

**Homework 7**  
**due Thurs., Apr. 2**

(1a) Show that if  $A : V \rightarrow V$  is a linear operator on an inner product space  $V$  satisfying  $A^* = A^{-1}$ , then  $A$  preserves lengths and inner products. Such matrices are called unitary.

(1b) Let  $V = S_N$  with inner product  $\langle y, z \rangle = \frac{1}{N} \sum_{j=0}^{N-1} y_j \bar{z}_j$ . Show that the adjoint of a matrix in this inner product is its conjugate transpose. Now use the fact that  $\frac{1}{N} F_N \bar{F}_N = I$  and  $F_N^T = F_N$  to conclude that  $\frac{1}{\sqrt{N}} F_N$  and  $\frac{1}{\sqrt{N}} \bar{F}_N$  are both unitary.

(1c) Show that the 2-norm of a diagonal matrix  $D$  acting on  $S_N$  is equal to the largest of the absolute values of the diagonal entries, i.e.

$$\sup_{y \neq 0} \frac{\|Dy\|_{S_N}}{\|y\|_{S_N}} = \max_{0 \leq j \leq N-1} |D_{jj}|.$$

(1d) Compute the 2-norm of the operator  $Ly = h * y$ , where  $h \in S_N$ .

(2) problem 15 page 153. To make your life simpler, assume that matrices are indexed from 0 to  $N-1$  so that  $a_\ell = A_{\ell,0}$ ,  $x_\ell = X_{\ell,0}$ ,  $y_\ell = Y_{\ell,0}$  for  $\ell = 0, \dots, n-1$ . In part (c), in addition to showing that  $n^{-1} F_n^T A \bar{F}_n = D$  is diagonal, show that  $n^{-1} \bar{F}_n^T A F_n = S$  is also diagonal. I consider this second version to be the more natural diagonalization.

(3) problem 16 page 153.

(4) Consider the heat equation on a periodic domain

$$u_t = u_{xx}, \quad u(x, 0) = u_0(x), \quad u(0, t) = u(2\pi, t).$$

(a) Check that  $u(x, t) = e^{-t} \cos x + e^{-4t} \sin 2x$  is the solution with initial condition  $u_0(x) = \cos x + \sin 2x$ . Use this initial condition in (b) and (d) below.

(b) code up the finite difference method  $u_j^{n+1} = \nu u_{j-1}^n + (1-2\nu)u_j^n + \nu u_{j+1}^n$  with  $\nu = \frac{\Delta t}{\Delta x^2}$  using  $\Delta t = 1/N$ ,  $\Delta x = 2\pi/M$ ,  $N = 1000$ ,  $M = 100$ . Note that periodicity requires that when  $j = 0$ ,  $u_{j-1}^n = u_{M-1}^n$  and when  $j = M-1$ ,  $u_{j+1}^n = u_0^n$ . Plot the numerical solution  $\{u_j^n\}_{j=0}^{M-1}$  at  $n = N$  (i.e.  $t = 1$ ). Also plot the error  $E_j^N = u_j^N - u(j\Delta x, 1)$ .

(c) we can write the above numerical scheme as  $u^{n+1} = L(u^n)$ , where  $Ly = h * y$  and  $h = (\dots, 0, \nu, 1-2\nu, \nu, 0, \dots)$ . Show that the transfer function of  $h$  is  $\hat{h}(\phi) = 1 - 4\nu \sin^2(\phi/2)$ .

(d) compute the numerical solution in (b) a different way: let

$$v = \text{fft}(u_0), \quad w_k = \hat{h}(\phi_k)^N v_k, \quad 0 \leq k \leq M-1, \quad U^N = \text{ifft}(w),$$

where  $\phi_k = 2\pi k/M$ . Plot the real and imaginary parts of the difference  $\{U_j^N - u_j^N\}_{j=0}^{M-1}$  to show that the two methods give the same answer (up to roundoff error). Be careful of “off by one” errors in the indices.