Math 228A, Fall 2006, Wilkening

Homework 3
due Thursday, Oct. 5

(1) (Chain rule). Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}^2$ and $u : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(y) = \begin{pmatrix} y_1 - e^{y_2} \\ y_2 \\ 2(y_1 - 2)^3 \end{pmatrix}, \qquad g(x) = \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ \ln(x_1 + 2x_2) \end{pmatrix}, \qquad u(x) = f(g(x)).$$

Let $x_0 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $y_0 = g(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Write out the formula for $u(x)$ in terms of x, compute the Jacobians $Df(y_0)$, $Dg(x_0)$ and $Du(x_0)$, and verify that $Du(x_0) = Df(g(x_0)) \cdot Dg(x_0)$.

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(2) The s-step Nystrom and Milne methods are derived exactly as the Adams-Bashforth and Adams-Moulton methods, except we integrate from t_{n-1} to t_{n+1} instead of t_n to t_{n+1}:

$$y_{n+1} = y_{n-1} + \int_{t_{n-1}}^{t_{n+1}} p(t)\,dt = h\big[b_0 f_{n+1} + b_1 f_n + \cdots + b_s f_{n-s+1}\big].$$

For the Nystrom method, we require $b_0 = 0$ and use

$$p(t) = p_1(t) f_n + \cdots + p_s(t) f_{n-s+1}, \qquad p_m(t) = \prod_{\substack{j=1 \\ j\neq m}}^{s} \frac{t - t_{n+1-j}}{t_{n+1-m} - t_{n+1-j}}, \qquad (1 \le m \le s).$$

For the Milne method, we allow $b_0 \neq 0$ and define

$$p(t) = p_0(t) f_{n+1} + \cdots + p_s(t) f_{n-s+1}, \qquad p_m(t) = \prod_{\substack{j=0 \\ j\neq m}}^{s} \frac{t - t_{n+1-j}}{t_{n+1-m} - t_{n+1-j}}, \qquad (0 \le m \le s).$$

(a) Compute the coefficients b_m for the three step Nystrom and two step Milne methods by integrating $p(t)$.

(b) Compute them using the fact that $\rho(z) = z^{s-2}(z^2 - 1)$ and

$$\sigma(z) = \frac{\rho(z)}{\ln z} + O(|z - 1|^p),$$

where $p = s$ for the Nystrom method and $p = s + 1$ for the Milne method.

(3) Determine the order of the three-step method

$$y_{n+1} - y_{n-2} = h\left[\frac{3}{8}f_{n+1} + \frac{9}{8}f_n + \frac{9}{8}f_{n-1} + \frac{3}{8}f_{n-2}\right],$$

the *three-eighths* scheme. Is it convergent?

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(4) Show that the multistep method

$$y_{n+1} + a_1 y_n + a_2 y_{n-1} + a_3 y_{n-2} = h\left[b_1 f_n + b_2 f_{n-1} + b_3 f_{n-2}\right]$$

is fourth order only if $a_1 + a_3 = 8$ and $a_2 = -9$. Hence, deduce that this method cannot be both fourth order and convergent.

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(5) By solving a three-term recurrence relation, calculate analytically the sequence of values y_2, y_3, y_4, $\ldots$ that is generated by the *midpoint rule*

$$y_{n+1} = y_{n-1} + 2hf_n$$

when it is applied to the differential equation $y' = -y$. Starting from the values $y_0 = 1$, $y_1 = 1 - h$, show that the sequence diverges as $n \to \infty$. Recall, however, that the root condition together with consistency of order $p \geq 1$ and suitable starting conditions imply convergence to the true solution over any *finite* interval as $h \to 0+$. Prove that this implementation of the midpoint rule (i.e. using these starting values) does converge over any finite interval $[0, T]$. [*Hint: Express the roots of the characteristic polynomial of the recurrence relation as* $\mp\exp(\pm \operatorname{arcsinh} h)$.]