Math 118 Midterm
Spring 2009, Wilkening

Answer only 4 of the following 5 questions. Indicate clearly which question you don't want graded (or I'll only grade the first four).

Name:

1. Let $f_n(x) = \left\{ \begin{array}{ll} 0, & 0 \leq x \leq 1 \\ \sqrt{n}, & 1 < x < 1 + n^{-1} \\ 0, & 1 + n^{-1} \leq x \leq 3 \end{array} \right\}$. Answer the following questions true or false and give a short proof justifying each answer.

a. (2.5 points) $f_n \to 0$ pointwise on $[0, 3]$.

b. (2.5 points) $f_n \to 0$ uniformly on $[0, 3]$.

c. (2.5 points) $f_n \to 0$ in $L^1[0, 3]$.

d. (2.5 points) $f_n \to 0$ in $L^2[0, 3]$.

2. Let V be the set of vectors in $\mathbb{C}^2$ with inner product $\langle x, y \rangle = x^T M \overline{y}$, $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

a. (4 points) Find an orthonormal basis for V.

b. (3 points) Let $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $W = \text{span}\{w\} = \{\alpha w \ : \ \alpha \in \mathbb{C}\}$. Find the closest point $v \in W$ to u in the norm of V, i.e. minimize $\|v - u\|^2 = \langle v - u, v - u \rangle$.

c. (3 points) Let $A = \begin{pmatrix} 1 & -i \\ 1 & 0 \end{pmatrix}$. Treating A as a linear operator $A : V \to V$, find the adjoint matrix A^*. In case it is helpful, $M^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.

3. Let $f(x) = \left\{ \begin{array}{ll} 1+x, & -1 \leq x \leq 0 \\ 1-x, & 0 \leq x \leq 1 \end{array} \right\} \in L^2(-1,1)$.

a. (6 points) Compute the coefficients c_k in the expansion $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x}$.

b. (4 points) Evaluate $\sum_{k=-\infty}^{\infty} |c_k|^2$.

4. Let $Lf = h * f$ be the first order (low pass) Butterworth filter with impulse response and system functions given by

$$h(t) = \left\{ \begin{array}{cc} \lambda_c e^{-\lambda_c t}, & t \geq 0 \\ 0 & t < 0 \end{array} \right\}, \qquad \hat{h}(\lambda) = \frac{\lambda_c}{\sqrt{2\pi}(\lambda_c + i\lambda)}.$$

a. (4 points) What impulse response function $h_1(t)$ corresponds to the following system function, which is intended to serve as a band pass filter to attenuate frequencies outside the ranges $(\lambda_0 - \lambda_c, \lambda_0 + \lambda_c) \cup (-\lambda_0 - \lambda_c, -\lambda_0 + \lambda_c)$?

$$\hat{h}_1(\lambda) = \frac{1}{2}\left[\frac{\lambda_c}{\sqrt{2\pi}[\lambda_c + i(\lambda - \lambda_0)]} + \frac{\lambda_c}{\sqrt{2\pi}[\lambda_c + i(\lambda + \lambda_0)]}\right].$$

b. (6 points) Now suppose we connect two identical Butterworth filters together in series to obtain a new filter $L_2 = L \cdot L$ (i.e. $L_2 f = L(Lf)$). Find an explicit formula for the impulse response function $h_2(t)$ and the system function $\hat{h}_2(\lambda)$ for L_2.

5a. (4 points) Suppose $P : V \to V$ is a bounded operator on an inner product space satisfying $P^* = P$ and $P^2 = P$, and let I be the identity map on V. Show that for any $u, v \in V$ we have $\langle (I - P)u, Pv \rangle = 0$.

5b. (6 pts) Recall that the uncertainty principle says that for any $a, \alpha \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$,

$$(\Delta_a f)(\Delta_\alpha \hat{f}) \geq \frac{1}{4}, \qquad \Delta_a f := \frac{\int (x-a)^2 |f(x)|^2 \, dx}{\int |f(x)|^2 \, dx}, \quad \Delta_\alpha \hat{f} := \frac{\int (\lambda - \alpha)^2 |\hat{f}(\lambda)|^2 \, d\lambda}{\int |\hat{f}(\lambda)|^2 \, d\lambda}.$$

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 3, \quad \int_{-\infty}^{\infty} x|f(x)|^2 \, dx = 6, \quad \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx = 15.$$

Show that

$$\int_{-\infty}^{\infty} \lambda^2 |\hat{f}(\lambda)|^2 \, d\lambda \geq \frac{3}{4}.$$