Computing Time-Periodic Solutions of Nonlinear Systems of Partial Differential Equations*

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Abstract

We describe an efficient numerical method for the computation of time-periodic solutions of nonlinear systems of partial differential equations. The strategy is to minimize a functional that measures how far solutions of the initial value problem are from being time-periodic by solving an adjoint problem to compute the gradient of the functional. We discuss applications to two different problems in interfacial fluid dynamics, the Benjamin-Ono equation and the vortex sheet with surface tension.

1 Introduction

Many evolutionary systems of partial differential equations possess special solutions such as traveling waves or time-periodic solutions. It can be useful to understand the set of all such solutions, but it can frequently be difficult to make either analytic or numerical studies of these solutions. In this paper, we will describe a numerical method for the computation of families of time-periodic solutions of nonlinear systems of partial

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differential equations which the authors have recently developed. The solutions we study are taken to also be spatially periodic.

The main idea of this numerical method is to define a functional (depending on the initial data and the presumed period) which is zero if the solution of the system is time-periodic with this period, and which is positive otherwise. This functional is then minimized; in order to do this, we need to be able to compute the variational derivative of the functional with respect to the initial data and the presumed period.

There are two primary types of alternative methods. Orthogonal collocation methods, such as AUTO [6], can be efficient when computing traveling solutions, but can be difficult to use for the much larger problem of finding time-periodic solutions. There are also shooting and multi-shooting methods [12], which require computing the variation for each of the unknowns individually (the unknowns being the discretized values of the initial data, or its leading Fourier modes). We believe that the present method is in many cases more efficient than these alternatives.

We have applied the method to two problems from interfacial fluid dynamics, finding time-periodic solutions of the Benjamin-Ono equation [1, 2], and finding time-periodic symmetric vortex sheets with surface tension [3, 4].

2 Formulation and Numerical Method

Consider the solution $u(x,t)$ of the initial value problem

$$\partial_t u = F(u), \quad u(x,0) = u_0(x). \tag{2.1}$$

No assumptions are made about the operator F; it can include nonlinear, nonlocal, and unbounded operators. We take the spatial domain to be 2π, with periodic boundary conditions, which is to say, $u(x+2\pi,t) = u(x,t)$. We consider the following functional:

$$G(u_0,T) = \frac{1}{2}\int_0^{2\pi} \left(u(x,T) - u_0(x)\right)^2 \, dx.$$

Clearly, a solution u of (2.1) with initial data u_0 is time-periodic with period T if and only if $G(u_0,T) = 0$.

We seek to minimize G numerically. To do this, we use the BFGS minimization algorithm [10]. This requires computing the derivative of G with respect to each of u_0 and T. It is clear how to compute the derivative with respect to T; we now describe how to calculate the derivative with respect to u_0. We remark that in applications, we may instead choose to minimize a different functional depending on the situation. In any case, computing the derivative with respect to the initial data will use the same ideas as the present calculation.

To begin, we compute the variation of G as follows:

$$\dot{G} = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} G(u_0 + \varepsilon\dot{u}_0, T) = \int_0^{2\pi} (u(x,T) - u_0(x))(\dot{u}(x,T) - \dot{u}_0(x))\, dx.$$

We seek to write this as $\dot{G} = \langle \frac{\delta G}{\delta u_0}, \dot{u}_0\rangle$. To this end, we define an auxiliary quantity $Q(x,s)$, where $Q(x,0) = u(x,T) - u_0(x)$ and $s = T - t$. We see then that $\dot{G} = \langle Q(\cdot,0), \dot{u}(\cdot,T) - \dot{u}_0\rangle$. We will define $Q(x,t)$ for positive s so that $\langle Q(\cdot,s), \dot{u}(\cdot,T-s)\rangle$ is constant.

Notice that since $u_t = F(u)$, we have $\dot{u}_t = (DF(u))\dot{u}$, where DF is the linearization of the operator F. Now, differentiating our inner product with respect to s, we have

$$\langle Q_s(\cdot,s), \dot{u}(\cdot,T-s)\rangle - \langle Q(\cdot,s), (DF(u(\cdot,T-s)))\dot{u}(\cdot,T-s)\rangle = 0.$$

Taking the adjoint of DF, we se that this equality can be accomplished by setting $Q_s = (DF(u(\cdot,T-s))^*Q$. We then have

$$\frac{\delta G}{\delta u_0}(x) = Q(x,T) - Q(x,0).$$

3 Results: Benjamin-Ono

The Benjamin-Ono equation is

$$u_t - Hu_{xx} + uu_x = 0,$$

where H is the Hilbert transform and subscripts denote differentiation. This is a well-known dispersive PDE which arises as a long-wave model in interfacial fluid dynamics. For $\beta \in (-1,1)$, there are the stationary solutions

$$u(x;\beta) = \frac{1-3\beta^2}{1-\beta^2} + \frac{4\beta(\cos(x)-\beta)}{1+\beta^2-2\beta\cos(x)}.$$

In one period, these solutions $u(x;\beta)$ have one hump; by suitably rescaling, we have N-hump solutions, given by $u(x;\beta,N) = Nu(Nx;\beta)$. Furthermore, if $u(x)$ is any such stationary solution of the Benjamin-Ono equation, then $u(x-ct)+c$ is a traveling wave solution. These are known to be all of the periodic traveling wave solutions of Benjamin-Ono [5].

Linearizing about the stationary solutions, we find the linearized evolution $v_t = iBAv$, where $A = H\partial_x - u$ and $B = \frac{1}{i}\partial_x$. We are able to solve for the eigenvalues and eigenvectors of the operator BA, and these give formulas for time-periodic solutions of the linearized equation. We then look for time-periodic solutions of the full problem, using the period and initial data for the time-periodic solutions of the linearized equations to determine the initial guess in the BFGS algorithm when minimizing G.

Our findings in [1] and [2] are that there are indeed bifurcations from stationary or traveling solutions of Benjamin-Ono, at all of the periods predicted by the linear theory. In fact, for each bifurcation, we have a four-parameter family of solutions emanating from the trivially time-periodic (i.e., stationary or traveling) solution. We furthermore find that every such four-parameter family emanating from a trivially time-periodic solution coincides with a four-parameter family emanating from a different trivially time-periodic solution; that is to say, we find that these stationary and traveling solutions are connected to each other through continua of nontrivial time-periodic solutions. One such nontrivial time-periodic solution is shown in Figure 3.1.

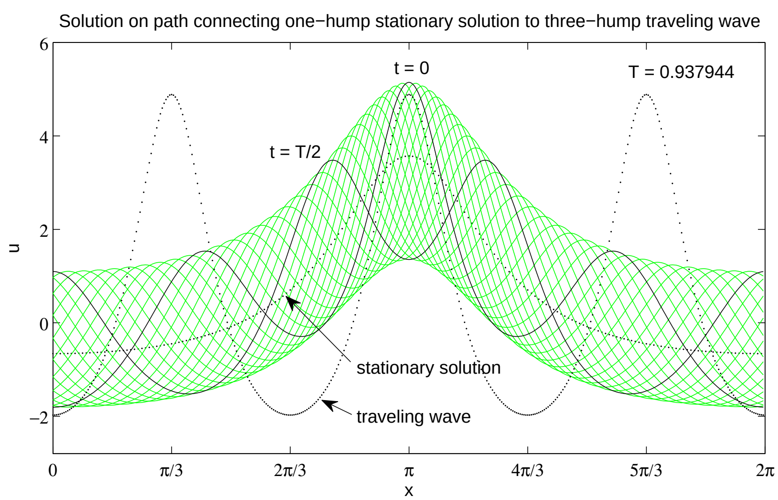

Figure 3.1: A time-periodic solution of the Benjamin-Ono equation. This solution belongs to a continuum of solutions which connect a stationary solution with one hump to a traveling solution with three humps; these trivially time-periodic solutions are shown with dotted curves.

For the Benjamin-Ono equation, several other authors have studied time-periodic solutions, finding explicit formulas for these solutions [9, 11]. It is in fact possible to use these explicit formulas to prove that the bifurcations we find in [1, 2] do in fact occur; in one case, this is proved in [2]. This is carried further by the second author in [13], and additionally, bifurcations which are not predicted by the linear theory are also shown to occur. The existence of explicit formulas for the time-periodic solutions of the Benjamin-Ono equation provides an important validation of our numerical method; we can confirm that every solution we have computed for this equation does in fact correspond to a time-periodic solution of the equation, and is not just a solution of the initial value problem which comes very close to returning to its initial state.

4 Results: Vortex Sheets

The vortex sheet with surface tension is the interface between two irrotational, immiscible fluids satisfying the incompressible Euler equations. Across the interface, the normal component of the velocity is continuous while the tangential velocity may jump. This implies that the vorticity is not identically zero, but is instead a Dirac measure supported on the interface. We describe the interface as a parameterized curve at each time, $(x(\alpha, t), y(\alpha, t))$, horizontally periodic with period 2π :

$$x(\alpha + 2\pi, t) = x(\alpha, t) + 2\pi, \qquad y(\alpha + 2\pi, t) = y(\alpha, t).$$

For this problem, an efficient numerical method for solution of the initial value problem was developed by Hou, Lowengrub, and Shelley [7, 8]. We use this same method for the numerical solution of the initial value problem when solving for time-periodic solutions with our algorithm.

The method of [7, 8] uses a natural geometric formulation of the problem. Rather than evolving (x, y), the tangent angle $\theta = \tan^{-1}(y_\alpha/x_\alpha)$ and the arclength element $s_\alpha = (x_\alpha^2 + y_\alpha^2)^{1/2}$ are evolved instead. Furthermore, the parameterization is taken to be a normalized arclength parameterization: if $L(t)$ is the length of one period of the interface at time t, then we insist on the condition $s_\alpha(\alpha, t) = L(t)/2\pi$. This condition can be maintained by properly choosing the tangential velocity of the interface; by contrast, the normal velocity is determined by the fluid dynamics. For a given 2π-periodic curve with corresponding tangent angle θ, the length L can be determined from θ. Therefore, the position of the interface can be determined from θ and the location of one point.

If we let $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ be unit tangent and normal vectors at each point of the interface at each time, then the velocity of the interface can be written $(x, y)_t = U\hat{\mathbf{n}} + V\hat{\mathbf{t}}$. It is sometimes convienient to denote the position of the interface as $z = x + iy$. The normal velocity is $U = \mathbf{W} \cdot \hat{\mathbf{n}}$, where $\mathbf{W} = (W_1, W_2)$ is the Birkhoff-Rott integral defined as

$$W_1 - iW_2 = \frac{1}{4\pi i}\mathrm{PV}\int_0^{2\pi} \gamma(\alpha') \cot\left(\frac{1}{2}(z(\alpha) - z(\alpha'))\right)\, d\alpha.$$

The quantity γ is the vortex sheet strength and measures the jump in tangential velocity across the interface.

We have the following evolution equations:

$$\theta_t = \frac{U_\alpha + V\theta_\alpha}{s_\alpha}, \qquad \gamma_t = \tau\frac{\theta_{\alpha\alpha}}{s_\alpha} + \left(\frac{\gamma(V - \mathbf{W}\cdot\hat{\mathbf{t}})}{s_\alpha}\right)_\alpha.$$

The constant τ here is the coefficient of surface tension, which is positive. With surface tension present, the system is dispersive and is well-posed.

If we were to attempt to minimize the functional G described above, we would need to compute solutions of the initial value problem from time $t = 0$ until time $t = T$, and we would then need to solve the adjoint problem backward in time over this same interval. Instead of doing this, we utilize symmetry and only perform one-fourth of the computations. We do this differently in the cases where the mean of γ is zero or non-zero. In the case that the mean of γ is zero, we take initial data $\theta_0(\alpha) = 0$, and we seek solutions for which

$$G_1(\gamma_0, T) = \int_0^{2\pi} \gamma^2 \left(\alpha, \frac{T}{4}\right)\, d\alpha = 0.$$

In the case that the mean of γ is non-zero, we again use initial data $\theta_0(\alpha) = 0$, and we seek solutions for which

$$\begin{aligned} G_2(\gamma_0, T) = \int_0^{2\pi} & \left(\gamma\left(\alpha, \frac{T}{4}\right) - \gamma\left(\alpha - \pi, \frac{T}{4}\right)\right)^2 \\ & + \left(\theta\left(\alpha, \frac{T}{4}\right) + \theta\left(\alpha - \pi, \frac{T}{4}\right)\right)^2\, d\alpha = 0 \qquad (4.1) \end{aligned}$$

We can argue that zeros of G_1 or G_2 lead to time-periodic solutions with period T. We now give the argument in the second case.

Consider γ_0 and T such that $G_2(\gamma_0, T) = 0$. We can then solve the initial value problem with data θ_0 and γ_0 to find θ and γ for $0 \leq t \leq \frac{T}{4}$, and we have $\theta\left(\alpha, \frac{T}{4}\right) = -\theta\left(\alpha - \pi, \frac{T}{4}\right)$ and $\gamma\left(\alpha, \frac{T}{4}\right) = \gamma\left(\alpha - \pi, \frac{T}{4}\right)$. If we define $\Theta(\alpha, t) = -\theta\left(\alpha - \pi, \frac{T}{4} - t\right)$ and $\Gamma(\alpha, t) = \gamma\left(\alpha - \pi, \frac{T}{4} - t\right)$, then Θ and Γ are also solutions of the vortex sheet evolution problem. Furthermore, $\Theta(\alpha, 0) = \theta\left(\alpha, \frac{T}{4}\right)$ and $\Gamma(\alpha, 0) = \gamma\left(\alpha, \frac{T}{4}\right)$. Therefore, Θ and Γ continue θ and γ. That is, for $\frac{T}{4} \leq t \leq \frac{T}{2}$, we can take $\theta(\alpha, t) = \Theta\left(\alpha, t - \frac{T}{4}\right) = -\theta\left(\alpha - \pi, \frac{T}{2} - t\right)$ and $\gamma(\alpha, t) = \Gamma\left(\alpha, t - \frac{T}{4}\right) = \gamma\left(\alpha - \pi, \frac{T}{2} - t\right)$. Similarly, we can continue the solution to $\frac{T}{2} \leq t \leq T$ by making the definitions $\theta(\alpha, t) = \theta\left(\alpha - \pi, t - \frac{T}{2}\right)$ and $\gamma(\alpha, t) = \gamma\left(\alpha - \pi, t - \frac{T}{2}\right)$. We see then that $\theta(\alpha, T) = \theta\left(\alpha - \pi, \frac{T}{2}\right) = -\theta(\alpha, 0) = \theta(\alpha, 0)$. Similarly, we have $\gamma(\alpha, T) = \gamma\left(\alpha - \pi, \frac{T}{2}\right) = \gamma(\alpha, 0)$. So, we have indeed found a time-periodic solution.

Using the BFGS minimization algorithm requires computing $\frac{\delta G_1}{\delta \gamma_0}$ or $\frac{\delta G_2}{\delta \gamma_0}$ as appropriate. As in the calculation of $\frac{\delta G}{\delta u_0}$ in Section 2, these variational derivatives can be found in terms of the solution of the adjoint PDE for the linearization of the equations of motion. Thus, we linearize the evolution equations about an arbitrary state, calculating the operator $DF(u)$ (using the notation of Section 2, with $u = (\theta, \gamma)$ here), and then

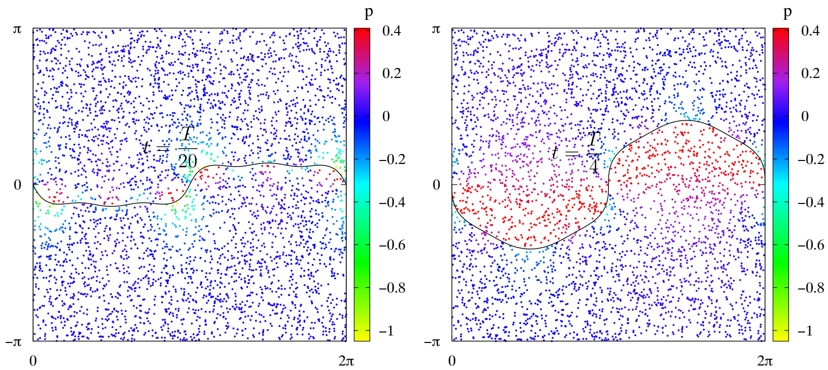

Figure 4.1: Two profiles of a time-periodic vortex sheet with surface tension with $\tau = 1$ and zero mean vortex sheet strength. The particles in the fluid are passive tracers added for visualization.

taking its adjoint $(DF(u))^*$. This is significantly complicated by the presence of nonlinear, nonlocal singular integral operators, such as the Birkhoff-Rott integral. Details of the calculation can be found in [3].

In [3] and [4], we compute bifurcations from the flat equilibrium $\theta = 0$ in the case of zero and nonzero mean vortex sheet strength, respectively. In both cases, we predict bifurcation from the flat vortex sheet using linear theory, and we then find numerically that the expected bifurcation does occur. We are thus able to compute many time-periodic solutions, continuing the bifurcation curves as far as possible. Two such time-periodic solutions are shown in Figures 4.1 and 4.2.

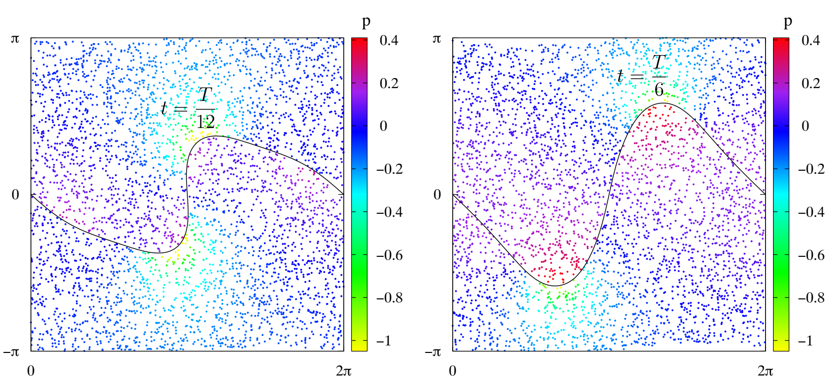

Figure 4.2: Two profiles of a time-periodic vortex sheet with surface tension with $\tau = 1$ and nonzero mean vortex sheet strength. Notice that the solution exhibits less symmetry than the solution in Figure 4.1.

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