228B HW 6 due Thurs. 4/19

① Consider the linear functional $\ell : \mathbb{R}^5 \to \mathbb{R}$ given by

$$\langle \ell, x\rangle = b^T x = -x_1 + 5x_2 + 4x_3 - 7x_4 + 3x_5, \quad b = \begin{pmatrix} -1 \\ 5 \\ 4 \\ -7 \\ 3 \end{pmatrix}$$

what is the norm of ℓ when $\mathbb{R}^5$ is equipped with the norm:

(a) $\|x\|_\infty = \max_{1\le i\le 5} |x_i|$ (b) $\|x\|_1 = \sum_{i=1}^{5} |x_i|$ (c) $\|x\|_2 = \sqrt{\sum_{i=1}^{5} x_i^2}$

(d) $\|x\|_A = \sqrt{x^T A x}$, $A = \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & 2 & 1 & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{pmatrix}$ hint: $|(x,y)_A| \le \|x\|_A \cdot \|y\|_A$ ← Cauchy-Schwarz

(e) $\|x\|_3 = \left(\sum_{i=1}^{5} |x_i|^3\right)^{1/3}$ hint: if $p>1, q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ then $|x^T y| \le \|x\|_p \|y\|_q$ ← Hölder inequality

② consider the bilinear form $a(x,y) = x^T A y$ and linear functional $\langle \ell, y\rangle = b^T y$

Solve $a(x,\cdot) = \langle \ell, \cdot\rangle$ for x (i.e. find all solutions if any exists):

(a) $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | (b) $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | (c) $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

useful theorem: Hahn-Banach theorem: Let M be a subspace of a normed linear space X and let ℓ be a linear functional on M s.t. $|\langle \ell, x\rangle| \le C\|x\|$ $\forall x \in M$. Then ℓ extends continuously to all of X, i.e. we can define $\langle \ell, x\rangle$ for $x \in X \setminus M$ s.t. $|\langle \ell, x\rangle| \le C\|x\|$ $\forall x \in X$.

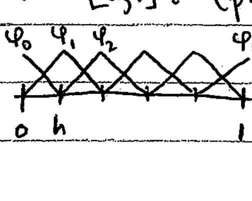
③ prove or give a counterexample:

(a) A linear functional ℓ defined on $C^\infty(\Omega) \cap H^1(\Omega)$ for which $\exists C < \infty$ s.t. $|\langle \ell, u\rangle| \le C\|u\|_0$ for all $u \in C^\infty(\Omega) \cap H^1(\Omega)$ extends continuously to a bounded linear functional on $H^1(\Omega)$.

(b) A linear functional ℓ defined on $C^\infty(\Omega) \cap H^1(\Omega)$ for which $\exists C < \infty$ s.t. $|\langle \ell, u\rangle| \le C\|u\|_1$ for all $u \in C^\infty(\Omega) \cap H^1(\Omega)$ extends continuously to a bounded linear functional on $L^2(\Omega)$.

④ Use finite elements to compute the first 10 eigenvalues and eigenfunctions of the Laplacian on $[0,1]$. (plot the latter on one graph)

$-\Delta u = \lambda u$



$A_{ij} = \int_0^1 \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial y}\, dx$, $M_{ij} = \int_0^1 \varphi_i \varphi_j\, dx$

$Au = \lambda M u$