

Math 256A. Alternative Version of Section 6 (Nonsingular Curves)

This treatment of Section 6 of the text (specifically, Lemma 6.5 through Theorem 6.9) avoids use of the “abstract nonsingular curve,” which is not used very much (if at all) in algebraic geometry.

We start with a replacement of Proposition 6.8.

Proposition. *Let $\phi: X \dashrightarrow Y$ be a rational map from a normal curve X to a projective variety Y . Then ϕ extends to a (unique) morphism $X \rightarrow Y$.*

Proof. We may assume that ϕ is dominant (replace Y with $\overline{\phi(X)}$), that $Y \subseteq \mathbb{P}^n$, and that Y is not contained in any of the coordinate hyperplanes. Let U be an open subset of X on which ϕ is represented by a morphism, and let $P \in X \setminus U$. Since X is normal, $\mathcal{O}_{P,X}$ is a dvr; let v be the associated discrete valuation on $K(X)$. We have that $1 = \phi^*(x_0/x_0), \phi^*(x_1/x_0), \dots, \phi^*(x_n/x_0)$ are nonzero elements of $K(X)$; pick i such that $v(\phi^*(x_i/x_0))$ is minimal. Then, for all j ,

$$v(\phi^*(x_j/x_i)) = v(\phi^*(x_j/x_0)) - v(\phi^*(x_i/x_0)) \geq 0,$$

so $\phi^*(x_j/x_i)$ lies in $\mathcal{O}_{P,X}$ for all j . Let V be an open neighborhood of P in X such that $\phi^*(x_j/x_i)$ is regular on V for all j . Letting $U_i \subseteq \mathbb{P}^n$ denote the open affine $\{x_i \neq 0\}$, we have that the image of the composition

$$A(Y \cap U_i) \hookrightarrow K(Y) \xrightarrow{\phi^*} K(X)$$

is contained in $\mathcal{O}(V)$, so ϕ^* determines a morphism $\phi': V \rightarrow Y$ that induces the same map $K(Y) \rightarrow K(X)$ as ϕ . Therefore ϕ' and ϕ are the same rational map, and therefore V is contained in the maximal domain of ϕ . In this way, ϕ can be extended to a morphism $X \rightarrow Y$. \square

Lemma. *Let W be a variety. Then its normalization map $\pi: \widetilde{W} \rightarrow W$ is surjective.*

Proof. Let $P \in W$. By the *lying-over theorem* (Atiyah-MacDonald 5.10 or Eisenbud 4.15), since $A(\widetilde{W})$ is integral over $A(W)$, it contains a prime ideal whose restriction to $A(W)$ is the maximal (hence prime) ideal $I(P)$. Replacing this prime with a maximal ideal containing it, if necessary, we have a maximal ideal \mathfrak{a} for which $\mathfrak{a} \cap A(W) = I(P)$. This maximal ideal corresponds to a point lying over P . \square

Theorem. *Every curve is birational to a nonsingular projective curve. Moreover, if C is a curve and $\phi: C \dashrightarrow \widetilde{C}$ is a birational map to a nonsingular projective curve \widetilde{C} , then ϕ is represented by a morphism on the (open) set of nonsingular points of C .*

Proof. Let C be a curve. We may assume that it is quasi-projective. It is birational to its projective closure, so we may assume that C is projective. Let $\{U_1, \dots, U_m\}$ be a covering of C by open affines. For each i let $\pi_i: V_i \rightarrow U_i$ be the normalization (Ex.

3.17). This map is birational (since $A(U_i) \subseteq A(V_i) \subseteq K(U_i)$), so there are nonempty open subsets $U'_i \subseteq U_i$ and $V'_i \subseteq V_i$ such that π_i induces an isomorphism $V'_i \xrightarrow{\sim} U'_i$.

Each V_i is affine, say V is closed in \mathbb{A}^{n_i} for all i . Let Y_i be its projective closure in \mathbb{P}^{n_i} . We have a rational map

$$\theta: C \dashrightarrow Y_1 \times \cdots \times Y_m \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m},$$

represented by a morphism on the set $U := \bigcap U'_i$. Let Y be the closure of the graph. Then Y is a projective curve, and we have morphisms $\phi: Y \rightarrow C$ and $\psi_i: Y \rightarrow Y_i$ for all i . As noted earlier, ϕ is birational, and is an isomorphism over U . In particular, Y is birational to C .

In addition, $Y \cap (U \times \prod \pi_i^{-1}(U))$ is the (closed) graph of the morphism

$$(\pi_1^{-1}, \dots, \pi_m^{-1}): U \rightarrow \prod Y_i,$$

so this nonempty open subset of Y is isomorphic to U and to $\pi_i^{-1}(U)$ for all i ; hence all ψ_i are also birational. Moreover, we have commutative diagrams

$$\begin{array}{ccc} Y \cap (U \times \prod \pi_i^{-1}(U)) & \xrightarrow{\psi_i} & V'_i & & K(Y) & \longleftarrow & K(V_i) \\ \phi \downarrow & & \downarrow \pi_i & \text{and} & \uparrow & & \uparrow \\ U & \hookrightarrow & U'_i & & K(U) & = & K(U_i) \end{array}$$

in which the maps on the left induce the maps on the right.

It remains only to show that Y is nonsingular.

Pick $P \in Y$, and choose i such that $\phi(P) \in U_i$. Let W be an open affine neighborhood of P in $\phi^{-1}(U_i)$, and let $\alpha: \widetilde{W} \rightarrow W$ be its normalization. By the lemma, there is a point $\widetilde{P} \in \widetilde{W}$ such that $\alpha(\widetilde{P}) = P$. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\beta} & V_i \\ \downarrow \alpha & & \downarrow \pi_i \\ W & \xrightarrow{\phi} & U_i, \end{array}$$

in which β exists because \widetilde{W} is normal, $\phi \circ \alpha$ is dominant, and π_i is the normalization. However, β coincides with $\psi_i \circ \alpha$ on some nonempty open set, since they are both associated to the same map of function fields. Therefore $\beta = \psi_i \circ \alpha$, so in particular $\psi_i(P) = \psi_i(\alpha(\widetilde{P})) = \beta(\widetilde{P})$ lies in V_i .

Therefore the local ring $\mathcal{O}_{P,Y}$ dominates $\mathcal{O}_{\psi_i(P),V_i}$. But the latter is normal, hence a dvr (by Theorem 6.2A), hence a valuation ring. By maximality of valuation rings with respect to dominance (Theorem 6.1A), $\mathcal{O}_{P,Y}$ equals $\mathcal{O}_{\psi_i(P),V_i}$; in particular it is regular (by 6.2A again).

This implies the first assertion of the theorem. The second assertion follows immediately from the proposition. \square