Homework 2
due Thursday, Sep. 20

(1) (Chain rule). Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}^2$ and $u : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(y) = \begin{pmatrix} y_1 - e^{y_2} \\ y_2 \\ 2(y_1 - 2)^3 \end{pmatrix}, \qquad g(x) = \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ \ln(x_1 + 2x_2) \end{pmatrix}, \qquad u(x) = f(g(x)).$$

Let $x_0 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $y_0 = g(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Write out the formula for $u(x)$ in terms of x, compute the Jacobians $Df(y_0)$, $Dg(x_0)$ and $Du(x_0)$, and verify that $Du(x_0) = Df(g(x_0)) \cdot Dg(x_0)$.

(2) In class we studied the behavior of Euler's method for the equation

$$\begin{aligned} y' &= f(y), \\ y(0) &= 0 \end{aligned} \qquad f(y) = 1 + |y - 1|$$

and showed that the numerical solution y_K at $t_K = 1$ satisfies

$$y_K = \frac{e}{2} + \frac{e}{2}\left(\ln 2 - \frac{1}{2}\right)h + \frac{e}{2}\left[\theta(1-\theta) + \frac{(\ln 2)^2}{2} - \ln 2 + \frac{11}{24}\right]h^2 + O(h^3),$$

where $e = \exp(1)$ and θ is a more or less random parameter between 0 and 1 that depends on where the numerical solution crosses $y = 1$. Thus, in spite of the discontinuity in $\frac{\partial f}{\partial y}$, it appears that there is a nice function $\varepsilon(t)$ such that

$$y_n = y(t_n) + h\varepsilon(t_n) + O(h^2), \qquad 0 \le n \le K = h^{-1}.$$

Solve the variational equation

$$\varepsilon'(t) = A(t)\varepsilon(t) + b(t), \qquad A(t) = f'(y(t)), \quad b(t) = -\frac{1}{2}y''(t)$$

by hand (i.e. analytically) to verify that $\varepsilon(1) = \frac{e}{2}\left(\ln 2 - \frac{1}{2}\right)$. Hint: solve up to $t = \ln 2$ and use the result as the initial condition for the new equation.

(3) Derive the four step Adams-Bashforth method and use it to solve the problems

(a) (2-d gravity) $x' = u$, $y' = v$, $u' = -x/(x^2+y^2)^{3/2}$, $v' = -y/(x^2+y^2)^{3/2}$, $x(0) = 1$, $y(0) = 0$, $u(0) = 0$, $v(0) = 1$, $T = 2\pi$

(b) (non-smooth f) $y' = 1 + |y-1|$, $y(0) = 0$, $T = 1$

Use the exact solutions to start the method going. Compute the error $\|e_K\| = \|y_K - y(t_K)\|$ at $t_K = T$ with $K = 12, 15, 19, 24, 30, 38, 48, 60, 76, \ldots, 6144, 7680, 9728$ using the exact solution for $y(T)$. (Note the pattern: $x_{k+2} = 2x_k$). Plot $\log_{10}\left(\|e_K\|\right)$ vs. $\log(h)$ (where $h = T/K$) and do a linear regression to find the order at which the numerical solution is converging to the exact solution.