Math 228A, Fall 2007, Wilkening

Homework 3
due Thursday, Oct. 4

(1) Determine the order of the three-step method

$$y_{n+3} - y_n = h\left[\frac{3}{8}f_{n+3} + \frac{9}{8}f_{n+2} + \frac{9}{8}f_{n+1} + \frac{3}{8}f_n\right],$$

the *three-eighths* scheme. Is it convergent?

---

(2) Show that the multistep method

$$y_{n+3} + a_1 y_{n+2} + a_2 y_{n+1} + a_3 y_n = h\big[b_1 f_{n+2} + b_2 f_{n+1} + b_3 f_n\big]$$

is fourth order only if $a_1 + a_3 = 8$ and $a_2 = -9$. Hence, deduce that this method cannot be both fourth order and convergent.

---

(3) Plot $\|A^n\|$ for $1 \le n \le 100$ for each of the following matrices and explain what you see:

$$A = \begin{pmatrix} 294.6 & 376.7 & -217.2 \\ -84.6 & -108.1 & 62.4 \\ 250.6 & 320.6 & -184.7 \end{pmatrix} \tag{1}$$

$$A = \begin{pmatrix} 0.740 & -0.2880 & -0.3840 \\ -0.288 & 0.2936 & 0.1248 \\ -0.384 & 0.1248 & 0.3664 \end{pmatrix} \tag{2}$$

$$A = \begin{pmatrix} 0 & -3 & 5 \\ -1 & -6 & 11 \\ 0 & -4 & 7 \end{pmatrix} \tag{3}$$

---

(4) By solving a three-term recurrence relation, calculate analytically the sequence of values $y_2, y_3, y_4, \ldots$ that is generated by the *midpoint rule*

$$y_{n+2} = y_n + 2hf_{n+1}$$

when it is applied to the differential equation $y' = -y$. Starting from the values $y_0 = 1$, $y_1 = 1-h$, show that the sequence diverges as $n \to \infty$. Recall, however, that the root condition together with consistency of order $p \geq 1$ and suitable starting conditions imply convergence to the true solution over any *finite* interval as $h \to 0$. Prove that this implementation of the midpoint rule (i.e. using these starting values) does converge over any finite interval $[0, T]$. *Hint: Express the roots of the characteristic polynomial of the recurrence relation as* $r_1 = -\exp(\operatorname{arcsinh} h)$, $r_2 = \exp(-\operatorname{arcsinh} h)$.