

## Math 185. Differentiating under the Integral

We begin with *Fubini's theorem*:

**Theorem** (Fubini). *Let  $a < b$  and  $c < d$  be real numbers, and let  $g: [a, b] \times [c, d] \rightarrow \mathbb{R}$  be a continuous function. Then*

$$\int_a^b \int_c^d g(x, y) dy dx = \int_c^d \int_a^b g(x, y) dx dy .$$

(In more general settings, such as those involving improper integrals, discontinuous functions, or more advanced concepts of integration, the theorem would assume that  $g$  is *absolutely integrable*:  $\int_a^b \int_c^d |g(x, y)| dy dx < \infty$ . In the situation here, though, this condition follows from continuity of  $g$  and compactness of its domain.)

This allows you to differentiate real integrals under the integral sign:

**Corollary.** *Let  $I$  be an open interval, and let  $\phi: [a, b] \times I \rightarrow \mathbb{R}$  be a continuous function. Assume that  $\phi$  is differentiable in its second variable  $y$ , and that the partial derivative  $\phi_y$  is continuous on  $[a, b] \times I$ . Let  $\psi: I \rightarrow \mathbb{R}$  be the function given by*

$$\psi(y) = \int_a^b \phi(t, y) dt, \quad y \in I .$$

Then  $\psi$  is differentiable, and

$$\psi'(y) = \int_a^b \phi_y(t, y) dt .$$

*Proof.* Let  $y \in I$  and fix  $c, d \in I$  with  $c < y < d$ . Applying Fubini's theorem to  $\phi_y$  on  $[a, b] \times [c, y]$  for  $y \in (c, d)$ , we have

$$\begin{aligned} \psi(y) &= \int_a^b (\phi(t, y) - \phi(t, c)) dt + \psi(c) \\ &= \int_a^b \int_c^y \phi_y(t, v) dv dt + \psi(c) \\ &= \int_c^y \int_a^b \phi_y(t, v) dt dv + \psi(c) . \end{aligned}$$

This last expression has derivative  $\int_a^b \phi_y(t, y) dt$ . □

Next, we need a result on continuity and integration.

**Lemma.** Let  $D$  be a domain in  $\mathbb{C}$ , let  $\phi: [a, b] \times D \rightarrow \mathbb{R}$  be a continuous function, and let  $\psi: D \rightarrow \mathbb{R}$  be the function

$$\psi(z) = \int_a^b \phi(t, z) dt .$$

Then  $\psi$  is continuous.

*Proof.* Pick  $z_0 \in D$  and  $\epsilon > 0$ . Pick  $\delta_0 > 0$  such that the closed disk

$$D' = \{z \in \mathbb{C} : |z - z_0| \leq \delta_0\}$$

is contained in  $D$ . Then, viewing  $D'$  as a subset of  $\mathbb{R}^2$ , we see that  $\phi$  is uniformly continuous on the compact set  $[a, b] \times D'$ , so in particular there is a  $\delta > 0$  with  $\delta < \delta_0$ , such that  $|z - z_0| < \delta$  implies  $|\phi(t, z) - \phi(t, z_0)| < \epsilon/(b - a)$  for all  $t$ . But then

$$|\psi(z) - \psi(z_0)| = \left| \int_a^b (\phi(t, z) - \phi(t, z_0)) dt \right| \leq \int_a^b |\phi(t, z) - \phi(t, z_0)| dt < \int_a^b \frac{\epsilon}{b - a} dt = \epsilon$$

for all  $z$  with  $|z - z_0| < \delta$ . This shows that  $\psi$  is continuous at  $z_0$ .  $\square$

Our main result is then the following.

**Theorem.** Let  $D$  be a domain in  $\mathbb{C}$ , and let  $F: [a, b] \times D \rightarrow \mathbb{C}$  be a continuous function. Assume that  $F$  is differentiable in its second variable  $z$ , and that  $F_z$  is continuous. Then the function  $f: D \rightarrow \mathbb{C}$  defined by

$$f(z) = \int_a^b F(t, z) dt$$

is analytic, and

$$f'(z) = \int_a^b F_z(t, z) dt .$$

*Proof.* Write  $F(t, z) = U(t, x, y) + iV(t, x, y)$ , where  $U$  and  $V$  are real-valued. Since  $F_z = U_x + iV_x$ ,  $U_x$  and  $V_x$  are continuous; by the Cauchy-Riemann equations, so are  $U_y$  and  $V_y$ . Let

$$u(x, y) = \int_a^b U(t, x, y) dt \quad \text{and} \quad v(x, y) = \int_a^b V(t, x, y) dt ,$$

so that  $f(z) = u(x, y) + iv(x, y)$ . By the corollary,  $u_x(x, y) = \int_a^b U_x(t, x, y) dt$ , and similarly for  $u_y$ ,  $v_x$ , and  $v_y$ . By the lemma, these partials are continuous, and they satisfy the Cauchy-Riemann equations (because  $U$  and  $V$  do), so  $f$  is analytic. Moreover,

$$f'(z) = u_x(x, y) + iv_x(x, y) = \int_a^b U_x(t, x, y) dt + i \int_a^b V_x(t, x, y) dt = \int_a^b F_z(t, z) dt . \quad \square$$