Math 185. Answers to Sample Second Midterm

1. (15 points) Carefully define the following. (In each definition you may use without defining them any terms or symbols that were used in the text prior to that definition.)

(a). Analytic continuation

Answer: If f_1 is analytic on a domain D_1, if f_2 is analytic on a domain D_2, if $D_1 \cap D_2 \neq \emptyset$, and if $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$, then we say that f_2 is an **analytic continuation** of f_1 to D_2. (Usually we'll have $D_2 \subseteq D_1$, but this not required in the definition.)

(b). Residue

Answer: Assume that z_0 is an isolated singularity for an analytic function $f(z)$, and that f has a Laurent series $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ in some annular neighborhood $0 < |z - z_0| < \epsilon$ (where $\epsilon > 0$). Then the **residue** $\operatorname{Res}_{z=z_0}(f)$ of f at z_0 is the coefficient c_{-1}.

(c). Pole of order m

Answer: A function f has a **pole of order** m at z_0 if its principal part at that point is of the form

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}$$

with $b_m \neq 0$.

(Alternatively, one could use the principal part in part (b), or the Laurent series on the annular domain $0 < |z - z_0| < \epsilon$ in part (c).)

2. (20 points) Find

$$\operatorname*{Res}_{z=0} \frac{\tan z - z}{(1-\cos z)^2} .$$

[**Hint:** Use power series.]

Answer: There is no simple formula for the Maclaurin series of $\tan z$, so it's better to work with sines and cosines. We have

$$\begin{aligned}
\frac{\tan z - z}{(1-\cos z)^2} &= \frac{\sin z - z\cos z}{(\cos z)(1-\cos z)^2} \\
&= \frac{(z - z^3/3! + \dots) - z(1 - z^2/2 + \dots)}{(z^2/2 - \dots)^2} \\
&= \frac{z^3/3 + \dots}{z^4/4 + \dots} .
\end{aligned}$$

The numerator has a zero of order 3, and the denominator has a zero of order 4, so the function overall has a simple pole at $z = 0$. Therefore, it is enough to look at only the leading terms of the power series and we see that the Laurent series expansion in a punctured neighborhood of $z = 0$ is

$$\frac{\tan z - z}{(1 - \cos z)^2} = \frac{4}{3z} + \dots ,$$

so the residue is $4/3$.

3. (20 points) Find

$$\int_C \frac{z\,dz}{e^z + 1} ,$$

where C is the positively-oriented circle $|z| = 4$.

Answer: The integrand is analytic except when $e^z = -1$, which happens if and only if $z = \pi i + 2\pi ik$ ($k \in \mathbb{Z}$); in other words iff $z = \pi in$ with n an odd integer. Since $\pi < 4 < 3\pi$, the poles inside the contour are $\pm\pi i$. By Theorem 2 on page 253, we have

$$\operatorname*{Res}_{z=\pi i} \frac{z}{e^z + 1} = \frac{\pi i}{e^{\pi i}} = \frac{\pi i}{-1} = -\pi i$$

and

$$\operatorname*{Res}_{z=-\pi i} \frac{z}{e^z + 1} = \frac{-\pi i}{e^{-\pi i}} = \frac{-\pi i}{-1} = \pi i .$$

Therefore

$$\int_C \frac{z\,dz}{e^z + 1} = 2\pi i \left(\operatorname*{Res}_{z=\pi i} \frac{z}{e^z + 1} + \operatorname*{Res}_{z=-\pi i} \frac{z}{e^z + 1} \right) = 2\pi i(-\pi i + \pi i) = 0 .$$

Well, we haven't gotten to page 253 yet (sorry), but the above residues can be computed using series as follows. Similarly to Exercise 2b on page 196, we can write

$$\frac{z}{e^z + 1} = \frac{(z - \pi i) + \pi i}{e^{\pi i} e^{z - \pi i} + 1} = \frac{\pi i + (z - \pi i)}{-(z - \pi i) - (z - \pi i)^2/2! - \dots} = \frac{-\pi i}{z - \pi i} + \dots ,$$

and so the residue at πi is $-\pi i$. The residue at $-\pi i$ is similar.

4. (25 points) Let D be the domain $\{z : 0 < |z| < \epsilon\}$ for some $\epsilon > 0$, and let f be an analytic function on D. Assume that f has an essential singularity at $z = 0$.

(a). Show that if g is a nonzero function on D with a removable singularity at $z = 0$, then the product fg has an essential singularity there.

Answer: Argue by contradiction.

Write $g(z) = \phi(z)(z - z_0)^n$ with $n \in \mathbb{N}$ and $\phi(z)$ analytic and nonzero at $z = 0$. If fg does not have an essential singularity at $z = 0$ then either:

(a) fg is analytic at $z = 0$, in which case $fg = \psi(g)(z - z_0)^m$ with $m \in \mathbb{N}$ and $\psi(z)$ analytic and nonzero at $z = 0$. Then

$$f(z) = \frac{f(z)g(z)}{g(z)} = \frac{\phi(z)}{\psi(z)}(z - z_0)^{m-n}$$

where ϕ/ψ is analytic and nonzero at $z = 0$. But this would imply that f has a removable singularity or pole at $z = 0$, contradiction.

or, (b) fg has a pole of order m at $z = 0$. In this case $fg = \psi(g)(z - z_0)^{-m}$ with $\psi(z)$ analytic and nonzero at $z = 0$, and as before we obtain a contradiction since $f(z) = (\phi(z)/\psi(z))(z - z_0)^{-m-n}$.

(b). Show that the same holds if g is any nonzero function not having an essential singularity at $z = 0$.

Answer: If g does not have an essential singularity at $z = 0$ then either it is analytic there, has a removable singularity there, or has a pole there. The first two cases are covered by part (a); the last case is similar to (a) except that $g(z) = \phi(z)(z - z_0)^n$ with $n \in \{-1, -2, \dots\}$.

(c). What can happen if g has an essential singularity at $z = 0$?

Answer: The product fg can have any type of (isolated) singularity at $z = 0$. For example, let $f(z) = e^{1/z}$. If $g(z) = e^{-1/z}$ then $fg = 1$ has a removable singularity at $z = 0$. If $g(z) = e^{-1/z}/z$ then $g(z)$ has an essential singularity at $z = 0$ (by part (b)) and $fg = 1/z$ has a pole. Finally, if $g(z) = e^{1/z}$ then $fg = e^{2/z}$ has an essential singularity at $z = 0$.

5. (20 points) Find

$$\int_0^\infty \frac{dx}{(x^2 + 1)^3}\,.$$

You may include residues in your answer without evaluating them.

Answer: The function $f(z) = 1/(z^2 + 1)^3$ is analytic everywhere except for poles at $z = \pm i$. Integrating around the contour given in Fig. 93 on page 263 (except for the residues drawn there) gives

$$\int_{-R}^{R} \frac{dx}{(x^2 + 1)^3} + \int_{C_R} \frac{dz}{(z^2 + 1)^3} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2 + 1)^3} \qquad (*)$$

for all (real) $R > 1$, where C_R is the semicircular contour $z(\theta) = Re^{i\theta}$, $0 \le \theta \le \pi$. We have $|z^2 + 1| \le R^2 - 1$ for all z with $|z| = R$, so

$$|f(z)| \le \frac{1}{(R^2 - 1)^3} \qquad \text{for all } z \in C_R$$

and therefore

$$\left| \int_{C_R} \frac{dz}{(z^2+1)^3} \right| \le \pi R \cdot \frac{1}{(R^2-1)^3} \ .$$

This latter expression $\to 0$ when $R \to \infty$, so

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{(z^2+1)^3} = 0 \ .$$

Taking the limit as $R \to \infty$ in (*) then gives

$$\text{P. V.} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^3}$$

and therefore, since $f(z)$ is even,

$$\int_0^{\infty} \frac{dx}{(x^2+1)^3} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^3} \ .$$