Math 185. Sample Answers to Problem Set #4

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1b. If $u(x,y) = 2x - x^3 + 3xy^2$, then $u_x = 2 - 3x^2 + 3y^2$, $u_{xx} = -6x$, $u_y = 6xy$, and $u_{yy} = 6x$, so $u_{xx}+u_{yy} = 0$. All of these partials are continuous (as is $u_{xy} = u_{yx} = 6x$), so $u(x,y)$ is harmonic.

To get a harmonic conjugate v , we have $v_y = u_x = 2 - 3x^2 + 3y^2$, so

$$v(x,y) = 2y - 3x^2y + y^3 + \phi(x)$$

on any open ball in the given domain. (It does not hold in general, since the intersection of the domain with a line $y =$ constant need not be connected.) Also, the equation $v_x = -u_y = -6xy$ gives $-6xy + \phi'(x) = -6xy$, so we may take $\phi(x) = 0$ and therefore $v(x,y) = 2y - 3x^2y + y^3$ is a harmonic conjugate of u in any open ball in the domain. It is then easy to check that this is a harmonic conjugate over the whole domain.

2. If v and V are both harmonic conjugates of the same function u , then $v_x = -u_y = V_x$, so $(\partial/\partial x)(v - V) = 0$ everywhere on D . Also $v_y = u_x = V_y$, so $(\partial/\partial y)(v - V)$ also vanishes everywhere on D . Therefore, as in the proof of Theorem 74, $v - V$ is constant on D .

Alternatively, we can note that $v - V$ is a harmonic conjugate of the zero function, so by Theorem 2 on page 80, $(v - V)i$ is analytic on D , and therefore so is $v - V$. By Exercise 7 on page 78, this function must be constant.

5. Taking partials of $ru_r = v_\theta$ with respect to r and of $rv_r = -u_\theta$ with respect to θ gives

$$u_r + ru_{rr} = v_{\theta r} \qquad \text{and} \qquad rv_{r\theta} = -u_{\theta\theta} \; .$$

Therefore, using also equality of the mixed partials, we have

$$r^2u_{rr} + ru_r = rv_{\theta r} = rv_{r\theta} = -u_{\theta\theta} \; .$$

Similarly, taking partials of $rv_r = -u_\theta$ with respect to r and of $ru_r = v_\theta$ with respect to θ gives

$$v_r + rv_{rr} = -u_{\theta r} \qquad \text{and} \qquad ru_{r\theta} = v_{\theta\theta} \; .$$

Therefore we have

$$r^2v_{rr} + rv_r = -ru_{\theta r} = -ru_{r\theta} = -v_{\theta\theta} \; .$$

7. If y can be written as a function of x on the curve $u(x,y)=c_1$ near (x_0,y_0), then by implicit differentiation we obtain $u_x+u_y y'=0$, so $y'=-u_x/u_y$. This condition is true whenever $u_y(x_0,y_0)\neq 0$, since then the gradient has nonzero vertical component, so the level curve is not vertical there (strictly speaking, this follows from the implicit function theorem).

Similarly, we obtain that if y can be written as a function of x on the curve $v(x,y)=c_2$ near (x_0,y_0), then implicit differentiation gives $y'=-v_x/v_y$. Again, this condition is satisfied if $v_y(x_0,y_0)\neq 0$.

These two slopes are negative reciprocals of each other, because their product is

$$\left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right)=\left(-\frac{u_x}{u_y}\right)\left(-\frac{-u_y}{u_x}\right)=-1$$

by the Cauchy-Riemann equations.

If $u_y(x_0,y_0)=0$, then the gradient is horizontal, so the level curve $u(x,y)=c_1$ is vertical. By Cauchy-Riemann we also have $v_x(x_0,y_0)=0$, and $f'(z_0)\neq 0$ implies $u_x(x_0,y_0)\neq 0$, hence $v_y(x_0,y_0)\neq 0$, so the level curve $v(x,y)=c_2$ has slope $-v_x(x_0,y_0)/v_x(x_0,y_0)=0$ at (x_0,y_0). Therefore the two level curves are perpendicular.

Similarly, if $u_x(x_0,y_0)=0$ then the level curves $u(x,y)=c_1$ and $v(x,y)=c_2$ are horizontal and vertical, respectively, so again they are perpendicular.

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1a. $\exp(2\pm 3\pi i)=e^2(\cos(\pm 3\pi)+i\sin(\pm 3\pi))=e^2(-1+0i)=-e^2$ (since $\cos(\pm 3\pi)=\cos\pi$ $=-1$ and similarly for $\sin(\pm 3\pi)$).

3. Since

$$\exp\bar{z}=e^x e^{-iy}=e^x\cos y-ie^x\sin y\ ,$$

we have $u(x,y)=e^x\cos y$ and $v(x,y)=-e^x\sin y$. Therefore

$$u_x=e^x\cos y,\qquad u_y=-e^x\sin y,\qquad v_x=-e^x\sin y,\qquad \text{and}\qquad v_y=-e^x\cos y\ .$$

The first Cauchy-Riemann equation $u_x=v_y$ holds if and only if $e^x\cos y=0$, which holds if and only if $\cos y=0$. The second Cauchy-Riemann equation $u_y=-v_x$ holds if and only if $e^x\sin y=0$, which holds if and only if $\sin y=0$. Since $\sin^2 y+\cos^2 y=1$, we never have $\sin y=\cos y=0$, so there is no point where $\exp\bar{z}$ is analytic.

6. By (7), we have

$$|\exp(z^2)|=\exp(\mathrm{Re}(z^2))=\exp(x^2-y^2)\leq\exp(x^2+y^2)=\exp(|z|^2)\ .$$

8a. By (7) on page 91, $e^x=|e^z|=2$, so $x=\ln 2$, and $y=\arg(e^z)+2\pi n=\pi+2\pi n$ ($n\in\mathbb{Z}$), so

$$z=\ln 2+(\pi+2\pi n)i=\ln 2+(2n+1)\pi i\ ,\qquad n\in\mathbb{Z}\ .$$

Conversely, if z is equal to one of these, then $e^z=-2$.

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1a. We have $\ln|-ei| = \ln e = 1$ and $\mathrm{Arg}(-ei) = \mathrm{Arg}(-i) = -\pi/2$, so since $-\pi/2$ falls in the range given by (6) on page 96, we have

$$\mathrm{Log}(-ei) = 1 - \frac{\pi}{2}i\ .$$

5. **a**. Since $((1+i)/\sqrt{2})^2 = i$, we have

$$i^{1/2} = \pm\frac{1+i}{\sqrt{2}} = \{e^{\pi i/4}, e^{-3\pi i/4}\}\ ;$$

therefore

$$\begin{aligned}\log(i^{1/2}) &= \left\{\left(2k+\frac{1}{4}\right)\pi i : k\in\mathbb{Z}\right\} \cup \left\{\left(2k-\frac{3}{4}\right)\pi i : k\in\mathbb{Z}\right\} \\ &= \left\{\left(n+\frac{1}{4}\right)\pi i : n\in\mathbb{Z}\right\}\ .\end{aligned}$$

The same is true of $(1/2)\log i$ by Exercise 2(b).

b. The set of values of $\log(i^2) = \log(-1)$ is

$$\{(2n+1)\pi i : n\in\mathbb{Z}\}$$

by Example 3 on page 94. On the other hand, by Exercise 2b, the set of values of $2\log i$ is

$$2\left\{\left(2n+\frac{1}{2}\right)\pi i : n\in\mathbb{Z}\right\} = \{(4n+1)\pi i : n\in\mathbb{Z}\}\ ,$$

which is not the same as for $\log(i^2)$, since for example $3\pi i \in \log(i^2)$ but $3\pi i \notin 2\log i$.

7. By (3) on page 93, taking $\exp$ of both sides of the equation $\log z = i\pi/2$ gives

$$z = e^{\log z} = e^{i\pi/2} = i\ .$$

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2. For the real parts, we have clearly that $|z_1 z_2| = |z_1||z_2|$, so

$$\mathrm{Re}\,\mathrm{Log}(z_1z_2) = \ln|z_1z_2| = \ln|z_1| + \ln|z_2| = \mathrm{Re}\,\mathrm{Log}\,z_1 + \mathrm{Re}\,\mathrm{Log}\,z_2\ .$$

For the imaginary parts, since $\mathrm{Im}\,\mathrm{Log}\,z_1 = \mathrm{Arg}\,z_1$ and similarly for z_2 and z_1z_2, all three values $\mathrm{Im}\,\mathrm{Log}\,z_1$, $\mathrm{Im}\,\mathrm{Log}\,z_2$, and $\mathrm{Im}\,\mathrm{Log}(z_1z_2)$ lie in the interval $(-\pi,\pi]$. Therefore,

$$\mathrm{Im}\,\mathrm{Log}(z_1z_2) - \mathrm{Im}\,\mathrm{Log}\,z_1 - \mathrm{Im}\,\mathrm{Log}\,z_2 \in (-3\pi, 3\pi)\ .$$

This quantity must also be an integer multiple of 2π, so it equals $2N\pi$ for some $N\in\{0,\pm1\}$.

Putting these together then gives that

$$\mathrm{Log}(z_1z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2 + 2N\pi i$$

with $N\in\{0,\pm1\}$.

6. By (1) on page 25, we have

$$z^{1/n} = \left\{\exp\left(\frac{1}{n}\ln r + i\left(\frac{\Theta}{n} + \frac{2\pi k}{n}\right)\right) : k = 0, 1, \ldots, n-1\right\}$$

and therefore all the values of $\log(z^{1/n})$ are given by

$$\log z^{1/n} = \frac{1}{n}\ln r + i\frac{\Theta + 2\pi k}{n} + 2\pi p = \frac{1}{n}\ln r + i\frac{\Theta + 2\pi(pn+k)}{n}$$

for $k = 0, 1, \ldots, n-1$ and $p \in \mathbb{Z}$.

On the other hand, dividing (2) on page 95 by n gives

$$\frac{1}{n}\log z = \frac{1}{n}\ln r + i\frac{\Theta + 2\pi q}{n}$$

with q ranging over $\mathbb{Z}$.

Now, given $w = (1/n)\ln r + i(\Theta + 2\pi(pn+k))/n \in \log z^{1/n}$, we can let $q = pn + k$ and see that $w = (1/n)\ln r + i(\Theta + 2\pi q)/n$ with $q \in \mathbb{Z}$; hence $w \in (1/n)\log z$. Conversely given $w = (1/n)\ln r + i(\Theta + 2\pi q) \in (1/n)\log z$ for some $q \in \mathbb{Z}$, we can write $q = pn + k$ for some $p \in \mathbb{Z}$ and $k \in \{0, 1, \ldots, n-1\}$ (as in the suggestion), and then see that $w = (1/n)\ln r + i(\Theta + 2\pi(pn+k))/n \in \log z^{1/n}$ lies in $\log z^{1/n}$. Thus we have shown the following equality of sets:

$$\log z^{1/n} = \frac{1}{n}\log z\,.$$