Math 275 Homework 4
Due: 12/2/05

In this assignment we will solve the equilibrium equations for a large deformation, small strain problem. Consider a thin sheet of aluminum which is wrapped around a cylinder of radius R. We'll assume plane strain to turn the problem into a 2 dimensional problem. Take the reference configuration to be

$$\Omega = (0,b) \times (0, 2\pi R), \qquad (b = .05 \text{ cm}, \ R = 10 \text{ cm}) \tag{1}$$

and assume the deformation is given by

$$\varphi(x,y) = \begin{pmatrix} r(x)\cos\frac{y}{R} \\ r(x)\sin\frac{y}{R} \end{pmatrix}, \qquad \nabla\varphi = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} r' & \\ & \frac{r}{R} \end{pmatrix}, \tag{2}$$

where $c = \cos(y/R)$, $s = \sin(y/R)$. Our goal is to find the function $r(x)$. Begin by computing $C = \nabla\varphi^T \nabla\varphi$ and $E = \sqrt{C} - I$. Then assume the 2nd Piola-Kirchhoff stress tensor satisfies the constitutive equation

$$\Sigma(x,y) = \hat{\Sigma}(\nabla\varphi(x,y)), \qquad \hat{\Sigma}(\nabla\varphi) = \lambda(\operatorname{tr} E)I + 2\mu E \tag{3}$$

exactly, i.e. with no error term. Such a material is called a St. Venant–Kirchhoff material. (As we saw in class, it is a good approximation when E is small). Take $\mu = 26\,\text{GPa}$, $\lambda = 58\,\text{GPa}$. The boundary conditions are that

$$r(0) = R, \qquad T(b,y)\mathbf{n} = \mathbf{0}, \tag{4}$$

where $T = \nabla\varphi\Sigma$ is the 1st Piola-Kirchhoff stress tensor and $\mathbf{n} = (1,0)^T$ is the unit normal in the reference configuration. (In other words, the outer boundary of the ring is a free surface with $\mathbf{g} = 0$.) The equilibrium equations are

$$\operatorname{div}(\nabla\varphi\Sigma) = \mathbf{0}. \tag{5}$$

Reduce them to an ODE for r of the form

$$\frac{d}{dx}\left[A(r')^2 - Br' - C\right] + D = 0. \qquad (B, C, D \text{ depend on } r) \tag{6}$$

Now define w to be the quantity in brackets and solve the first order ODE

$$\frac{d}{dx}\begin{pmatrix} r \\ w \end{pmatrix} = \begin{pmatrix} \frac{B}{2A}\left[1 + \left(1 + \frac{4A(C+w)}{B}\right)^{1/2}\right] \\ -D \end{pmatrix} \tag{7}$$

with boundary conditions given by (4). The traction boundary condition looks like $q := w(b) + C|_{x=b} = 0$. Solve this ODE by a shooting method; i.e. solve the initial value problem (using RK4) with initial conditions $r(0) = R$, $w(0) = \gamma$ and vary γ until the boundary condition at $x = b$ is satisfied. (The easiest way to do this is via bracketing; find γ_0 and γ_1 so that $q(\gamma_0) < 0$, $q(\gamma_1) > 0$. Then repeatedly bisect until $|q| < 10^{-8}$, say. I expect γ to be slightly less than $-C|_{x=0}$.) Plot $r(x,0)$, $\Sigma_{11}(x,0)$, $\Sigma_{12}(x,0)$ and $\Sigma_{22}(x,0)$ on the interval $(0,b)$. What is the thickness $(r(b) - R)$ of the deformed sheet?