Math 228A, Fall 2006, Wilkening

Homework 2
due Thursday, Sep. 21

(1) In class we studied the behavior of Euler's method for the equation

$$\begin{aligned} y' &= f(y), \\ y(0) &= 0 \end{aligned} \qquad f(y) = 1 + |y-1|$$

and showed that the numerical solution y_K at $t_K = 1$ satisfies

$$y_K = \frac{e}{2} + \frac{e}{2}\left(\ln 2 - \frac{1}{2}\right) h + \frac{e}{2}\theta(1-\theta)h^2 + O(h^3),$$

where $e = \exp(1)$ and θ is a more or less random parameter between 0 and 1 that depends on where the numerical solution crosses $y = 1$. Thus, in spite of the discontinuity in $\frac{\partial f}{\partial y}$, it appears that there is a nice function $\tilde{e}(t)$ such that

$$y_n = y(t_n) + h\tilde{e}(t_n) + O(h^2), \qquad 0 \le n \le K = h^{-1}.$$

Solve the variational equation

$$\tilde{e}'(t) = A(t)\tilde{e}(t) + b(t), \qquad A(t) = f'(y(t)), \quad b(t) = -\frac{1}{2}y''(t)$$

by hand (i.e. analytically) to verify that $\tilde{e}(1) = \frac{e}{2}\left(\ln 2 - \frac{1}{2}\right)$. Hint: solve up to $t = \ln 2$ and use the result as the initial condition for the new equation.

---

(2) Derive the four step Adams-Bashforth method and use it to solve the problems

(a) $x' = u$, $y' = v$, $u' = -x/(x^2+y^2)$, $v' = -y/(x^2+y^2)$,
$x(0) = 1$, $y(0) = 0$, $u(0) = 0$, $v(0) = 1$, $T = 2\pi$

(b) $y' = 1 + |y-1|$, $y(0) = 0$, $T = 1$

Use the exact solutions to start the method going. Compute the error $\|e_K\| = \|y_K - y(t_K)\|$ at $t_K = T$ with $K = 12, 15, 19, 24, 30, 38, 48, 60, 76, \ldots, 6144, 7680, 9728$ using the exact solution for $y(T)$. Plot $\log_{10}\left(\|e_K\|\right)$ vs. $\log(h)$ (where $h = T/K$) and do a linear regression to find the order at which the numerical solution is converging to the exact solution.