

## Math 185. Sample Answers to Problem Set #12

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1. Suppose that  $f$  does have a constant value  $w_0$  in some nonempty open subset of  $D$ . Then the function  $f(z) - w_0$  is analytic on  $D$  and vanishes on a domain contained in  $D$ . By the lemma on page 83,  $f(z) - w_0$  must vanish everywhere on  $D$ , so  $f(z)$  has the constant value  $w_0$  on  $D$ . This proves the contrapositive of the statement that the problem asks for.

4. If  $x + iy \in \mathbb{R}$ , then  $y = 0$ , so since  $e^{i0} = \cos 0 + i \sin 0 = 1$ , we have  $f(z) = e^x \cdot 1 \in \mathbb{R}$ . Thus the reflection principle applies, and we have  $\overline{f(z)} = f(\bar{z})$  for all  $z \in \mathbb{C}$ .

To see this directly, we have

$$\begin{aligned}\overline{f(z)} &= \overline{e^x(\cos y + i \sin y)} = e^x(\cos y - i \sin y) \\ &= e^x(\cos(-y) + i \sin(-y)) = e^x e^{-iy} = f(\bar{z}).\end{aligned}$$

5. Let  $D$  be a domain as in the reflection principle, and let  $f(z)$  be an analytic function on  $D$ . Let  $g(z) = if(z)$ . Then  $g(z)$  is analytic on  $D$ . Also,  $f(x)$  is purely imaginary for each  $x \in D \cap \mathbb{R}$  if and only if  $g(x)$  is real for each such  $x$ , if and only if  $\overline{g(z)} = g(\bar{z})$  for all  $z \in D$ . This holds if and only if  $\overline{if(z)} = if(\bar{z})$ , which in turn is equivalent to  $\overline{f(z)} = -f(\bar{z})$  for all  $z \in D$ .

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1. The function  $1/(z^2 + 1)$  is even, is defined everywhere on the real axis, and has only a pole at  $z = i$  in the upper half plane. By Theorem 2 on page 253, the residue there is  $1/2i$ . For all  $z$  in the semicircle  $C_R$  (defined on pages 263–264), we have

$$\left| \frac{1}{z^2 + 1} \right| \leq \frac{1}{R^2 - 1},$$

so

$$\left| \int_{C_R} \frac{dz}{z^2 + 1} \right| \leq \pi R \cdot \frac{1}{R^2 - 1} = \frac{\pi R}{R^2 - 1}$$

for all  $R > 1$ . This expression goes to 0 as  $R \rightarrow \infty$ , so by the method of Section 78, we have

$$\int_0^\infty \frac{dx}{x^2 + 1} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{x^2 + 1} = \frac{1}{2} \cdot 2\pi i \cdot \frac{1}{2i} = \frac{\pi}{2}.$$

5. We first compute some residues. Let  $f(z) = \frac{z^2}{(z^2 + 9)(z^2 + 4)^2}$ .

By Theorem 2 on page 253 (with  $p(z) = z^2/(z^2 + 4)^2$ ), we have

$$\operatorname{Res}_{z=3i} f(z) = \frac{(3i)^2}{2(3i)((3i)^2 + 4)^2} = \frac{-9}{6i(-5)^2} = \frac{3i}{50}.$$

By the theorem on page 244, we have

$$\begin{aligned} \operatorname{Res}_{z=2i} f(z) &= \left. \frac{d}{dz} \left( \frac{z^2}{(z^2 + 9)(z + 2i)^2} \right) \right|_{z=2i} \\ &= \left. \frac{2z(z^2 + 9)(z + 2i)^2 - z^2(2z(z + 2i)^2 + 2(z^2 + 9)(z + 2i))}{(z^2 + 9)^2(z + 2i)^4} \right|_{z=2i} \\ &= \frac{4i(5)(-16) - (-4)(4i(-16) + 2(5)(4i))}{5^2 \cdot 16^2} \\ &= \frac{4i(-80 - 64 + 40)}{80^2} \\ &= -\frac{13i}{200}. \end{aligned}$$

If  $R > 3$ , then for all  $z \in C_R$  (defined on page 263–264), we have

$$|f(z)| \leq \frac{R^2}{(R^2 - 9)(R^2 - 4)^2},$$

so letting  $R \rightarrow \infty$  in (3) on page 266, we have  $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$ . Thus

$$\begin{aligned} \int_0^\infty \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2} &= \frac{1}{2} \int_{-\infty}^\infty \frac{x^2 dx}{(x^2 + 9)(x^2 + 4)^2} = \frac{1}{2} \cdot 2\pi i \left( \operatorname{Res}_{z=2i} f(z) + \operatorname{Res}_{z=3i} f(z) \right) \\ &= \pi i \left( \frac{3i}{50} - \frac{13i}{200} \right) = -\pi \left( \frac{12 - 13}{200} \right) = \frac{\pi}{200}. \end{aligned}$$

6. For  $R$  large ( $R > 3$  suffices) we have

$$\left| \frac{1}{z^2 + 2z + 2} \right| \leq \frac{1}{R^2 - 2R - 2}$$

for all  $z \in C_R$ , where  $C_R$  is as defined on page 263–264. As usual, we have

$$\int_{-R}^R \frac{dx}{x^2 + 2x + 2} + \int_{C_R} \frac{dz}{z^2 + 2z + 2} = 2\pi i \operatorname{Res}_{z=-1+i} \frac{1}{z^2 + 2z + 2},$$

we can take the limit as  $R \rightarrow \infty$ . The integral over  $C_R$  drops out, leaving

$$\text{P.V.} \int_{-\infty}^\infty \frac{dx}{x^2 + 2x + 2} = 2\pi i \operatorname{Res}_{z=-1+i} \frac{1}{z^2 + 2z + 2} = \frac{2\pi i}{2(-1+i) + 2} = \pi.$$

(Note that  $f(x)$  does not need to be even for (10) on page 264 to hold.)

8. If  $R > 1$  then the integrand is bounded in absolute value by  $1/(R^3 - 1)$  along the circular part of the given contour, so the integral along that circular part is bounded in absolute value by

$$\frac{2\pi R}{3} \cdot \frac{1}{R^3 - 1} = \frac{2\pi R}{3(R^3 - 1)}.$$

This bound goes to zero when  $R \rightarrow \infty$ , so we have

$$\lim_{R \rightarrow \infty} \left( \int_0^R \frac{dx}{x^3 + 1} - \int_0^R \frac{e^{2\pi i/3} dx}{(e^{2\pi i/3} x)^3 + 1} \right) = 2\pi i \operatorname{Res}_{z=(1+\sqrt{3}i)/2} \frac{1}{z^3 + 1} = \frac{2\pi i}{3\left(\frac{1+\sqrt{3}i}{2}\right)^2}.$$

On the other hand, we have

$$\begin{aligned} \lim_{R \rightarrow \infty} \left( \int_0^R \frac{dx}{x^3 + 1} - \int_0^R \frac{e^{2\pi i/3} dx}{(e^{2\pi i/3} x)^3 + 1} \right) &= \lim_{R \rightarrow \infty} (1 - e^{2\pi i/3}) \int_0^R \frac{dx}{x^3 + 1} \\ &= \left( \frac{3 - \sqrt{3}i}{2} \right) \int_0^\infty \frac{dx}{x^3 + 1}. \end{aligned}$$

Combining these two equations gives

$$\int_0^\infty \frac{dx}{x^3 + 1} = \frac{2\pi i}{3\left(\frac{1+\sqrt{3}i}{2}\right)^2 \left(\frac{3-\sqrt{3}i}{2}\right)} = \frac{2\pi i}{3\left(\frac{-1+\sqrt{3}i}{2}\right)\left(\frac{3-\sqrt{3}i}{2}\right)} = \frac{2\pi i}{3\sqrt{3}i} = \frac{2\pi}{3\sqrt{3}}.$$