

Math 256A. Glueing Results

This handout collects some various results on glueing in various contexts.

Theorem 1 (Glueing of morphisms of sheaves). *Let X be a topological space, let \mathcal{F} and \mathcal{G} be sheaves on X , and let $\{U_i\}$ be an open cover of X . Suppose we are given morphisms $\phi_i: \mathcal{F}|_{U_i} \rightarrow \mathcal{G}|_{U_i}$ of sheaves for all i , such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ for all i, j . Then there exists a unique morphism $\phi: \mathcal{F} \rightarrow \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$ for all i .*

Proof. Let V be an open subset of X . Then we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \prod_i \mathcal{F}(V \cap U_i) & \longrightarrow & \prod_{i,j} \mathcal{F}(V \cap U_i \cap U_j) \\ & & \downarrow \exists \phi(V) & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{G}(V) & \longrightarrow & \prod_i \mathcal{G}(V \cap U_i) & \longrightarrow & \prod_{i,j} \mathcal{G}(V \cap U_i \cap U_j) \end{array}$$

in which the rows are exact (as in the equivalent condition defining a sheaf), and in which the rightmost two vertical maps are determined by ϕ_i . The compatibility of ϕ_i and ϕ_j on $U_i \cap U_j$ implies that the rightmost square commutes; therefore by a standard arrow chase there exists a map $\phi(V): \mathcal{F}(V) \rightarrow \mathcal{G}(V)$ making the leftmost square commute.

To see that this map defines a morphism of sheaves, imagine a three-dimensional commutative diagram in which one plane is the above diagram, and the second plane is identical except that all occurrences of V have been replaced by some open subset W of V . Then the square lying over the middle vertical map commutes, so the square over the leftmost vertical map also must commute; hence ϕ is a morphism of sheaves.

If $V \subseteq U_i$ for some i , then $V \cap U_i = V$, so the map $\phi(V)$ coincides with the i^{th} factor of the middle vertical map; this shows that $\phi|_{U_i} = \phi_i$, as was to be shown.

Finally, ϕ is unique since the maps $\phi(V)$ are uniquely determined by the condition that the leftmost square of the diagram commutes. \square

Corollary 2 (Comparison of sheaf morphisms). *Let X be a topological space, let \mathcal{F} and \mathcal{G} be sheaves on X , let ϕ and ψ be morphisms of sheaves $\mathcal{F} \rightarrow \mathcal{G}$, and let $\{U_i\}$ be an open cover of X . If $\phi|_{U_i} = \psi|_{U_i}$ for all i , then $\phi = \psi$.*

Proof. This follows immediately from the uniqueness assertion of Theorem 1. \square

Theorem 3 (Glueing of sheaves; Ex. 1.22). *Let X be a topological space, and let $\{U_i\}_{i \in I}$ be an open cover of X . Suppose we are given sheaves \mathcal{F}_i on U_i for all i , and isomorphisms $\phi_{ij}: \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ for all i, j , satisfying: (1) ϕ_{ii} is the identity morphism for all i ; and (2) $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$ for all i, j, k . Then there exists a sheaf \mathcal{F} on X and isomorphisms $\psi_i: \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ for all i , such that $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$ for all i, j . Moreover the data $(\mathcal{F}, \{\psi_i\})$ are unique up to unique isomorphism.*

Proof. Done in class on Thursday 22 October. \square

Proposition 4 (Comparison of morphisms of schemes). *Let $f: X \rightarrow Y$ and $g: X \rightarrow Y$ be morphisms of schemes, and let $\{U_i\}$ be an open cover of X . If $f|_{U_i} = g|_{U_i}$ for all i , then $f = g$.*

Proof. Suppose $f \neq g$. If the maps on topological spaces differ, then they do so on some U_i , so the result is obvious in that case. Otherwise, $f^\#$ and $g^\#$ are different maps $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X = g_*\mathcal{O}_X$, so there exists an open $U \subseteq X$ and a section $s \in \mathcal{O}_Y(U)$ such that $f^\#(U)(s) \neq g^\#(U)(s)$ in $\mathcal{O}_X(f^{-1}(U))$. Since \mathcal{O}_X is a sheaf, there is an index i such that $f^\#(U)(s)|_{f^{-1}(U) \cap U_i} \neq g^\#(U)(s)|_{f^{-1}(U) \cap U_i}$, and therefore $(f|_{U_i})^\#(s) \neq (g|_{U_i})^\#(s)$. Thus $f|_{U_i} \neq g|_{U_i}$. \square

Theorem 5 (Glueing of morphisms of schemes). *Let X and Y be schemes, and let $\{U_i\}$ be an open cover of X . Suppose we are given morphisms (of schemes) $f_i: U_i \rightarrow Y$ for each i , such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j . Then there exists a unique morphism $f: X \rightarrow Y$ such that $f|_{U_i} = f_i$ for all i .*

Proof. On the topological level, this is obvious. Now given an open subset $V \subseteq Y$ and a section $s \in \mathcal{O}_Y(V)$, the sheaf morphisms $f_i^\#$ give sections

$$s_i \in (f_i)_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V) \cap U_i).$$

Since $f_i = f_j$ on $U_i \cap U_j$, the sections s_i and s_j agree on $f^{-1}(V) \cap U_i \cap U_j$, hence they glue to give a section of $\mathcal{O}_X(f^{-1}(V)) = f_*\mathcal{O}_X(V)$. This defines $f^\#(s)$, and it is easy to see that this is a ring homomorphism and is compatible with the restriction maps. That the maps $\mathcal{O}_{Y, f(P)} \rightarrow \mathcal{O}_{X, P}$ are local homomorphisms for all $P \in X$ follows from the same condition for the f_i . Therefore f is a scheme morphism, and it is clear from the construction that $f|_{U_i} = f_i$ for all i . \square

Theorem 6 (Glueing of schemes; Ex. 2.12). *Let $\{X_i\}$ be a family of schemes. Assume that we are given, for each $i \neq j$, an open subset $U_{ij} \subseteq X_i$ and an isomorphism $\phi_{ij}: U_{ij} \xrightarrow{\sim} U_{ji}$, satisfying (1) $\phi_{ij} = \phi_{ji}^{-1}$ for all $i \neq j$; (2) $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all distinct i, j, k ; and (3) $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$ for all distinct i, j, k . Then there exists a scheme X , an open cover $\{V_i\}$ of X , and isomorphisms $\psi_i: X_i \rightarrow V_i$ for all i , such that (i) $\psi_i(U_{ij}) = V_i \cap V_j$ and (ii) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij} , for all $i \neq j$. Moreover, the data $(X, \{\psi_i\})$ are unique up to unique isomorphism.*

Proof. To construct $\text{sp}(X)$, let the set X be the disjoint union $\coprod X_i$, modulo the equivalence relation $x \sim x$, $x \in U_{ij} \sim \phi_{ij}(x)$. Conditions (1) and (2) imply that this relation is symmetric and transitive, respectively. Define $\psi_i: X_i \rightarrow X$ to be the composition $X_i \rightarrow \coprod X_j \rightarrow X$, let $V_i = \text{im } \psi_i$, and give X the topology in which $U \subseteq X$ is open if and only if $\psi_i^{-1}(U)$ is open for all i . Then each V_i is open (by openness of each U_{ji}) and ψ_i is a homeomorphism onto its image (since the ϕ_{ij} are all homeomorphisms). Moreover, (i) holds by construction.

To construct \mathcal{O}_X , consider the sheaves $(\psi_i)_*\mathcal{O}_{X_i}$ on each V_i . The compositions $\psi_j \circ \phi_{ij} \circ \psi_i^{-1}$ give isomorphisms $(\psi_i)_*\mathcal{O}_{X_i} \xrightarrow{\sim} (\psi_j)_*\mathcal{O}_{X_j}$ on $V_i \cap V_j$ for all $i \neq j$, and we have the identity isomorphisms on V_i of course. These isomorphisms satisfy condition (1) of Theorem 3 by construction, and they satisfy (2) of Theorem 3 by (1)–(3) of this theorem. Therefore, these sheaves glue to give a sheaf \mathcal{O}_X on X , which gives X the structure of a scheme by the local nature of the definition. Also, the ψ_i are defined to be isomorphisms of schemes in the obvious way, and they satisfy (ii) by construction and by the similar condition from Theorem 3.

The uniqueness condition holds because if $(X', \{\psi'_i\})$ is another collection of such data, then $\psi'_i \circ \psi_i^{-1}$ define isomorphisms $V_i \xrightarrow{\sim} V'_i := \text{im } \psi'_i$, and these are compatible on $V_i \cap V_j$ since $\psi'_i \circ \psi_i^{-1} = \psi'_j \circ \phi_{ij} \circ \phi_{ij}^{-1} \circ \psi_j^{-1} = \psi'_j \circ \psi_j^{-1}$. Therefore these isomorphisms glue to give a morphism $\alpha: X \rightarrow X'$. Applying similar logic gives another morphism $X' \rightarrow X$, and the compositions of these morphisms are the identity morphisms on X and X' , respectively, so α is an isomorphism. This map is unique by Proposition 4 because its restriction to each V_i is uniquely determined. \square

Comparison of S -morphisms is immediate from Proposition 4.

Theorem 7 (Glueing of S -morphisms of S -schemes). *Let X and Y be S -schemes, and let $\{U_i\}$ be an open cover of X . Suppose we are given S -morphisms $f_i: U_i \rightarrow Y$ for each i , such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j . Then there exists a unique S -morphism $f: X \rightarrow Y$ such that $f|_{U_i} = f_i$ for all i .*

Proof. Applying Theorem 5 gives a morphism $f: X \rightarrow Y$ such that $f|_{U_i} = f_i$ for all i . The fact that this is an S -morphism follows from Proposition 4 and the fact that the f_i are S -morphisms. \square

Theorem 8 (Glueing of S -schemes). *Let $\{X_i\}$ be a family of S -schemes. Assume that we are given, for each $i \neq j$, an open subset $U_{ij} \subseteq X_i$ and an S -isomorphism $\phi_{ij}: U_{ij} \xrightarrow{\sim} U_{ji}$, satisfying (1) $\phi_{ij} = \phi_{ji}^{-1}$ for all $i \neq j$; (2) $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ for all distinct i, j, k ; and (3) $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$ for all distinct i, j, k . Then there exists an S -scheme X , an open cover $\{V_i\}$ of X , and S -isomorphisms $\psi_i: X_i \rightarrow V_i$ for all i , such that (i) $\psi_i(U_{ij}) = V_i \cap V_j$ and (ii) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij} , for all $i \neq j$. Moreover, the data $(X, \{\psi_i\})$ are unique up to unique isomorphism.*

Proof. Forgetting about S for a moment, apply Theorem 6 to the given data. This produces a scheme X , an open cover $\{V_i\}$ of X , and isomorphisms $\psi_i: X_i \rightarrow V_i$ for all i . Then conditions (i) and (ii) and the uniqueness condition of this theorem hold; it remains only to give X the structure of an S -scheme and to check that the ψ_i are S -morphisms.

For each i let $f_i: V_i \rightarrow S$ be the morphism $g_i \circ \psi_i^{-1}$, where $g_i: X_i \rightarrow S$ is the morphism associated to X_i . Then the V_i have structures as S -schemes and the ψ_i are S -morphisms. For each $i \neq j$ we have $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$; this holds by condition (ii), condition (3), and the fact that the ϕ_{ij} are S -morphisms. We then glue the f_i to give $f: X \rightarrow S$, as desired. \square