Math 185. Sample Answers to Second Midterm

1. (15 points) Carefully define the following. (In each definition you may use without defining them any terms or symbols that were used in the text prior to that definition.)

   (a). Contour integral $\int_C f(z)\,dz$ (give a formula)

   Answer (4 points): $\displaystyle\int_a^b f(z(t))z'(t)\,dt$, where $z = z(t)$, $a \le t \le b$, is the contour.

   (b). Circle of convergence, or radius of convergence (choose one, and define it without referring to the other one)

   Answer (3 points): Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Then the **circle of convergence** of this power series is the largest circle centered on z_0 such that the series converges for all z inside of the circle. Or, the **radius of convergence** is the least upper bound of the set $\{|z - z_0| : \text{the series converges at } z\}$ (or infinity, if the set is unbounded).

   (c). Uniformly convergent series

   Answer (4 points): A series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is **uniformly convergent** on a set S if for all $\epsilon > 0$ there is an $N = N_\epsilon$ such that

$$\left| \sum_{n=0}^{\infty} a_n(z - z_0)^n - \sum_{n=0}^{N} a_n(z - z_0)^n \right| < \epsilon$$

   for all $N > N_\epsilon$ and *(this is important)* for all $z \in S$.

   (d). Essential singularity

   Answer (4 points): An **essential singularity** is an isolated singular point for a function where the principal part has infinitely many (nonzero) terms.

2. (20 points) Prove Liouville's theorem: a bounded, entire function must be constant.

   In proving this, you should be careful not to use any results from later in the book that rely on this result. You may also not quote the theorem of which this result forms a part, but may use any results from earlier in the book.

   Answer:
   Since f is bounded, there is a real number $M \ge 0$ such that $|f(z)| \le M$ for all $z \in \mathbb{C}$. By Cauchy's inequality ((2) on page 170) with $n = 1$ we then have

$$|f'(z_0)| \le \frac{M}{R} ,$$

   where R is the radius of a circle such that f is analytic on and within the circle of radius R centered on z_0 . Since f is entire we can let $R \to \infty$, hence $|f'(z_0)| = 0$, so $f'(z_0) = 0$. Since this is true for all $z_0 \in \mathbb{C}$, it follows by a theorem in the book (page 74) that f must be constant.

3. (22 points) Let D be the domain $|z| < 1$. Suppose that $f\colon D \to \mathbb{C}$ is analytic on D, and that $|f^{(n)}(0)| \le 1$ for all $n = 0, 1, 2, \ldots$. Show that f can be extended to an entire function.

   (Justify any claims about convergence.)

   Answer:
   Since f is analytic at $z = 0$, it has a Taylor series there:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ ,$$

where $a_n = f^{(n)}(0)/n!$. Since $|f^{(n)}(0)| \le 1$ for all $n \in \mathbb{N}$, we have $|a_n| \le 1/n!$, so for all $z \in \mathbb{C}$ the Taylor series converges since it converges absolutely:

$$\sum_{n=0}^{\infty} |a_n z^n| \le \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty \ .$$

Therefore the radius of convergence of the series $\sum a_n z^n$ is infinite, so the analytic function it defines is entire, and extends f by Taylor's theorem.

4. (20 points) Find the Laurent series that represents the function

$$f(z) = \frac{1}{z(z-3)}$$

in the domain $1 < |z-1| < 2$.

   Answer:
   First, break the function down into partial fractions. If we write

$$\frac{1}{z(z-3)} = \frac{A}{z} + \frac{B}{z-3} \ ,$$

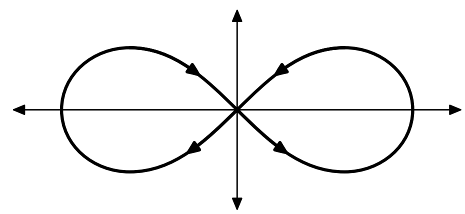
then $1 = A(z-3) + Bz$. Plugging in $z = 0$ gives $A = -1/3$; using $z = 3$ gives $B = 1/3$. Thus we have

$$\begin{aligned}
\frac{1}{z(z-3)} &= \frac{1}{3}\left(\frac{1}{z-3} - \frac{1}{z}\right) \\
&= \frac{1}{3}\left(\frac{1}{(z-1)-2} - \frac{1}{(z-1)+1}\right) \\
&= \frac{1}{3}\left(-\frac{1}{2}\cdot\frac{1}{1-\frac{z-1}{2}} - \frac{1}{z-1}\cdot\frac{1}{1+\frac{1}{z-1}}\right) \\
&= -\frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n - \frac{1}{3(z-1)}\sum_{n=0}^{\infty}\left(\frac{-1}{z-1}\right)^n \\
&= -\frac{1}{3}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}} + \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(z-1)^n} \ .
\end{aligned}$$

This uses the series $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$, noting that $\left|\dfrac{z-1}{2}\right| < 1$ and $\left|\dfrac{-1}{z-1}\right| < 1$.

5. (23 points) Let C be the "figure-eight" contour

$$(x^2+y^2)^2 = 2(x^2-y^2)\ ,$$



illustrated on the right.

(a). Why shouldn't one use the residue at infinity to evaluate the integral

$$\int_C \frac{e^z\,dz}{z^2-1}\ ?$$

Answer (8 points): Because C is not a simple closed contour: it crosses itself. (See the theorem on page 238.)

(b). Compute this integral. Explain your steps carefully.

(If you're stuck, compute residues for partial credit.)

Answer (15 points): The function $f(z) = \dfrac{e^z}{z^2-1}$ has isolated singular points at $z = \pm 1$. The given curve intersects the x-axis when $x^4 = 2x^2$, so $x = \pm\sqrt{2}$ (and, of course, $x = 0$). Therefore, these singular points are inside of the contour. (Strictly speaking, I shouldn't say *inside* when C is not a simple closed contour, but $z = 1$ is inside of C_1 and $z = -1$ is inside of C_2, where the simple closed contours C_1 and C_2 are defined below.)

First, the residues:

$$\operatorname*{Res}_{z=1} \frac{e^z}{z^2-1} = \operatorname*{Res}_{z=1} \frac{e^z/(z+1)}{z-1} = \left.\frac{e^z}{z+1}\right|_{z=1} = \frac{e}{2}$$

and

$$\operatorname*{Res}_{z=-1} \frac{e^z}{z^2-1} = \operatorname*{Res}_{z=-1} \frac{e^z/(z-1)}{z+1} = \left.\frac{e^z}{z-1}\right|_{z=-1} = -\frac{1}{2e}\ .$$

These use either Problem 3 on page 243, or the theorem on page 244.

Now let C_1 be the right half of C, and let C_2 be the left half, both positively oriented. Since C goes around its left half *clockwise,* it is negatively oriented there, so

$$C = C_1 - C_2\ .$$

Therefore

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=1} f(z) - 2\pi i \operatorname*{Res}_{z=-1} f(z) = 2\pi i \left(\frac{e}{2} + \frac{1}{2e}\right)\ .$$