Some Analytic Results for Kimura Diffusion Operators

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Abstract

In this note we prove several analytical results about generalized Kimura diffusion operators, L, defined on compact manifolds with corners, P. It is shown that the $\mathcal{C}^0(P)$-graph closure of L acting on $\mathcal{C}^2(P)$ always has a compact resolvent. In the $1d$-case, where $P = [0,1]$, we also establish a gradient estimate $\|\partial_x f\|_{\mathcal{C}^0([0,1])} \leq C\|Lf\|_{\mathcal{C}^0([0,1])}$, provided that L has strictly positive weights at $\partial[0,1] = \{0,1\}$. This in turn leads to a precise characterization of the domain of the $\mathcal{C}^0$-graph closure in this case.

1 Introduction

The Kimura diffusion operator is used to defined the standard "forward time model" in Population Genetics and other fields in evolutionary Biology. The classical Kimura diffusion operator is defined on the simplex, $\Sigma_N \subset \mathbb{R}^N$, by the second order differential operator

$$L_{\mathrm{Kim}} u = \sum_{1\leq i,j\leq N} x_i(\delta_{ij} - x_j)\partial_{x_i}\partial_{x_j} + \sum_{1\leq i\leq N} b_i(x)\partial_{x_i}, \tag{1.1}$$

where $\Sigma_N = \{(x_1, \ldots, x_N) : 0 \leq x_i, i = 1, \ldots, N \text{ and } x_1 + \cdots + x_N \leq 1\}$. The first order part

$$\sum_{1\leq i\leq N} b_i(x)\partial_{x_i}, \tag{1.2}$$

is required to be inward pointing along the boundary of Σ_N. In most applications to Biology the coefficients $b_i(x)$ are polynomials. The linear terms in $\{b_i(x)\}$ model mutation, and the higher order terms model effects like selection and migration. In a recent monograph [2] this class of operators on simplices has been generalized to a

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broad class of second order operators, called "generalized Kimura diffusions," which are defined on manifolds-with-corners. These are briefly defined below; we refer the reader to the cited monograph for complete definitions.

A topological space, P, is an N-dimensional manifold with corners if every point, p, has a coordinate chart isomorphic to $[0,1)^n \times (-1,1)^m$, where $n+m=N$. Clearly the ∂P is not usually a smooth manifold, but is a stratified space. We denote these coordinates by $(x;y)$. If p corresponds to $(0;y)$, then we say that p lies on a boundary stratum of codimension n.

A second order partial differential operator, L, defined on P, is a generalized Kimura diffusion operator, if, in such a coordinate chart, L is given by

$$Lu = \sum_{i=1}^{n}[x_i a_i(x;y)\partial_{x_i}^2 + b_i(x;y)\partial_{x_i}]u + \sum_{i,j=1}^{n} + x_i x_j a_{ij}(x;y)\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{n}\sum_{l=1}^{m} x_i c_{il}\partial_{x_i}\partial_{y_l}u + \sum_{k,l=1}^{m} d_{kl}(x;y)\partial_{y_k}\partial_{y_l}u + \sum_{l=1}^{m} d_l(x;y)\partial_{y_l}u, \quad (1.3)$$

where the second order part is strictly elliptic in the interior, and the coefficients satisfy

$$a_i(0;y) > 0, \quad \sum_{l,m=1}^{m} d_{kl}(0;y)\xi_k\xi_l \geq C|\xi|^2, \quad (1.4)$$

for some $C>0$, and $b_i(x;y) \geq 0$ where $x_i = 0$. More detailed definitions can be found in [2]. The principal symbol, $\sigma_2(L)$, defines an incomplete Riemannian metric on P. The WF-Hölder spaces are defined with respect to the distance function defined by this metric.

If P is a compact manifold with corners and L is a generalized Kimura diffusion on P, then it is shown that the initial value problem,

$$\partial_t u - Lu = 0 \text{ with } u(x,0) = f(x), \quad (1.5)$$

has a unique solution that is in $\mathcal{C}^\infty(P\times(0,\infty)) \cap \mathcal{C}^0(P\times[0,\infty))$ provided that f belongs to a WF-Hölder space. This is called the *regular solution*. Optimal results in these spaces are also obtained for the inhomogeneous problem $\partial_t u - Lu = g$. These results have been extended to data in $\mathcal{C}^0(P)$ in [5] and [3]. At least as regards initial data in $\mathcal{C}^0(P)$ there are several points that have not yet been satisfactorily understood: 1. Estimates on ∇u, for the solution to (1.5), in terms of the initial data do not seem to be optimal. 2. It has not been shown that the $\mathcal{C}^0(P)$-graph closure of L acting on $\mathcal{C}^2(P)$ has a compact resolvent. This short note is directed towards improving our state of knowledge on these points.

We show below that the $\mathcal{C}^0(P)$-graph closure of L acting on $\mathcal{C}^2(P)$ always has a compact resolvent. We also consider the gradient estimate in 1-dimension for the elliptic problem $Lu = h$ on $[0,1]$. Under the additional assumption that the first order part of L is strictly inward pointing at 0 and 1, we show that the $\mathcal{C}^0$-norm of $\partial_x u$ is bounded by a constant times $\|Lu\|_{\mathcal{C}^0([0,1])}$. We also show that this is false if the first order term vanishes at one or both of the boundary points.

2 The Graph Closure in the $\mathcal{C}^0$-topology

One of the goals in this paper is to describe the domain of the $\mathcal{C}^0$-graph closure of a Kimura diffusion in $1d$. The key to this result is a gradient estimate for the elliptic problem. This is also a step towards proving sharp estimates on the gradient for the solution of the parabolic problem as well as proving estimates in higher dimensional cases. These analytic results are also useful in the development of spectrally accurate numerical methods for solving variants of the Kimura diffusion equation, see [4].

In $1d$ we work on the unit interval, $[0,1]$, where the simplest Kimura diffusions are given by

$$L_{\alpha,\beta}u = \frac{x(1-x)}{2\rho}\partial_x^2 u + \alpha(1-x)\partial_x u - \beta x \partial_x u, \qquad (0 < x < 1), \tag{2.1}$$

with $\rho > 0$, $\alpha \geq 0$ and $\beta \geq 0$. For the parabolic problem $u_t = L_{\alpha,\beta}u$, the operator $L_{\alpha,\beta}$ can be generalized by allowing ρ, α, β to depend on time; by imposing singular, possibly time-dependent boundary conditions (when $\alpha < 1$ or $\beta < 1$); or by adding additional terms that vanish at the boundary, such as $\gamma(x,t)x(1-x)\partial_x u$. In this section we assume that α, β, ρ are constant. Scaling time and adjusting α and β accordingly, we may assume $2\rho = 1$. The rescaled coefficients α, β are called the *weights* of L at the boundary of $[0,1]$.

In this section we consider the $\mathcal{C}^0$-graph closure of a Kimura diffusion operator in 1-dimension. The main result here is a precise characterization of the domain of the graph closure of a Kimura operator with respect to the $\mathcal{C}^0$-norm, which is a consequence of an estimate of $\partial_x u$ in terms of Lu.

2.1 The definition of WF-Hölder-spaces

In this section we collect some basic definitions and review some analytical results. We begin with the definitions of the $1d$ WF-Hölder spaces. A complete discussion can be found in [2], which contains a thorough treatment of the analysis of generalized Kimura diffusions in these specially adapted Hölder spaces.

We define the WF-distance on the interval to be

$$\rho_{\mathrm{WF}}(x_1, x_2) = \int_{x_1}^{x_2} \frac{dx}{\sqrt{x(1-x)}}. \tag{2.2}$$

For $0 \leq x_1, x_2 \leq \frac{1}{2}$ this behaves like $\rho_{\mathrm{WF}}(x_1,x_2) \propto |\sqrt{x_1} - \sqrt{x_2}|$, and if $\frac{1}{2} \leq x_1, x_2 \leq 1$, then $\rho_{\mathrm{WF}}(x_1,x_2) \propto |\sqrt{1-x_1} - \sqrt{1-x_2}|$. For $0 < \gamma \leq 1$, the WF-Hölder space, $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}([0,1])$, consists of functions $f \in \mathcal{C}^0([0,1])$ for which the semi-norm

$$[f]_{\mathrm{WF},\gamma} = \sup_{x_1 \neq x_2 \in [0,1]} \frac{|f(x_1) - f(x_2)|}{\rho^{\gamma}_{\mathrm{WF}}(x_1,x_2)} \tag{2.3}$$

is finite. The norm on $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}([0,1])$ is defined by

$$\|f\|_{\mathrm{WF},0,\gamma} = \|f\|_\infty + [f]_{\mathrm{WF},\gamma}. \tag{2.4}$$

We also define another, ladder of spaces, denoted by $\mathcal{C}^{0,2+\gamma}_{\mathrm{WF}}([0,1])$, with $0 < \gamma \leq 1$. A function $f \in \mathcal{C}^1([0,1]) \cap \mathcal{C}^2((0,1))$, belongs to this space if the norm

$$\|f\|_{\mathrm{WF},0,2+\gamma} = \|f\|_{\mathrm{WF},0,\gamma} + \|\partial_x f\|_{\mathrm{WF},0,\gamma} + \|x(1-x)\partial_x^2 f\|_{\mathrm{WF},0,\gamma} \tag{2.5}$$

is finite. These spaces play a central role in the analysis of Kimura diffusion operators, as the maps $(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1} : \mathcal{C}^{0,\gamma}_{\mathrm{WF}}([0,1]) \to \mathcal{C}^{0,2+\gamma}_{\mathrm{WF}}([0,1])$, are bounded for all $0 < \gamma <$

1, and $\lambda \in (0, \infty)$, see [2]. This provides a precise characterization of the domain of a Kimura operator acting on a WF-Hölder space. Missing from this treatment is the classical case of data in $\mathcal{C}^0$, which we consider below.

2.2 Domain of L acting on $C^0([0,1])$

We now analyze the precise domain of the $\mathcal{C}^0$-graph closure of a Kimura operator, L. It is defined using $\mathcal{C}^2([0,1])$ as a core and taking the closure with respect to the graph norm

$$\|f\|_\infty + \|Lf\|_\infty. \tag{2.6}$$

We denote the domain of the closure by $\overline{\mathcal{D}}$. It depends only on the weights, (α, β) in (2.1). The easiest case to treat is when both α and β are positive. Since it is no more difficult, we treat the case of

$$L = L_{\alpha,\beta} + x(1-x)s(x)\partial_x,$$

for $s(x)$ a continuous function on $[0,1]$.

Theorem 2.1. *Suppose that α and β are positive. The domain, $\overline{\mathcal{D}}$ of the $\mathcal{C}^0$-graph closure of the operator $L = L_{\alpha,\beta} + x(1-x)s(x)\partial_x$ is given by*

$$\overline{\mathcal{D}} = \{f \in \mathcal{C}^2((0,1)) \cap \mathcal{C}^1([0,1]) : \lim_{x\to 0^+,1^-} x(1-x)\partial_x^2 f(x) = 0\}. \tag{2.7}$$

Proof. We first show that $\overline{\mathcal{D}} \subset \mathcal{C}^1([0,1])$, which will imply that $\overline{\mathcal{D}} \subset \mathcal{C}^2((0,1))$. To that end, suppose that $f \in \mathcal{C}^2([0,1])$ and $Lf = h$. This can be expressed as an equation for $g(x) = \partial_x f(x)$:

$$x(1-x)\partial_x g + (\alpha(1-x) - \beta x + x(1-x)s(x))g(x) = h(x). \tag{2.8}$$

An elementary calculation shows that this equation can be rewritten as

$$\partial_x(\sigma(x)g(x)) = \frac{\sigma(x)h(x)}{x(1-x)}, \tag{2.9}$$

where $\sigma(x) = x^\alpha(1-x)^\beta S(x)$, with

$$S(x) = \exp\left[\int_0^x s(y)dy\right]. \tag{2.10}$$

As α and β are both positive, integrating this relation gives two Volterra operators expressing g in terms of h :

$$g(x) = \int_0^x \frac{\sigma(y)h(y)dy}{\sigma(x)y(1-y)} \stackrel{d}{=} K_0 h(x) \quad \text{and} \quad g(x) = -\int_x^1 \frac{\sigma(y)h(y)dy}{\sigma(x)y(1-y)} \stackrel{d}{=} K_1 h(x). \tag{2.11}$$

Lemma 2.2. *If α and β are positive, then $K_0 : \mathcal{C}^0([0,\frac{1}{2}]) \to \mathcal{C}^0([0,\frac{1}{2}])$, and $K_1 : \mathcal{C}^0([\frac{1}{2},1]) \to \mathcal{C}^0([\frac{1}{2},1])$ are bounded operators.*

Proof. The proof follows easily from the facts that

$$\max_{x\in[0,\frac{1}{2}]}\int_0^x \frac{\sigma(y)dy}{\sigma(x)y(1-y)} < \infty. \tag{2.12}$$

and

$$\max_{x\in[\frac{1}{2},1]}\int_x^1 \frac{\sigma(y)dy}{\sigma(x)y(1-y)} < \infty. \tag{2.13}$$

□

Remark 2.3. It is interesting to note that

$$\lim_{x\to 0^+} K_0 h(x) = \frac{h(0)}{\alpha} \text{ and } \lim_{x\to 1^-} K_1 h(x) = \frac{h(1)}{\beta}. \tag{2.14}$$

Since $K_0 h = K_1 h$ for h in the range of L, it follows that

$$\int_0^1 \frac{\sigma(y)h(y)dy}{y(1-y)} = 0, \tag{2.15}$$

which also follows from the fact that $L1 = 0$, and the Fredholm alternative. A simple calculation shows that the function $\eta_0(x) = K_0 1$ is real analytic in $[0,1)$ and $\eta_1(x) = K_1 1$ is real analytic in $(0,1]$.

From these observations it follows easily that $\overline{\mathcal{D}} \subset \mathcal{C}^1([0,1])$: let $< f_n > \subset \mathcal{C}^2([0,1])$ be a sequence converging in the $\mathcal{C}^0$-norm to f, for which $h_n = Lf_n$ converges in the $\mathcal{C}^0$-norm to h. From the continuity of K_0 (or K_1) on $\mathcal{C}^0$ it follows that

$$\partial_x f_n(x) = \begin{cases} K_0 h_n(x) \text{ for } x \in [0,\frac{1}{2}] \text{ and} \\ K_1 h_n(x) \text{ for } x \in [\frac{1}{2},1] \end{cases} \tag{2.16}$$

converge uniformly to some $g \in \mathcal{C}^0([0,1])$. Since

$$f_n(x) - f_n(0) = \int_0^x \partial_x f_n(y)dy, \tag{2.17}$$

it therefore follows that

$$f(x) - f(0) = \int_0^x g(y)dy, \tag{2.18}$$

from which it is immediate that $\partial_x f = g$, and therefore $f \in \mathcal{C}^1([0,1])$.

For any function $f \in \mathcal{C}^2([0,1])$ the $\lim_{x\to 0^+,1^-} x(1-x)\partial_x^2 f(x) = 0$. Under the limit considered in the previous paragraph we know that the pair $(\partial_x f_n(0), \partial_x f_n(1))$ converges to the pair $(\partial_x f(0), \partial_x f(1))$; as $(Lf_n(0), Lf_n(1))$ also converges to $(Lf(0), Lf(1))$, and $Lf(x) \in \mathcal{C}^0([0,1])$, it is clear that

$$\lim_{x\to 0^+,1^-} x(1-x)\partial_x^2 f(x) = 0.$$

This shows that the set on the right hand side of (2.7) contains $\overline{\mathcal{D}}$. To prove the opposite inclusion we need to show that for a function f satisfying the conditions on the right hand side of (2.7), there is a sequence $< f_n > \subset \mathcal{C}^2([0,1])$, so that (f_n, Lf_n) converges uniformly to (f, Lf). It is left to the reader to show that the sequence

$$f_n(x) = f\left(\frac{1+(n-2)x}{n}\right) \tag{2.19}$$

has the desired properties. This completes the proof of the proposition. □

Remark 2.4. The precise characterization of the domain of the $\mathcal{C}^0$-graph closure in higher dimensions remains an open problem.

As a corollary of the proof we see that, if α and β are both positive, then the operator $f \mapsto \partial_x f$ is relatively bounded with respect to $f \mapsto Lf$.

Corollary 2.5. *If α and β are positive, then there exists a constant C_L so that, for all $f \in \overline{\mathcal{D}}$,*

$$\|\partial_x f\|_{\mathcal{C}^0([0,1])} \leq C_L \|Lf\|_{\mathcal{C}^0([0,1])}. \tag{2.20}$$

If α is positive, but $\beta = 0$, then such an estimate holds on $[0, x]$ for any $0 < x < 1$, and if $\alpha = 0$, but β is positive then such an estimate holds on $[x, 1]$ for $0 < x < 1$.

Remark 2.6. This result is very useful if α and β depend continuously on time. In this case we can write

$$\begin{aligned} &L_{\alpha(t),\beta(t)} - \lambda = \\ &\left[\mathrm{Id} + [(\alpha(t) - \alpha(0))(1-x)\partial_x - (\beta(t) - \beta(0))x\partial_x](L_{\alpha(0),\beta(0)} - \lambda)^{-1}\right] \cdot (L_{\alpha(0),\beta(0)} - \lambda). \end{aligned} \tag{2.21}$$

From the corollary it follows that, as an operator from $\mathcal{C}^0([0,1])$ to itself, the term

$$[(\alpha(t) - \alpha(0))(1-x)\partial_x - (\beta(t) - \beta(0))x\partial_x](L_{\alpha(0),\beta(0)} - \lambda)^{-1} \tag{2.22}$$

has norm bounded by $C(|\alpha(t) - \alpha(0)| + |\beta(t) - \beta(0)|)$, suggesting a simple perturbative approach to solve the Kimura equation with time dependent coefficients. Such a method is detailed in [4].

Remark 2.7. By analogy with Euclidean potential theory one might expect that if $Lf = h$, then some WF-Hölder norm of $\partial_x f$ would be bounded by the $\mathcal{C}^0$-norm of h. This, in fact cannot be the case. If there were a $\gamma > 0$ and a C for which $\|\partial_x f\|_{\mathrm{WF},0,\gamma} \leq C\|Lf\|_{\mathcal{C}^0}$, then, using the observations in Remark 2.3, we could show that $K_0 : \mathcal{C}^0([0, \frac{1}{2}]) \to \mathcal{C}^{0,\gamma}_{\mathrm{WF}}([0, \frac{1}{2}])$ is a bounded operator. This would imply that K_0 is compact as a map from $\mathcal{C}^0([0, \frac{1}{2}])$ to itself. Assuming that α and β are positive and $s(x) = 0$, it is not difficult to show that, for $\lambda \in [0, \frac{1}{\alpha}]$, we have

$$K_0\left[x^{\frac{1-\lambda\alpha}{\lambda}}(1-x)^{\frac{1+\lambda\beta}{\lambda}}\right] = \lambda x^{\frac{1-\lambda\alpha}{\lambda}}(1-x)^{\frac{1+\lambda\beta}{\lambda}}. \tag{2.23}$$

For λ in this range these functions belong to $\mathcal{C}^0([0,1])$, hence K_0 does not have a discrete spectrum, and is therefore not compact. A similar argument applies using K_1 for $x \in [\frac{1}{2}, 1]$.

This analysis leaves open the cases where one or both of the weights α, β vanish. We let $\overline{\mathcal{D}}_{\alpha,\beta}$ denote the domain of the $\mathcal{C}^0$-graph norm closure of $L_{\alpha,\beta}$. From the proof of this proposition it is clear that the estimates for the first derivative at 0 or 1 only depends on the coefficient of the first order term at the respective endpoint. The remaining question is whether or not the first derivative of $f \in \overline{\mathcal{D}}_{\alpha,\beta}$ remains bounded even if the relevant coefficient is zero. This, in fact, is not the case. We demonstrate this for the simpler model operator $L_0 = x\partial_x^2$, acting on $\mathcal{C}^0([0,\infty))$. We let $\overline{\mathcal{D}}_0$ denote the $\mathcal{C}^0$-graph closure starting with the core $\mathcal{C}^2_c([0,\infty))$.

Choosing $\varphi(x) \in \mathcal{C}_c^\infty([0,1))$, with $\varphi(x) = 1$ for $x \in [0, \frac{1}{2}]$, and $0 < \nu < 1$, we define the function:

$$l_\nu(x) = x[-\log x]^{1-\nu}\varphi(x). \tag{2.24}$$

Applying ∂_x and L_0 to l_ν for x the interval $[0, \frac{1}{2}]$ gives:

$$\begin{aligned} \partial_x l_\nu(x) &= [-\log x]^{1-\nu} - (1-\nu)[-\log x]^{-\nu}, \\ L_0 l_\nu(x) &= -(1-\nu)[-\log x]^{-\nu}(1 + \nu[-\log x]^{-1}). \end{aligned} \tag{2.25}$$

This shows that $\partial_x l_\nu(x)$ diverges as $x \to 0^+$,

$$L_0 l_\nu(x) \in \mathcal{C}^0([0,\infty)), \text{ and } \lim_{x\to 0^+} L_0 l_\nu(x) = 0. \tag{2.26}$$

It is not difficult to show that $(l_\nu(\cdot + \frac{1}{n}), L_0 l_\nu(\cdot + \frac{1}{n}))$ converges uniformly to $(l_\nu, L_0 l_\nu)$, and hence $l_\nu \in \overline{\mathcal{D}}_0$. If $f \in \overline{\mathcal{D}}_0$, then clearly $\lim_{x\to 0^+} x\partial_x^2 f(x) = 0$, which implies that

$$\partial_x f(x) = o(|\log x|). \tag{2.27}$$

From this analysis, it is clear that the domains $\overline{\mathcal{D}}$ depend only on the weights, (α, β), and that the descriptions at the two endpoints are entirely independent. It is difficult to give a precise description of the domain at a boundary point where a weight vanishes, and so we do not state a result describing $\overline{\mathcal{D}}_{0,\beta}, \overline{\mathcal{D}}_{\alpha,0}, \overline{\mathcal{D}}_{0,0}$, beyond the statement that for $f \in \overline{\mathcal{D}}_{0,\beta}$ (resp. $f \in \overline{\mathcal{D}}_{\alpha,0}$)

$$\lim_{x\to 0^+} x\partial_x^2 f(x) = 0, \quad \left(\lim_{x\to 1^-} (1-x)\partial_x^2 f(x) = 0 \text{ resp.}\right), \tag{2.28}$$

which in turn implies that (2.27) (or its analogue for $x = 1$) holds for $f \in \overline{\mathcal{D}}_{0,\beta}$ (or $f \in \overline{\mathcal{D}}_{\alpha,0}$). The examples l_ν show that this estimate is essentially sharp. We leave the details of these (elementary) arguments to the reader.

3 General Vector Fields as Perturbations

As noted above,in many applications the first order part of the Kimura operator is of the form

$$\alpha(1-x)\partial_x u - \beta x\partial_x u + s(x)x(1-x)\partial_x, \tag{3.1}$$

where $s(x)$ is a smooth function on $[0,1]$. In Population Genetics the vector field $s(x)x(1-x)\partial_x$ models the effects of selection. We denote this vector field by

$$V = s(x)x(1-x)\partial_x. \tag{3.2}$$

In the simplest model of selection, s is a constant, which quantifies the selective difference between the two types. An examination of the arguments that follow shows that it suffices to treat the case that $s(x) = 1$, which we henceforth assume.

In the previous section we proved that any vector field is a relatively *bounded* operator with respect to L provided that α and β are both positive. In this section we show that a vector field with coefficient vanishing at 0 and 1 is a relatively *compact* perturbation of the operator

$$L_{\alpha,\beta} = x(1-x)\partial_x^2 + \alpha(1-x)\partial_x - \beta x\partial_x. \tag{3.3}$$

More precisely, for $0 < \lambda$, the operators $V(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1}$, $(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1}V$ are compact maps from $\mathcal{C}^0([0,1])$ to itself. The truth of this statement only requires that α and β are non-negative. We begin with the easier result:

Proposition 3.1. *For λ in the resolvent set of $L_{\alpha,\beta}$ the operator $V(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1}$ defines a compact map from $\mathcal{C}^0([0,1])$ to itself. There is a constant C so that, for $\lambda \in (0,\infty)$ we have the estimate:*

$$\|V(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1} f\|_\infty \leq \frac{C}{\sqrt{\lambda}}\|f\|_\infty. \tag{3.4}$$

Proof. This result follows from the basic estimate for the 1d-Cauchy problem. There is a constant C_0 so that if $u(x,t)$ is the solution of

$$(\partial_t - L_{\alpha,\beta})u = 0 \quad \text{with} \quad u(x,0) = f(x), \tag{3.5}$$

then

$$|\partial_x u(x,t)| \leq \frac{C_0\|f\|_\infty}{\sqrt{tx(1-x)}}, \tag{3.6}$$

which follows from Lemma 6.1.10 in [2], and the construction of the resolvent kernel in [1]. By integrating this estimate in x we can easily show that there is a constant C_1 so that

$$|u(x_1,t) - u(x_2,t)| \leq \frac{C_1}{\sqrt{t}}\rho_{\mathrm{WF}}(x_1,x_2)\|f\|_\infty, \tag{3.7}$$

where ρ_{WF} is defined in (2.2). This estimate implies that for $t > 0$ the operator $e^{tL_{\alpha,\beta}} : L^\infty([0,1]) \to \mathcal{C}^{0,1}_{\mathrm{WF}}([0,1])$ is bounded, with norm bounded by $C/\sqrt{t}$.

The resolvent operator can be expressed as the Laplace transform of the heat kernel:

$$R(\lambda) = (L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1} = \int_0^\infty e^{tL_{\alpha,\beta}} e^{-t\lambda} dt. \tag{3.8}$$

From this representation and the estimate in (3.6) we easily derive the estimate

$$\begin{aligned} |x(1-x)\partial_x(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1} f| &\leq C_0\|f\|_\infty\sqrt{x(1-x)}\int_0^\infty \frac{e^{-t\lambda}}{\sqrt{t}}dt \\ &= \frac{C_0\|f\|_\infty\sqrt{x(1-x)}}{\sqrt{\lambda}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}ds. \end{aligned} \tag{3.9}$$

The estimate in (3.4) follows easily from this.

From (3.6) we see that, for any fixed λ, and finite $\epsilon > 0$, there is a constant C_λ so that

$$\left\| x(1-x)\partial_x \int_0^\epsilon e^{tL_{\alpha,\beta}} e^{-t\lambda} f dt \right\|_\infty \leq C_\lambda\|f\|_\infty\sqrt{\epsilon}. \tag{3.10}$$

From this it follows that to prove the compactness of $x(1-x)\partial_x(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1}$ it suffices to show that, for sufficiently large positive λ, the operator

$$VR_\epsilon(\lambda)f = x(1-x)\partial_x \int_\epsilon^\infty e^{tL_{\alpha,\beta}} e^{-t\lambda} f dt \tag{3.11}$$

is compact for any $\epsilon > 0$. The operator $R_\epsilon(\lambda)$ is defined by

$$R_\epsilon(\lambda) = \int_\epsilon^\infty e^{tL_{\alpha,\beta}} e^{-t\lambda} dt. \tag{3.12}$$

This statement follows from Theorem 11.2.1 in [2], which shows that there is a $k > 0$ so that, for any $0 < \gamma < 1$ we have

$$\|e^{tL_{\alpha,\beta}} f\|_{\mathrm{WF},0,2+\gamma} \leq C_\gamma \left[\frac{1}{t} + e^{kt}\right] \|f\|_{\mathrm{WF},0,\gamma}. \tag{3.13}$$

We rewrite $VR_\epsilon(\lambda)f = Ve^{-\frac{\lambda\epsilon}{2}} R_{\frac{\epsilon}{2}}(\lambda) e^{\frac{\epsilon}{2}L_{\alpha,\beta}} f$. The estimate in (3.7) shows that $e^{\frac{\epsilon}{2}L_{\alpha,\beta}}$ maps the space $C^0([0,1])$ into $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}([0,1])$. Since $V : \mathcal{C}^{0,2+\gamma}_{\mathrm{WF}}([0,1]) \to \mathcal{C}^{0,\gamma}_{\mathrm{WF}}([0,1])$ it follows easily from the estimate in (3.13) that, for $k < \lambda$, the composition $VR_\epsilon(\lambda) = Ve^{-\frac{\lambda\epsilon}{2}} R_{\frac{\epsilon}{2}}(\lambda) e^{\frac{\epsilon}{2}L_{\alpha,\beta}}$ maps $\mathcal{C}^0([0,1])$ to $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}([0,1])$, and is therefore a compact operator from $\mathcal{C}^0([0,1])$ to itself. This in turn shows that $R(\lambda)$ is a compact operator if $k < \lambda$. The resolvent identity, $R(\mu) = R(\lambda) + (\mu - \lambda)R(\mu)R(\lambda)$, then implies that this is true throughout the resolvent set. $\square$

For the transition from $VR(\lambda)$ to $R(\lambda)V$ we use an identity satisfied by the operators $L_{\alpha,\beta}$ and $x(1-x)\partial_x$. Simple calculations show that

$$x(1-x)\partial_x L_{1,1} = L_{0,0}x(1-x)\partial_x, \tag{3.14}$$

and

$$x(1-x)\partial_x[\alpha(1-x)\partial_x - \beta x\partial_x] = [\alpha(1-x)\partial_x - \beta x\partial_x]x(1-x)\partial_x - [\alpha(1-x)^2 + \beta x^2]\partial_x. \tag{3.15}$$

Combining these results, we see that

$$x(1-x)\partial_x L_{\alpha+1,\beta+1} = L_{\alpha,\beta}x(1-x)\partial_x - [\alpha(1-x)^2 + \beta x^2]\partial_x. \tag{3.16}$$

From this identity we easily derive that, for $\lambda \in (0,\infty)$ we have the relation

$$\begin{aligned}(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1}x(1-x)\partial_x &= x(1-x)\partial_x(L_{\alpha+1,\beta+1} - \lambda\,\mathrm{Id})^{-1} - \\ &(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1}[(\alpha(1-x)^2 + \beta x^2)\partial_x](L_{\alpha+1,\beta+1} - \lambda\,\mathrm{Id})^{-1}\end{aligned} \tag{3.17}$$

With this formula and Proposition 3.1 we establish

Proposition 3.2. *The operator $(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1}x(1-x)\partial_x$ is compact on $\mathcal{C}^0([0,1])$. There is a constant $C_{\alpha,\beta}$ so that its norm, as an operator from $\mathcal{C}^0$ to itself, is bounded by $C_{\alpha,\beta}/\sqrt{\lambda}$, for $\lambda \in (0,\infty)$.*

Proof. Proposition 3.1 shows that the first term on the right hand side of (3.17) is compact and satisfies the desired norm bound. From Theorem 2.1 it follows that

$$[\alpha(1-x)^2 + \beta x^2]\partial_x(L_{\alpha+1,\beta+1} - \lambda\,\mathrm{Id})^{-1}$$

is bounded from $\mathcal{C}^0([0,1])$ to itself; the estimate in (3.7) implies that

$$(L_{\alpha,\beta} - \lambda\,\mathrm{Id})^{-1} : \mathcal{C}^0([0,1]) \to \mathcal{C}^{0,1}_{\mathrm{WF}}([0,1])$$

is bounded with norm bounded by $C/\sqrt{\lambda}$.

Using the operators K_0, K_1 defined in (2.11), with parameters $\alpha+1, \beta+1$, we see that, for $x \in [0, \frac{1}{2}]$,

$$\begin{aligned}[]
[\partial_x(L_{\alpha+1,\beta+1} - \lambda \operatorname{Id})^{-1} f](x) =& [K_0 L_{\alpha+1,\beta+1}(L_{\alpha+1,\beta+1} - \lambda \operatorname{Id})^{-1} f](x) \\
=& \left(K_0[\operatorname{Id} + \lambda(L_{\alpha+1,\beta+1} - \lambda \operatorname{Id})^{-1}]f\right)(x),
\end{aligned} \tag{3.18}$$

with a similar formula employing K_1 for $x \in [\frac{1}{2}, 1]$. Combining these observations with the estimate (3.4), the observations at the end of the proof of Proposition 3.1, and with the fact that $(L_{\alpha+1,\beta+1} - \lambda \operatorname{Id})^{-1}$ as a map from $\mathcal{C}^0$ to itself has norm $1/\lambda$, we obtain the norm bound

$$\|(L_{\alpha,\beta} - \lambda \operatorname{Id})^{-1} x(1-x)\partial_x f\|_\infty \leq \frac{C_{\alpha,\beta}}{\sqrt{\lambda}} \|f\|_\infty, \text{ for } \lambda \in (0, \infty). \tag{3.19}$$

□

4 Compact Resolvents in Arbitrary Dimension

The WF-Hölder theory of a Kimura diffusion, L, on arbitrary compact manifold with corners, P, is systematically developed in [2]. A fundamental question left unanswered there is whether the resolvent $(L - \lambda \operatorname{Id})^{-1}$ is always a compact operator on $\mathcal{C}^0(P)$. This is proved in [3] under the additional assumption that the weights are strictly positive. i.e., the first order part of L is uniformly transverse to ∂P. In this final section we see that the argument used to prove Proposition 3.1 is easily adapted to prove this statement in general. Let P denote a compact manifold with corners, as defined in Chapter 2.1 of [2] and L a generalized Kimura diffusion operator, as defined in Definition 2.2.1 of [2].

As above, ρ_{WF} denotes the incomplete metric defined on P by $\sigma_2(L)$, the principal symbol of L. For $0 < \gamma < 1$ we let $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}(P)$ denote the subset of $\mathcal{C}^0(P)$ consisting of functions for which the semi-norm

$$[f]_{\mathrm{WF},\gamma} = \sup_{\{x_1 \neq x_2 \in P} \frac{|f(x_1) - f(x_2)|}{\rho^{\gamma}_{\mathrm{WF}}(x_1, x_2)} \tag{4.1}$$

is finite. It is easy to see that the unit ball in $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}(P)$ is a compact subset of $\mathcal{C}^0(P)$.

With these preliminaries we prove the following result:

Theorem 4.1. *If P is a compact manifold with corners and L is a generalized Kimura diffusion operator defined on P, then the resolvent $(L - \lambda \operatorname{Id})^{-1} : \mathcal{C}^0(P) \to \mathcal{C}^0(P)$ is a compact operator.*

Proof. For $f \in \mathcal{C}^{0,\gamma}_{\mathrm{WF}}(P)$ let $u(x,t)$ denote the unique regular solution to the initial value problem $(\partial_t - L)u = 0$ and $u(p, 0) = f(p)$. Theorem 11.2.1 in [2] shows that there are constants C, K so that

$$\|u(\cdot, t)\|_{\mathrm{WF},0,\gamma} \leq Ce^{Kt} \|f\|_{\mathrm{WF},0,\gamma}. \tag{4.2}$$

Theorem 1.5 in [5] shows that, for any $0 < \gamma < 1$, and $\tau > 0$, there is a constant $C_{\tau,\gamma}$ so that

$$\|u(\cdot, \tau)\|_{\mathrm{WF},0,\gamma} \leq C_{\tau,\gamma} \|f\|_{\mathcal{C}^0}. \tag{4.3}$$

As before, for $\epsilon, \lambda > 0$, let

$$R_\epsilon(\lambda) f = \int_\epsilon^\infty e^{-\lambda t} e^{Lt} f dt. \tag{4.4}$$

From the estimates above we see that, for $K < \lambda$, we have the estimate

$$\begin{aligned} \|R_\epsilon(\lambda) f\|_{\mathrm{WF},0,\gamma} &\leq \frac{C}{\lambda - K} \|e^{\epsilon L} f\|_{\mathrm{WF},0,\gamma} \\ &\leq \frac{C C_{\epsilon,\gamma}}{\lambda - K} \|f\|_{\mathcal{C}^0}. \end{aligned} \tag{4.5}$$

This shows that, for any $\epsilon > 0$, and $\lambda > K$, the operator $R_\epsilon(\lambda) : \mathcal{C}^0(P) \to \mathcal{C}^0(P)$ is compact. Since e^{Lt} is a contraction on $\mathcal{C}^0$ we easily derive that

$$\|[R(\lambda) - R_\epsilon(\lambda)] f\|_{\mathcal{C}^0} \leq \epsilon \|f\|_{\mathcal{C}^0}. \tag{4.6}$$

Hence, for $K < \lambda$, the resolvent, $R(\lambda)$, is the uniform limit of a sequence of compact operators and therefore compact. Once again, the resolvent identity implies that this is true throughout the resolvent set. $\square$

Pop's results in [5] easily imply that the eigenfunctions of the $\mathcal{C}^0(P)$-graph closure of L are in $C^{0,\gamma}_{\mathrm{WF}}(P)$, and are therefore automatically smooth. Hence the spectrum of L acting on $\mathcal{C}^0(P)$ is the same as its spectrum acting on $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}(P)$. Theorem 11.1.1 in [2] shows that, other than 0, this spectrum lies in a conic neighborhood of the negative real axis, contained in the open left half plane. It is also the case that generalized eigenfunctions (the null-space of $(L - \lambda \,\mathrm{Id})^k$ for $k > 1$) belong to $\mathcal{C}^{0,\gamma}_{\mathrm{WF}}(P)$, and hence are smooth.

This analysis leaves open the question of characterizing the precise domain of the $\mathcal{C}^0(P)$-graph closure of L acting on $\mathcal{C}^2(P)$, when $\dim P > 1$. The result in the 1d-case, proved above, shows that this domain can be expected to depend on the transversality properties of the first order part of L along ∂P.

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