Math 55 Lecture 22 §9.4, §9.5

Recall: Let A, B be sets. A binary relation R from A to B is

Notation: Use aRb to

And a relation on A is a

A relation R on A is transitive if whenever

Suppose R is not trans. What do we add to R to make it trans.?

Say $R =$ what to add?

Attempt 1: Is it enough to add a new arrow $a \to c$ whenever $a \to b$ and $b \to c$?

Attempt 2:

Def: A path of length n from a to b in the directed graph G is a sequence of edges

Also: there is path from a to b in relation R if there is sequence of elements

What are paths of length 2

Recall: If R relation on A,

$R \circ R = \{(a,c) \mid \}$.

So $R \circ R \circ R = \{(a,d) \mid \}$,

Notation: $R^2 =$, $R^3 =$ ...

Theorem: Let R be relation on A. There is path of length n, where $n \in \mathbb{Z}^+$, from a to b, iff

Pf:

Def: Let R be relation on A. The connectivity relation $R^* = \{(a,b) \mid \exists$ path $\}$

So $R^* =$

Ex: Let $R = \{(a,b) \mid \}$

$R^* = ??$

Thm: The transitive closure of relation R is R^*

Pf: What do we need to show?

"R^* is the smallest transitive relation that contains R"

(1)

(2)

(3)

(1) $R^* \supset R$ by def.

(2) To show R^* is transitive: need to show that

(3)

§9.5 Equivalence Relations

Def: A relation R on set A called equivalence relation if

Recall: R is reflexive if $\forall\, a \in A$,

R is symmetric if whenever $(a,b) \in R$, we also have

Ex: Choose $n \in \mathbb{Z}^+$ & let R be relation on $\mathbb{Z}$ defined by

$R = \{ (a,b) \mid \qquad \}$

Then R is

(We showed

Def: If $(a,b) \in R$ and R an equiv. relation, we say a and b are

Sometimes use notation

Ex: $R = \{(a,b) \mid a \equiv b \mod n\}$ is

+ we use notation

Ex: Are these equiv. relations on $\{0,1,2\}$?

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Ex: Which of these relations on set of all functions $\mathbb{Z} \to \mathbb{Z}$ are equiv. relations?

- $R = \{(f,g) \mid \qquad\qquad \}$

  Reflex:

  Symm:

  Trans:

- $R = \{(f,g) \mid \qquad\qquad \}$

  Reflex:

  Symm:

  Trans:

Ex:

Ex: Let R be relation on the set $\mathbb{Z} \times \mathbb{Z} = \{(a,b) \mid a,b \in \mathbb{Z}^+\}$ s.t.
$((a,b),(c,d)) \in R$ iff

Pf:

Def: Let R be an equiv. relation on set A.
Choose $a \in A$. Define
$[a]_R =$ Called the <u>equivalence class</u> of a. Sometimes denoted $[a]$.
Any element $b \in [a]_R$ called a

Ex: What is equiv. class of 1, 2, for

Ex: How many equiv. classes are there

Sol:

Note:

Theorem: Let R be an equiv. relation on set A.

TFAE: (i) (ii) (iii)

Pf: Show (i) $\Rightarrow$ (ii).

Show (ii) $\Rightarrow$ (iii)

Show (iii) $\Rightarrow$ (i).

__Theorem__: Let R be equiv. relation on set A.
__Then__: The union of all equiv. classes is
Two equiv. classes are either

__Def__: A __partition__ of a set S is collection of disjoint nonempty subsets of S,

Ex: