Homework 4
due Thursday, Mar. 15

(1) Consider the equation $u_t = -u_x$ with initial condition $u(x,0) = \sin^{40}(\pi x)$ on the unit interval $0 \le x \le 1$ with periodic boundary conditions.
(a) Use the upwind, Lax-Friedrichs, Lax-Wendroff, and Leapfrog methods and solve to time $T = 1$. For leapfrog, use the centered scheme to take the first step. Make plots of the solutions at the final time.
(b) What is the maximum value of ν for which each scheme is stable? Plot the energy loss and phase shift in the solution at time $T = 1$ as a function of ν up to ν_{max} for each scheme. Is it better to take smaller timesteps k for a given h?

(2) Consider the hyperbolic system $\mathbf{u}_t = A\mathbf{u}_x$ on the unit interval with periodic boundary conditions.
(a) Compute the local truncation error for the Lax-Wendroff and Leapfrog schemes.
(b) Use both schemes to solve

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 3 & -15 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

with initial conditions $u(x,0) = \sin(2\pi x)$, $v(x,0) = \sin(4\pi x)$. (Use the Lax-Wendroff scheme for the first step of the Leapfrog scheme). What is the exact solution? At what rates are your numerical methods converging? Do you get the same convergence rates if you use a more complicated initial condition, such as $u(x,0) = 0$, $v(x,0) = \frac{0.1}{1-0.9\sin^{10}(\pi x)}$? What happens when you take $\frac{k}{h} = \frac{1}{3}$ in the leapfrog scheme?

(3) (vibrating string). Consider the wave equation $u_{tt} = u_{xx}$ on the unit interval with boundary conditions $u(0,t) = 0$, $u(1,t) = 0$ and initial conditions:

$$(i) \quad \begin{aligned} u(x,0) &= \sin^{40}(2\pi x), \\ u_t(x,0) &= 0, \end{aligned} \qquad (ii) \quad \begin{aligned} u(x,0) &= 0, \\ u_t(x,0) &= \sin^{40}(2\pi x). \end{aligned}$$

Implement Lax-Friedrichs, Lax-Wendroff, and Leapfrog and solve over one period of the solution. Make 3d plots of $u(x,t)$.

(4) Choose one of the schemes and repeat (3) for that scheme with the right boundary condition replaced by $u_x(1,t) = 0$.