$$+\frac{b_j^n}{\Delta x}(U_{j+1}^n - U_j^n) + c_j^n U_j^n + d_j^n, \qquad (2.147)$$

which leads to

$$\begin{aligned} e_j^{n+1} &= e_j^n + \nu a_j^n(e_{j+1}^n - 2e_j^n + e_{j-1}^n) + \frac{\Delta t}{\Delta x} b_j^n (e_{j+1}^n - e_j^n) - \Delta t T_j^n \\ &= (1 - 2\nu a_j^n - \mu b_j^n) e_j^n \\ &\quad + (\nu a_j^n + \mu b_j^n) e_{j+1}^n + \nu a_j^n e_{j-1}^n - \Delta t T_j^n. \end{aligned} \qquad (2.148)$$

In order to ensure that all the coefficients on the right of this equation are non-negative, we now need only

$$2\nu a_j^n + \mu b_j^n \le 1. \qquad (2.149)$$

This requires a more severe restriction on the size of the time step when $b \neq 0$, but no restriction on the size of Δx.

If the function $b(x,t)$ changes sign, we can use the forward difference where it is positive, and the backward difference where it is negative; this idea is known as *upwind differencing*. Unfortunately we have to pay a price for this lifting of the restriction needed to ensure a maximum principle. The truncation error is now of lower order; the forward difference introduces an error of order Δx, instead of the order $(\Delta x)^2$ given by the central difference. However, we shall discuss this issue in the chapter on hyperbolic equations.

A general parabolic equation may also often appear in the self-adjoint form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(p(x,t)\frac{\partial u}{\partial x}\right) \qquad (2.150)$$

where, as usual, we assume that the function $p(x,t)$ is strictly positive. It is possible to write this equation in the form just considered, as

$$\frac{\partial u}{\partial t} = p\frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x}\frac{\partial u}{\partial x}, \qquad (2.151)$$

but it is usually better to construct a difference approximation to the equation in its original form. We can write

$$\left[p\frac{\partial u}{\partial x}\right]_{j+1/2}^n \approx p_{j+1/2}^n \left(\frac{u_{j+1}^n - u_j^n}{\Delta x}\right), \qquad (2.152)$$

and a similar approximation with j replaced by $j-1$ throughout. If we subtract these two, and divide by Δx, we obtain an approximation to the right-hand side of the equation, giving the explicit difference scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{(\Delta x)^2}\left[p_{j+1/2}^n (U_{j+1}^n - U_j^n) - p_{j-1/2}^n (U_j^n - U_{j-1}^n)\right], \qquad (2.153)$$

which gives in explicit form

$$U_j^{n+1} = \left(1 - \nu(p_{j+1/2}^n + p_{j-1/2}^n)\right) U_j^n + \nu p_{j+1/2}^n U_{j+1}^n + \nu p_{j-1/2}^n U_{j-1}^n. \qquad (2.154)$$

This shows that the form of error analysis which we have used before will again apply here, with each of the coefficients on the right-hand side being non-negative provided that

$$\nu P \le \tfrac{1}{2}, \qquad (2.155)$$

where P is an upper bound for the function $p(x,t)$ in the region. So this scheme gives just the sort of time step restriction which we should expect, without any restriction on the size of Δx.

The same type of difference approximation can be applied to give an obvious generalisation of the θ-method. The details are left as an exercise, as is the calculation of the leading terms of the truncation error (see Exercises 7 and 8).

2.16 Polar co-ordinates

One-dimensional problems often result from physical systems in three dimensions which have cylindrical or spherical symmetry. In polar co-ordinates the simple heat equation becomes

$$\frac{\partial u}{\partial t} = \frac{1}{r^\alpha}\frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u}{\partial r}\right) \qquad (2.156)$$

or

$$u_t = u_{rr} + \frac{\alpha}{r}u_r, \qquad (2.157)$$

where $\alpha = 0$ reduces to the case of plane symmetry which we have considered so far, while $\alpha = 1$ corresponds to cylindrical symmetry and $\alpha = 2$ to spherical symmetry. The methods just described can easily be applied to this equation, either in the form (2.156), or in the form (2.157). Examination of the stability restrictions in the two cases shows that there is not much to choose between them in this particular situation. However, in each case there is clearly a problem at the origin $r = 0$.

A consideration of the symmetry of the solution, in either two or three dimensions, shows that $\partial u/\partial r = 0$ at the origin; alternatively, (2.157) shows that either u_{rr} or u_t, or both, would be infinite at $r = 0$, were u_r non-zero. Now keep t constant, treating u as a function of r only, and expand in a Taylor series around $r = 0$, giving

$$\begin{aligned} u(r) &= u(0) + r u_r(0) + \tfrac{1}{2} r^2 u_{rr}(0) + \ldots \\ &= u(0) + \tfrac{1}{2} r^2 u_{rr}(0) + \ldots \end{aligned} \qquad (2.158)$$

and

$$\begin{aligned} \frac{1}{r^\alpha}\frac{\partial}{\partial r}\left(r^\alpha \frac{\partial u}{\partial r}\right) &= \frac{1}{r^\alpha}\frac{\partial}{\partial r}\left[r^\alpha u_r(0) + r^{\alpha+1} u_{rr}(0) + \ldots\right] \\ &= \frac{1}{r^\alpha}\left[(\alpha+1) r^\alpha u_{rr}(0) + \ldots\right] \\ &= (\alpha+1) u_{rr}(0) + \ldots . \end{aligned} \qquad (2.159)$$

Writing Δr for r in (2.158) we get

$$u(\Delta r) - u(0) = \tfrac{1}{2}(\Delta r)^2 u_{rr}(0) + \ldots \qquad (2.160)$$

and we thus obtain a difference approximation to be used at the left end of the domain,

$$\frac{U_0^{n+1} - U_0^n}{\Delta t} = \frac{2(\alpha+1)}{(\Delta r)^2}(U_1^n - U_0^n). \qquad (2.161)$$

This would also allow any of the θ-methods to be applied.

An alternative, more physical, viewpoint springs directly from the form (2.156). Consider the heat balance for an annular region between two surfaces at $r = r_{j-1/2}$ and $r = r_{j+1/2}$ as in Fig. 2.11: the term $r^\alpha \partial u/\partial r$ on the right-hand side of (2.156) is proportional to a heat flux times a surface area; and the difference between the fluxes at surfaces with radii $r_{j-1/2}$ and $r_{j+1/2}$ is applied to raising the temperature in a volume which is proportional to $(r_{j+1/2}^{\alpha+1} - r_{j-1/2}^{\alpha+1})/(\alpha+1)$. Thus on a uniform mesh of spacing Δr, a direct differencing of the right-hand side of (2.156) gives

$$\begin{aligned} \frac{\partial U_j}{\partial t} &\approx \frac{\alpha+1}{r_{j+1/2}^{\alpha+1} - r_{j-1/2}^{\alpha+1}} \delta_r \left(r_j^\alpha \frac{\delta_r U_j}{\Delta r}\right) \\ &= \frac{(\alpha+1)\left[r_{j+1/2}^\alpha U_{j+1} - \left(r_{j+1/2}^\alpha + r_{j-1/2}^\alpha\right) U_j + r_{j-1/2}^\alpha U j - 1\right]}{\left[r_{j+1/2}^\alpha + r_{j+1/2}^{\alpha-1} r_{j-1/2} + \ldots + r_{j-1/2}^\alpha\right](\Delta r)^2} \\ &\qquad \text{for } j = 1, 2, \ldots . \end{aligned} \qquad (2.162a)$$

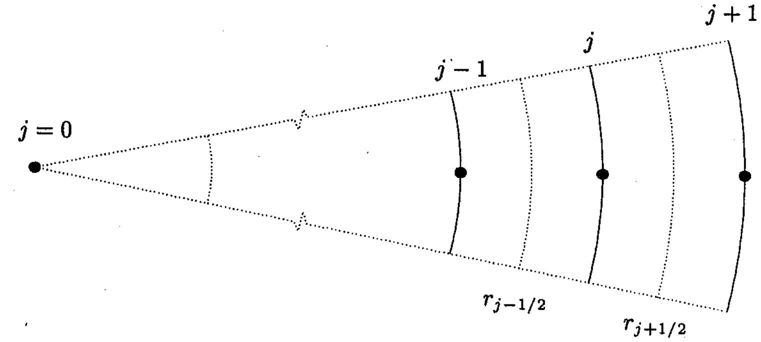

Fig. 2.11. Polar co-ordinates.

At the origin where there is only one surface (a cylinder of radius $r_{1/2} = \frac{1}{2}\Delta r$ when $\alpha = 1$, a sphere of radius $r_{1/2}$ when $\alpha = 2$) one has immediately

$$\frac{\partial U_0}{\partial t} \approx \frac{\alpha+1}{r_{1/2}^{\alpha+1}} r_{1/2}^\alpha \frac{U_1 - U_0}{\Delta r} = 2(\alpha+1)\frac{U_1 - U_0}{(\Delta r)^2}, \qquad (2.162b)$$

which is in agreement with (2.161). Note also that (2.162a) is identical with difference schemes obtained from either (2.156) or (2.157) in the case of cylindrical symmetry ($\alpha = 1$); but there is a difference in the spherical case because $r_{j+1/2}^2 + r_{j+1/2} r_{j-1/2} + r_{j-1/2}^2$ is not the same as $3r_j^2$.

The form (2.162a) and (2.162b) is simplest for considering the condition that a maximum principle should hold. From calculating the coefficient of U_j^n in the θ-method, one readily deduces that the worst case occurs at the origin and leads to the condition

$$2(\alpha+1)(1-\theta)\Delta t \le (\Delta r)^2. \qquad (2.162c)$$

This becomes more restrictive as the number of space dimensions increases in a way which is consistent with what we shall see in Chapter 3.