sample midterm questions (the actual midterm will be shorter and include a mix of easier homework style questions)

(1) Show that if $x \in l^2(\mathbb{N})$ and $y \in l^2(\mathbb{N})$ then $\sum_{k=1}^{\infty} x_k y_k$ converges.

(2) Only one of the following 4 statements is true. Identify it and give a proof. (you do not need to exhibit counterexamples for the others).

$$L^1(0,1) \subset L^2(0,1), \quad l^1(\mathbb{N}) \subset l^2(\mathbb{N}), \quad L^1(\mathbb{R}) \subset L^2(\mathbb{R}), \quad L^2(\mathbb{R}) \subset L^1(\mathbb{R}).$$

(3) Let $T : l^2(\mathbb{Z}) \to L^2(-\pi,\pi)$ map Fourier coefficients to the functions they represent, i.e.

$$Tc = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

Give a formula defining the adjoint function $T^* : L^2(-\pi,\pi) \to l^2(\mathbb{Z})$, i.e. compute $(T^*f)_k$ for any $f \in L^2$ and $k \in \mathbb{Z}$.

(4) Show that $\displaystyle\int_{-\pi}^{\pi} \frac{\sin\big((N+1/2)x\big)}{\sin(x/2)}\,dx = 2\pi$.

(5) Use Plancherel's theorem to compute $\displaystyle\int_{-\infty}^{\infty} \operatorname{sinc}^2(\lambda)\,d\lambda$, where $\operatorname{sinc}(\lambda) = \frac{\sin(\lambda)}{\lambda}$.

(6) Compute $\mathcal{F}^{-1}\left[\frac{2}{\pi}\operatorname{sinc}^2\lambda\right](x)$. (Evaluate the convolution explicitly).

(7) Let $f(x) = x(1-x)$ for $0 \le x \le 1$. We can expand f in a Fourier cosine or sine series as follows:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x), \qquad f(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x).$$

State the rate at which the coefficients a_k and b_k decay (i.e. what are the largest values of r_1 and r_2 such that there exist constants C_1 and C_2 such that $|a_k| \le C_1 k^{-r_1}$ and $|b_k| \le C_2 k^{-r_2}$ for $k \ge 1$? You do not need to justify your answer.)

(8) Are the following filters (a) linear, (b) time-invariant, (c) causal?

1. $(L_1 f)(t) = f(t) - \int_t^{t^2} f(x)\,dx$
2. $(L_2 f)(t) = \int_{-\infty}^{\infty} f(x) e^{-(x-t)^2}\,dx$

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First Midterm Exam

State your answers clearly and fully (with whole sentences, please). Include all your work. (Total points = 40.)

9 1.a) Give a precise definition of what is meant by an orthonormal **basis** for an infinite-dimensional vector-space with inner product. Make clear how you use the inner product (beyond for "orthonormal").

5 b) Give a specific example of an orthonormal basis for $L^2([0, 1])$.

5 2.a) Give the definition of the convolution of two functions in $L^1(R)$.

9 b) Show that if g and h are filter functions in $L^1(R)$, and if L_g and L_h are the corresponding filter operators, then $L_g L_h = L_{g*h}$.

12 3. Let f be the function in $L^2([0, 1])$ defined on [0,1] by $f(t) = t^5$. Explain precisely what will happen for convergence of the Fourier series for f for

a) mean-square convergence,
b) uniform convergence, and
c) pointwise convergence.

Give reasons for your answers.