228B Homework 2
due Thurs, Feb 8

1. Consider the scheme $D_t^+ u = D_x^+ D_x^- u + 10 D_x^0 u$

for the equation $\begin{cases} u_t = u_{xx} + 10 u_x \\ u(x,0) = g(x) \end{cases}$ $\quad 0 \le t \le T$, $\; -\infty < x < \infty$

a. write down the operator B such that $u^{n+1} = B u^n$

b. show that $\|B\|_{\infty,h} = 1$ if $\nu = \frac{k}{h^2} \le \frac{1}{2}$ and $h \le \frac{1}{5}$

c. show that $\|B\|_{1,h} = 1$ if $\nu \le 1/2$ and $h \le 1/5$

d. compute the amplification factor $G(\xi)$ for this scheme and plot it for $\begin{cases} \nu = 1/2, \; h = 1/2 \\ \nu = 1/2, \; h = 1/5 \\ \nu = 1/2, \; h = 1/8 \end{cases}$

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2. use the above scheme to solve the equation numerically on the finite interval $0 \le x \le 1$ with periodic boundary conditions $u(1) = u(0)$, $u_x(1) = u_x(0)$ and initial conditions $g(x) = \sin(2\pi x)$ up to time $T = 0.1$

Compute the order of convergence with $\nu = 1/2$ and $\nu = 1/6$.

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3. A circulant matrix is constant along diagonals with entries that "wrap around":

$$A_{k\ell} = \begin{cases} r_{k-\ell} & k \ge \ell \\ r_{N+k-\ell} & k < \ell \end{cases}$$

For convenience, let's index our matrices starting at zero. Show that $AU = U\Lambda$

with $U_{\ell m} = e^{\frac{2\pi i \ell m}{N}}$ $\quad 0 \le \ell \le N-1$, $\; 0 \le m \le N-1$, $\quad \Lambda = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} \end{pmatrix}$

and $\lambda_m = \sum_{j=0}^{N-1} r_j e^{-\frac{2\pi i j m}{N}}$

example ($N=4$):

$$\begin{pmatrix} r_0 & r_3 & r_2 & r_1 \\ r_1 & r_0 & r_3 & r_2 \\ r_2 & r_1 & r_0 & r_3 \\ r_3 & r_2 & r_1 & r_0 \end{pmatrix}$$