries are positive. The only negative entries appearing are the lower-left entries of the first and third matrices in (5.7); these entries do not affect the upper-right entry of this triple product. In the case when γ is a loop, we use (5.8) instead, where no negative entries appear. \square

If ρ is a *non-standard* M -path for the generalized arc or loop γ , then ρ can be deformed into a standard M -path ρ_γ by the local adjustments utilized in the proof of Lemma 4.8:

- (1) going from u to v in Figure 9 clockwise instead of counterclockwise,
- (2) reversing the order of a step of type 2 crossing an arc $\tau_i \in T$ and a step of type 3 moving along the arc τ_i ,
- (3) rotating around an h_m using a combination of steps of types 1 and 2,
- (4) reselecting the starting or ending point of the M -path corresponding to a closed loop.

Each local adjustment of type (1) or (2) changes the sign of the corresponding matrix but does not otherwise affect the matrix. The local adjustments of type (3) and (4) does not affect ur or tr of the resulting matrix product.

6. SKEIN RELATIONS FOR GENERALIZED ARCS AND CLOSED LOOPS

In this section we prove certain *skein relations*, which give multiplication formulas for the cluster algebra elements corresponding to generalized arcs and loops, i.e. arcs and loops which are allowed to have self-intersections. We work in the setting of principal

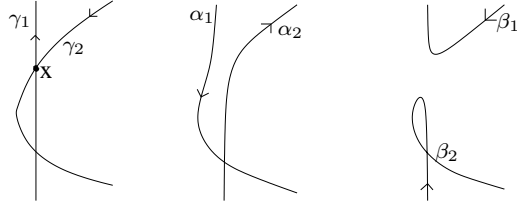


FIGURE 19. Skein relation for two generalized arcs crossing



FIGURE 20. Skein relation involving a closed loop



FIGURE 21. Skein relation resolving a self-intersection

coefficients. To prove these results, we use the matrix formulas from Section 4. The proofs of skein relations then follow from the matrix identities in Lemma 6.11. Throughout this section we fix a marked surface (S, M) , an ideal triangulation $T = (\tau_1, \dots, \tau_n)$, and the cluster algebra $\mathcal{A} = \mathcal{A}_\bullet(B_T)$ which has principal coefficients with respect to T .

Definition 6.1 (*Smoothing*). Let γ, γ_1 , and γ_2 be generalized arcs or closed loops such that we have one of the following two cases:

- (1) γ_1 crosses γ_2 at a point x ,
- (2) γ has a self-intersection at a point x .

We define the *smoothing* of $C = \{\gamma_1, \gamma_2\}$ or $C = \{\gamma\}$ at the point x to be the pair of configurations C_+ and C_- , where C_+ (respectively, C_-) is the same as C except for the local change that replaces the crossing or self-intersection \times with the pair of segments \cup (resp., $\supset C$).

Note that in the case that two generalized arcs cross, each resulting configuration contains two generalized arcs; if two closed loops cross, each resulting configuration contains only a single closed loop (necessarily with self-intersections); and if a generalized arc and a closed loop cross, then each configuration is a single generalized arc. When a self-intersection is smoothed, one of the two resulting configurations has an extra closed loop. See Figures 19, 20, and 21.

Remark 6.2. Since we consider generalized arcs up to isotopy, we may assume that each intersection is transverse. In particular, there are no triple intersections.

6.1. Skein relations for unpunctured surfaces. When (S, M) is a surface without punctures, there are simple formulas for the skein relations. However, the formulas are somewhat more complicated to write down in the presence of punctures. For this reason, we start by writing down the formula in the unpunctured case, and give the appropriate generalization (together with proofs of all results) in Section 6.2.

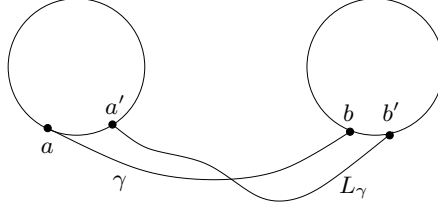


FIGURE 22. Elementary lamination for an arc whose endpoints are on the boundary of S

Definition 6.3. Let γ be an arc in an unpunctured surface (S, M) . Denote by L_γ the curve which runs along γ within a small neighborhood of it. Such a curve is known as the *elementary lamination* associated to γ , see [FT, Definition 16.2]. If γ has an endpoint a on a (circular) component C of the boundary of S , then L_γ begins at a point $a' \in C$ located near a in the counterclockwise direction, and proceeds along γ as shown in Figure 22. In particular, if $T = (\tau_1, \dots, \tau_n)$, then we let L_i denote L_{τ_i} .

Proposition 6.4. Let γ_1 and γ_2 be two generalized arcs which intersect each other at least once; let x be a point of intersection; and let α_1, α_2 and β_1, β_2 be the two pairs of arcs obtained by smoothing γ_1 and γ_2 at x . We then obtain the following identity:

$$(6.1) \quad \chi_{\gamma_1, T} \chi_{\gamma_2, T} = \pm \chi_{\alpha_1, T} \chi_{\alpha_2, T} \prod_{i=1}^n y_i^{(c_i - a_i)/2} \pm \chi_{\beta_1, T} \chi_{\beta_2, T} \prod_{i=1}^n y_i^{(c_i - b_i)/2},$$

where $c_i = e(\gamma_1, L_i) + e(\gamma_2, L_i)$, $a_i = e(\alpha_1, L_i) + e(\alpha_2, L_i)$, and $b_i = e(\beta_1, L_i) + e(\beta_2, L_i)$.

Proposition 6.5. Let γ_1 be a generalized arc or loop and let γ_2 be a generalized loop, such that γ_1 and γ_2 intersect each other; let x be a point of intersection. Let α and β be the two oriented generalized arcs or loops obtained by smoothing γ_1 and γ_2 at x , as in Figure 20.

$$(6.2) \quad \chi_{\gamma_1, T} \chi_{\gamma_2, T} = \pm \chi_{\alpha, T} \prod_{i=1}^n y_i^{(c_i - a_i)/2} \pm \chi_{\beta, T} \prod_{i=1}^n y_i^{(c_i - b_i)/2},$$

where $c_i = e(\gamma_1, L_i) + e(\gamma_2, L_i)$, $a_i = e(\alpha, L_i)$, and $b_i = e(\beta, L_i)$.

Proposition 6.6. Let γ be a generalized arc or closed loop with a self-intersection at x . Let α_1, α_2 , and β be the generalized arcs and loops obtained by smoothing at x , as in Figure 21. Then we have the identity

$$(6.3) \quad \chi_{\gamma, T} = \pm \chi_{\alpha_1, T} \chi_{\alpha_2, T} \prod_{i=1}^n y_i^{(c_i - a_i)/2} \pm \chi_{\beta, T} \prod_{i=1}^n y_i^{(c_i - b_i)/2},$$

where $c_i = e(\gamma_1, L_i) + e(\gamma_2, L_i)$, $a_i = e(\alpha_1, L_i) + e(\alpha_2, L_i)$, and $b_i = e(\beta, L_i)$.

6.2. Skein relations for surfaces with punctures. We now give the skein relations for surfaces with punctures.

Remark 6.7. For technical reasons we assume in this section that the ideal triangulation T has no self-folded triangles. In particular, when there is a self-folded triangle, we cannot use the dictionary between signed intersection numbers and transverse measures that is described below in Remark 6.17.

Before stating the relations, we need to define some terminology. The following definition is a slight variant of the M -paths defined in Section 4.

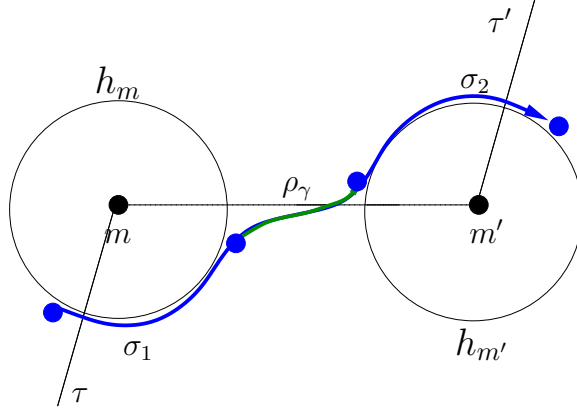


FIGURE 23. A loosened M -path with a positive signed excess with respect to both τ and τ'

Definition 6.8. A loosened M -path $\tilde{\rho}_\gamma$ for the (oriented) generalized arc γ from the marked point m to m' , is a concatenation of oriented curves $\sigma_2 \circ \rho_\gamma \circ \sigma_1$ such that:

- ρ_γ is an M -path for γ
- σ_1 (respectively, σ_2) is a concatenation of elementary steps of types 1 and 2 traveling along h_m (resp., $h_{m'}$).

Definition 6.9. Let $\tilde{\rho}_\gamma = \sigma_2 \circ \rho_\gamma \circ \sigma_1$ be a loosened M -path. We define the signed excess of $\tilde{\rho}_\gamma$ with respect to the arc $\tau_i \in T$ as

$$\begin{aligned} \tilde{\ell}(\tilde{\rho}_\gamma, \tau_i) &= \#\{\text{intersections in } \tau_i \cap \sigma_1 \text{ if } \sigma_1 \text{ travels counterclockwise along } h_m\} \\ &+ \#\{\text{intersections in } \tau_i \cap \sigma_2 \text{ if } \sigma_2 \text{ travels clockwise along } h_{m'}\} \\ &- \#\{\text{intersections in } \tau_i \cap \sigma_1 \text{ if } \sigma_1 \text{ travels clockwise along } h_m\} \\ &- \#\{\text{intersections in } \tau_i \cap \sigma_2 \text{ if } \sigma_2 \text{ travels counterclockwise along } h_{m'}\}. \end{aligned}$$

We also make the convention that if α is a closed loop, then $\tilde{\ell}(\tilde{\rho}_\alpha, \tau_i) = 0$.

Lemma 6.10. We use the notations of Definitions 6.8 and 6.9. Then we have that

$$|\text{ur}(\overline{M}(\tilde{\rho}_\gamma))| = \overline{\chi}_{\gamma, T} \prod_{i=1}^n y_i^{\frac{-\tilde{\ell}(\tilde{\rho}_\gamma, \tau_i)}{2}}.$$

Proof. By Definition 6.8, the matrix $\overline{M}(\tilde{\rho}_\gamma)$ associated to a loosened M -path $\tilde{\rho}_\gamma = \sigma_2 \circ \rho_\gamma \circ \sigma_1$ can be decomposed as a product $\overline{M}(\tilde{\rho}_\gamma) = \overline{M}(\sigma_2) \overline{M}(\rho_\gamma) \overline{M}(\sigma_1)$, where σ_1 and σ_2 consist only of steps of types 1 and 2. Therefore $\overline{M}(\sigma_1)$ and $\overline{M}(\sigma_2)$ are lower-triangular matrices in $SL_2(\mathbb{R})$, so we obtain

$$\begin{aligned} |\text{ur}(\overline{M}(\tilde{\rho}_\gamma))| &= \left| \text{ur} \left(\begin{bmatrix} \prod_{i=1}^n y_i^{b_i/2} & 0 \\ * & \prod_{i=1}^n y_i^{-b_i/2} \end{bmatrix} \overline{M}(\rho_\gamma) \begin{bmatrix} \prod_{i=1}^n y_i^{a_i/2} & 0 \\ * & \prod_{i=1}^n y_i^{-a_i/2} \end{bmatrix} \right) \right| \\ &= |\text{ur}(\overline{M}(\rho_\gamma))| \prod_{i=1}^n y_i^{b_i/2 - a_i/2} = \overline{\chi}_{\gamma, T} \prod_{i=1}^n y_i^{\frac{-\tilde{\ell}(\tilde{\rho}_\gamma, \tau_i)}{2}}. \end{aligned}$$

Here, the last equality follows from Definition 6.9. \square

For convenience, we also record the following identities involving 2×2 matrices. If m is a 2×2 matrix, let $\text{ur}(m)$ denote its upper right entry, and $\text{tr}(m)$ denote its trace. These

identities will be used to prove skein relations involving reduced matrices associated to generalized arcs and loops (as in Definition 4.5).

Lemma 6.11. *Let m_1, m_2, m_3 be 2×2 matrices where $\det(m_1) = 1$. Then*

$$(6.4) \quad \text{ur}(m_2 m_1) \text{ur}(m_1 m_3) = \text{ur}(m_1) \text{ur}(m_2 m_1 m_3) + \text{ur}(m_2) \text{ur}(m_3),$$

$$(6.5) \quad \text{ur}(m_3 m_2) \text{tr}(m_1) = \text{ur}(m_3 m_1 m_2) + \text{ur}(m_3 m_1^{-1} m_2), \text{ and}$$

$$(6.6) \quad \text{tr}(m_2) \text{tr}(m_1) = \text{tr}(m_1 m_2) + \text{tr}(m_1^{-1} m_2).$$

Proof. These identities can be easily checked by hand or computer. \square

In what follows, the notation \sim denotes isotopy of curves.

Proposition 6.12. *Let γ_1 and γ_2 be two generalized arcs which intersect each other at least once; let x be a point of intersection; and let α_1, α_2 and β_1, β_2 be the two pairs of arcs obtained by smoothing γ_1 and γ_2 at x , as in Figure 19. In particular, we choose orientations and label these four arcs so that:*

$$\gamma_1 \sim \beta_1 \circ \alpha_2, \quad \gamma_2 \sim \alpha_1 \circ \beta_1, \quad \text{and} \quad \beta_2 \sim \alpha_1 \circ \beta_1 \circ \alpha_2.$$

We then construct standard M -paths (Definition 5.9) ρ_{α_1} from s_1 to t_1 and ρ_{α_2} from s_2 to t_2 . We let $\tilde{\rho}_{\beta_1}$ be a loosened M -path from t_2 to s_1 that extends a standard M -path corresponding to β_1 . Finally, we choose loosened M -paths $\tilde{\rho}_{\gamma_1}$, $\tilde{\rho}_{\gamma_2}$, and $\tilde{\rho}_{\beta_2}$ so that

$$(6.7) \quad \tilde{\rho}_{\gamma_1} \sim \tilde{\rho}_{\beta_1} \circ \rho_{\alpha_2}, \quad \tilde{\rho}_{\gamma_2} \sim \rho_{\alpha_1} \circ \tilde{\rho}_{\beta_1}, \quad \text{and} \quad \tilde{\rho}_{\beta_2} \sim \rho_{\alpha_1} \circ \tilde{\rho}_{\beta_1} \circ \rho_{\alpha_2}.$$

We then obtain the following identity:

$$(6.8) \quad \bar{\chi}_{\gamma_1, T} \bar{\chi}_{\gamma_2, T} = \pm \bar{\chi}_{\alpha_1, T} \bar{\chi}_{\alpha_2, T} \prod_{i=1}^n y_i^{\frac{\tilde{\ell}(\tilde{\rho}_{\gamma_1, \tau_i}) + \tilde{\ell}(\tilde{\rho}_{\gamma_2, \tau_i})}{2}} \pm \bar{\chi}_{\beta_1, T} \bar{\chi}_{\beta_2, T} \prod_{i=1}^n y_i^{\frac{\tilde{\ell}(\tilde{\rho}_{\gamma_1, \tau_i}) + \tilde{\ell}(\tilde{\rho}_{\gamma_2, \tau_i}) - \tilde{\ell}(\tilde{\rho}_{\beta_1, \tau_i}) - \tilde{\ell}(\tilde{\rho}_{\beta_2, \tau_i})}{2}}.$$

Proposition 6.13. *Let γ_1 be a generalized arc or loop and let γ_2 be a generalized loop, such that γ_1 and γ_2 intersect each other; let x be a point of intersection. Let α and β be the two oriented generalized arcs or loops obtained by smoothing γ_1 and γ_2 at x , as in Figure 20. We construct standard M -paths ρ_{γ_1} and ρ_{γ_2} , respectively, so that they intersect at some $v_{m, \tau}^{\pm}$. We let $\rho_{\gamma_1}^{(1)}$ be the first portion of the M -path ρ_{γ_1} (up until $v_{m, \tau}^{\pm}$), and $\rho_{\gamma_1}^{(2)}$ be the second portion; if γ_1 is a loop, then we let $\rho_{\gamma_1}^{(2)}$ denote the empty path. We then choose loosened M -paths (or just M -paths when γ_1 is a closed loop) $\tilde{\rho}_{\alpha}$ and $\tilde{\rho}_{\beta}$ for α and β such that $\tilde{\rho}_{\alpha} \sim \rho_{\gamma_1}^{(2)} \circ \rho_{\gamma_2} \circ \rho_{\gamma_1}^{(1)}$, and $\tilde{\rho}_{\beta} \sim \rho_{\gamma_1}^{(2)} \circ (\rho_{\gamma_2})^{-1} \circ \rho_{\gamma_1}^{(1)}$. We have the identity*

$$(6.9) \quad \bar{\chi}_{\gamma_1, T} \bar{\chi}_{\gamma_2, T} = \pm \bar{\chi}_{\alpha, T} \prod_{i=1}^n y_i^{\frac{-\tilde{\ell}(\tilde{\rho}_{\alpha, \tau_i})}{2}} \pm \bar{\chi}_{\beta, T} \prod_{i=1}^n y_i^{\frac{-\tilde{\ell}(\tilde{\rho}_{\beta, \tau_i})}{2}}.$$

Proposition 6.14. *Let γ be a generalized arc or closed loop with a self-intersection at x . Let α_1, α_2 , and β be the generalized arcs and loops obtained by smoothing at x , as in Figure 21. Construct the standard M -path ρ_{γ} for γ , and write it as $\rho_{\gamma}^{(3)} \circ \rho_{\gamma}^{(2)} \circ \rho_{\gamma}^{(1)}$, where the subpaths meet at point x . (If γ is a loop, we let $\rho_{\gamma}^{(3)}$ be the empty path.) We define an M -path $\tilde{\rho}_{\alpha_1} = \rho_{\gamma}^{(2)}$, and choose loosened M -paths $\tilde{\rho}_{\alpha_2}$ and $\tilde{\rho}_{\beta}$ which are isotopic to $\rho_{\gamma}^{(3)} \circ \rho_{\gamma}^{(1)}$ and $\rho_{\gamma}^{(3)} \circ (\rho_{\gamma}^{(2)})^{-1} \circ \rho_{\gamma}^{(1)}$, respectively. Then we have the identity*

$$(6.10) \quad \bar{\chi}_{\gamma, T} = \pm \bar{\chi}_{\alpha_1, T} \bar{\chi}_{\alpha_2, T} \prod_{i=1}^n y_i^{\frac{-\tilde{\ell}(\tilde{\rho}_{\alpha_2, \tau_i})}{2}} \pm \bar{\chi}_{\beta, T} \prod_{i=1}^n y_i^{\frac{-\tilde{\ell}(\tilde{\rho}_{\beta, \tau_i})}{2}}.$$

Remark 6.15. In the above three propositions, as well as Propositions 6.4, 6.5, 6.6, and Corollary 6.18, the signs should be positive; see [FG1, Section 12] and [Th].

Proof. We start by proving Proposition 6.12. Let $m_1 = \overline{M}(\tilde{\rho}_{\beta_1})$, $m_2 = \overline{M}(\rho_{\alpha_1})$, and $m_3 = \overline{M}(\rho_{\alpha_2})$. By (6.7) and Remark 4.10, we have that

$$(6.11) \quad m_1 m_3 = \pm \overline{M}(\tilde{\rho}_{\gamma_1}), \quad m_2 m_1 = \pm \overline{M}(\tilde{\rho}_{\gamma_2}), \quad \text{and} \quad m_2 m_1 m_3 = \pm \overline{M}(\tilde{\rho}_{\beta_2}).$$

By (6.4), it follows that

$$|\text{ur}(\overline{M}(\tilde{\rho}_{\gamma_1})) \text{ur}(\overline{M}(\tilde{\rho}_{\gamma_2}))| = \pm |\text{ur}(\overline{M}(\rho_{\alpha_1})) \text{ur}(\overline{M}(\rho_{\alpha_2}))| \pm |\text{ur}(\overline{M}(\tilde{\rho}_{\beta_1})) \text{ur}(\overline{M}(\tilde{\rho}_{\beta_2}))|.$$

Applying Lemma 6.10 to each of $\tilde{\rho}_{\gamma_1}$, $\tilde{\rho}_{\gamma_2}$, and $\tilde{\rho}_{\beta_2}$, and substituting into the above equation, we obtain (6.8), as desired.

We now prove Proposition 6.13. Let $m_1 = \overline{M}(\rho_{\gamma_2})$, $m_2 = \overline{M}(\rho_{\gamma_1}^{(1)})$, and $m_3 = \overline{M}(\rho_{\gamma_1}^{(2)})$. In the case that γ_1 is a loop, α and β are also loops, and $\tilde{\ell}(\rho_\alpha, \tau_i) = \tilde{\ell}(\rho_\beta, \tau_i) = 0$ for any arc $\tau_i \in T$. Since in this case, $\tilde{\rho}_\alpha$ and $\tilde{\rho}_\beta$ are M -paths (not loosened M -paths), we obtain (6.9) immediately from (6.6). In the case that γ_1 is a generalized arc, we have that $\tilde{\rho}_\alpha$ and $\tilde{\rho}_\beta$ are loosened M -paths, so we apply Lemma 6.10 to $|\text{ur}(\overline{M}(\tilde{\rho}_\alpha))|$ and $|\text{ur}(\overline{M}(\tilde{\rho}_\beta))|$. We then obtain (6.9) from (6.5).

The proof of Proposition 6.14 is analogous. We set $m_1 = \overline{M}(\rho_\gamma^{(2)})$, $m_2 = \overline{M}(\rho_\gamma^{(1)})$, and $m_3 = \overline{M}(\rho_\gamma^{(3)})$. If γ is a generalized arc, then from (6.5) we have that $\text{ur}(m_3 m_1 m_2) = \text{ur}(m_3 m_2) \text{tr}(m_1) - \text{ur}(m_3 m_1^{-1} m_2)$. And if γ is a closed loop, then from (6.6) we have that $\text{tr}(m_1 m_2) = \text{tr}(m_2) \text{tr}(m_1) - \text{tr}(m_1^{-1} m_2)$. Applying Lemma 6.10 gives (6.10). \square

We can use the previous three propositions to obtain formulas for the quantities $\hat{\chi}_{\gamma, T}$. Before stating these formulas, we introduce a variant of intersection numbers that we refer to as the *signed intersection number* between a (loosened) M -path ρ and an arc $\tau \in T$.

Recall from Definition 3.11 that $e(\gamma, \tau)$ denotes the *intersection number* between γ and τ , i.e. the number of crossings between the generalized arc γ and the arc τ .

Definition 6.16. Given a (loosened) M -path $\tilde{\rho}_\gamma$ for a generalized arc or loop γ , and an arc τ , we define the *signed intersection number* between $\tilde{\rho}_\gamma$ and τ to be

$$\ell(\tilde{\rho}_\gamma, \tau) = e(\gamma, \tau) + \tilde{\ell}(\tilde{\rho}_\gamma, \tau).$$

Remark 6.17. Our definitions of signed intersection numbers and signed excess for loosened M -paths are motivated by the definition of transverse measures appearing in [FT, Sec. 13]. In particular, when T has no self-folded triangles, the signed intersection number $\ell(\tilde{\rho}_\gamma, \tau)$ between a loosened M -path $\tilde{\rho}_\gamma$ corresponding to an arc γ and an arc $\tau \in T$ is equal to the *transverse measure* $\ell_{\overline{L}_\tau}(\overline{\gamma})$ between a *lift* $\overline{\gamma}$ of the arc γ to an *opened surface* and a lift \overline{L}_τ of the elementary lamination corresponding to arc $\tau \in T$. There is a direct connection between these two quantities if T has self-folded triangles as well, but the formula relating the two is more complicated. In particular, in this case we have a multi-lamination containing two elementary laminations spiralling into a puncture (one clockwise, one counterclockwise), but the transverse measures associated to such laminations do not match up as simply to the signed intersection numbers which are defined in this paper.

Corollary 6.18. *Using the notation of Proposition 6.12, we have*

$$\begin{aligned} \chi_{\gamma_1, T} \chi_{\gamma_2, T} &= \pm \chi_{\alpha_1, T} \chi_{\alpha_2, T} \prod_{i=1}^n y_i^{\frac{\ell(\tilde{\rho}_{\gamma_1}, \tau_i) + \ell(\tilde{\rho}_{\gamma_2}, \tau_i) - \ell(\rho_{\alpha_1}, \tau_i) - \ell(\rho_{\alpha_2}, \tau_i)}{2}} \\ &\quad \pm \chi_{\beta_1, T} \chi_{\beta_2, T} \prod_{i=1}^n y_i^{\frac{\ell(\tilde{\rho}_{\gamma_1}, \tau_i) + \ell(\tilde{\rho}_{\gamma_2}, \tau_i) - \ell(\tilde{\rho}_{\beta_1}, \tau_i) - \ell(\tilde{\rho}_{\beta_2}, \tau_i)}{2}}. \end{aligned}$$

Using the notation of Proposition 6.13, we have

$$\chi_{\gamma_1, T} \chi_{\gamma_2, T} = \pm \chi_{\alpha, T} \prod_{i=1}^n y_i^{\frac{\ell(\rho_{\gamma_1, \tau_i}) + \ell(\rho_{\gamma_2, \tau_i}) - \ell(\tilde{\rho}_{\alpha, \tau_i})}{2}} \pm \chi_{\beta, T} \prod_{i=1}^n y_i^{\frac{\ell(\rho_{\gamma_1, \tau_i}) + \ell(\rho_{\gamma_2, \tau_i}) - \ell(\tilde{\rho}_{\beta, \tau_i})}{2}}.$$

Using the notation of Proposition 6.14, we have

$$\chi_{\gamma, T} = \pm \chi_{\alpha_1, T} \chi_{\alpha_2, T} \prod_{i=1}^n y_i^{\frac{\ell(\rho_{\gamma, \tau_i}) - \ell(\tilde{\rho}_{\alpha_1, \tau_i}) - \ell(\tilde{\rho}_{\alpha_2, \tau_i})}{2}} \pm \chi_{\beta, T} \prod_{i=1}^n y_i^{\frac{\ell(\rho_{\gamma, \tau_i}) - \ell(\tilde{\rho}_{\beta, \tau_i})}{2}}.$$

Proof. Since we have assumed that T has no self-folded triangles, we have that $\chi_{\gamma, T} = \hat{\chi}_{\gamma, T}$. Now it follows from Definition 4.9 that when we substitute the quantities $\hat{\chi}_{\gamma, T}$ for $\bar{\chi}_{\gamma, T}$ in Propositions 6.12, 6.13 and 6.14, we get an extra factor of $\sqrt{y_{\tau_i}}$ for every elementary step of type 2 in the corresponding M -path. This explains the appearance of the $e(\gamma, \tau_i)/2$'s (implicitly via the signed intersection numbers) in the exponents of the new identities. Corollary 6.18 then follows from (6.8), (6.9), and (6.10). \square

Remark 6.19. Note that the coefficients in Corollary 6.18 seem to depend on our choices of M -paths and loosened M -paths. However, it follows from [FT, (13.10)] (which cites [FG1, FG3] and well-known facts about intersection numbers) that in fact the coefficients do not depend on our choices. To apply [FT, (13.10)] to our situation, we use the dictionary indicated in Remark 6.17 in the case when T has no self-folded triangles. Though the signed intersection numbers or signed excesses *do* depend on the choices of loosened M -paths, the differences between the signed intersection numbers appearing in the exponents of these expressions are independent of these choices.

Lemma 6.20. *The coefficients appearing in Corollary 6.18 are Laurent monomials in the variables y_1, \dots, y_n .*

Proof. It suffices to show that the exponent of each y_i in Corollary 6.18 is an integer (not a half-integer). We first claim that for any loosened M -path $\tilde{\rho}_{\gamma}$, $\ell(\tilde{\rho}_{\gamma}, \tau) \equiv e(\tilde{\rho}_{\gamma}, \tau) \pmod{2}$. To see this, write $\tilde{\rho}_{\gamma} = \sigma_2 \circ \rho_{\gamma} \circ \sigma_1$, and let $a = e(\sigma_1, \tau)$, $b = e(\sigma_2, \tau)$, and $x = e(\rho_{\gamma}, \tau)$. Then $e(\tilde{\rho}_{\gamma}, \tau) = x + a + b$ and $\ell(\tilde{\rho}_{\gamma}, \tau) = x \pm a \pm b$. It's clear that both $x + a + b$ and $x \pm a \pm b$ have the same parity.

Now observe that if C (as in Definition 6.1) is a pair of curves with an intersection or one curve with a self-intersection, and we smooth C at a point x , obtaining C_+ , then $e(C, \tau) - e(C_+, \tau)$ is a (positive) even integer. Here, if $C = \{\gamma_1, \gamma_2\}$, then $e(C, \tau)$ denotes $e(\gamma_1, \tau) + e(\gamma_2, \tau)$. The lemma follows. \square

We now turn back to the unpunctured surface case, and Propositions 6.4, 6.5, and 6.6. In this situation, every marked point $m \in M$ is on the boundary of S , and thus we have an arc segment $h_m \cap S$ (rather than a circle h_m) associated to each $m \in M$. We define some new notation that will be useful in the proofs of these propositions.

Given a boundary component C_m containing m , we let b_m denote the boundary segment immediately before and incident to m when traveling along C_m in the counterclockwise direction. We let V_m^+ denote the clockwise-most vertex on $h_m \cap S$, which also has the form v_{m, b_m}^- . See Figure 24. With this notation in mind, we now prove Propositions 6.4, 6.5, and 6.6 together, using Corollary 6.18.

Proof. The main idea of this proof is that for each arc α involved in one of the skein relations, we can choose a loosened M -path $\tilde{\rho}_{\alpha}$ such that $\ell(\tilde{\rho}_{\alpha}, \tau_i) = e(\alpha, L_i)$. In particular, we choose loosened M -paths $\tilde{\rho}$ so that they start and end at a point in the set $\{V_m^+\}_{m \in M}$. For a generalized arc γ , this completely determines the starting and ending points of the

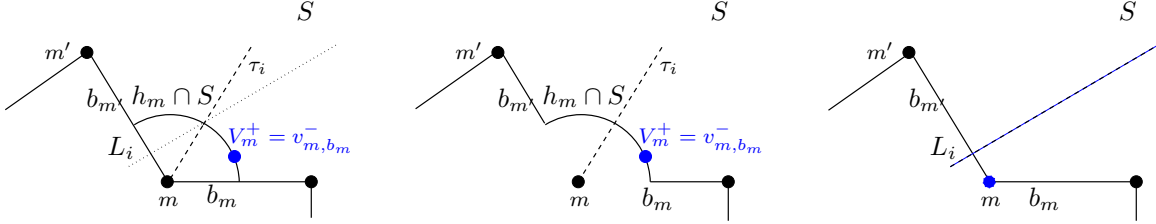


FIGURE 24. (Left): Illustrating a horocyclic arc segment at a boundary. (Middle and Right): Comparing the local configuration of τ_i and V_m^+ with the local configuration of L_i and the marked point m .

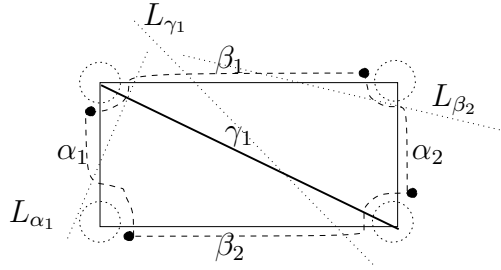


FIGURE 25. Comparing shear coordinates with respect to elementary laminations to intersections with loosened M -paths

associated loosened M -path $\tilde{\rho}_\gamma$. As above, we decompose such a loosened M -path $\tilde{\rho}_\gamma$ as $\sigma_2 \circ \rho_\gamma \circ \sigma_1$ where ρ_γ is an M -path, and σ_i consists only of steps of type 1 and 2.

First, observe that in this situation the signed excess $\tilde{\ell}(\tilde{\rho}_\gamma, \tau_i)$ is nonnegative and equal to $e(\sigma_1, \tau_i) + e(\sigma_2, \tau_i)$ for any generalized arc γ and any arc $\tau_i \in T$; see Figure 23. Second, observe from Figure 24 that the relationship between V_m^+ and τ_i is analogous to the relationship between m and L_i : in particular, any arc $\tau_i \in T$ incident to m lies in the counterclockwise direction (along C_m) from V_m^+ just as the corresponding elementary lamination L_i lies in the counterclockwise direction from m . Consequently, for any generalized arc or loop γ and arc $\tau_i \in T$, we have the identity

$$(6.12) \quad \ell(\tilde{\rho}_\gamma, \tau_i) = \tilde{\ell}(\tilde{\rho}_\gamma, \tau_i) + e(\gamma, \tau_i) = e(\sigma_1, \tau_i) + e(\sigma_2, \tau_i) + e(\gamma, \tau_i) = e(\gamma, L_i).$$

Plugging (6.12) into Corollary 6.18 completes the proof of all three propositions. \square

See Figure 25 for an example; note that of the three laminations listed, L_{γ_1} is the only lamination that crosses $\{\gamma_1, \gamma_2\}$ and $\{\beta_1, \beta_2\}$ but not $\{\alpha_1, \alpha_2\}$. Analogously the concatenation $\tilde{\rho}_{\alpha_2} \circ \tilde{\rho}_{\beta_1}$ can be pulled taut as to avoid intersections with β_1 and β_2 . This yields the loosened M -path $\tilde{\rho}_{\gamma_1}$. However, the loosened M -paths $\tilde{\rho}_{\alpha_1}$ and $\tilde{\rho}_{\beta_1}$ must intersect both the pairs $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$. This is consistent with the exchange relation

$$x_{\gamma_1} x_{\gamma_1'} = y_{\gamma_1} x_{\alpha_1} x_{\alpha_2} + x_{\beta_1} x_{\beta_2}$$

that we obtain by shear coordinates in this example.

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