MSRI, subfactors, knots and the development of quantum topology.

Vaughan Jones

25th Anniversary celebration of MSRI, Jan 2008

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And a trace tr on the algebra generated by the e_i's uniquely defined by

$$tr(we_{n+1}) = \tau tr(w) \quad \text{if } w \text{ is a word on } e_1, e_2, \ldots, e_n.$$

Restrictions on τ

For infinitely many e_i's to exist as above, $\frac{1}{\tau}$ must be either ≥ 4 or one of the numbers

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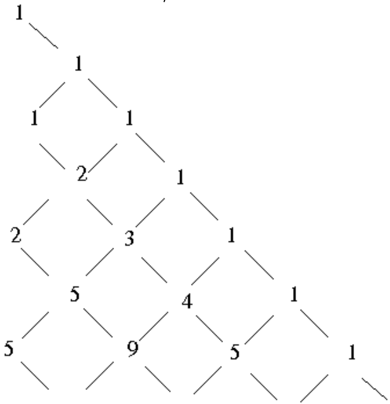
Representation theory of the e_i's.
Given by the following diagram:

For $0 < \tau < 1/4$:



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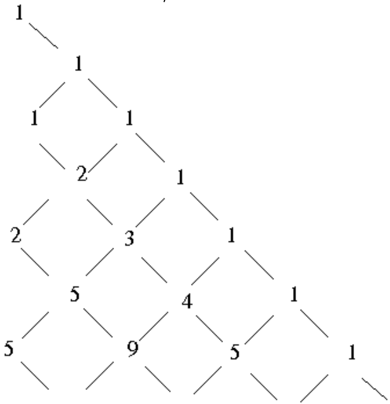
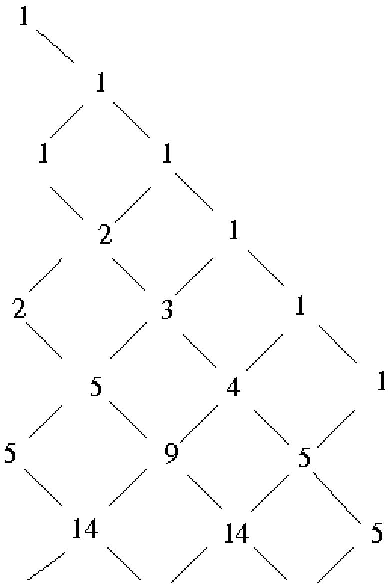


or for $1/\tau = 4\cos^2 \pi/k$:



(here $1/\tau = 4\cos^2 \pi/7$)

Braid group: on generators $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$,

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With $\tau = \dfrac{t}{(1+t)^2}$. If e_i were diagonalised, σ_i look like:

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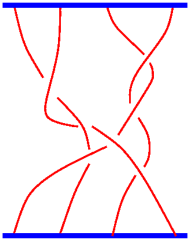

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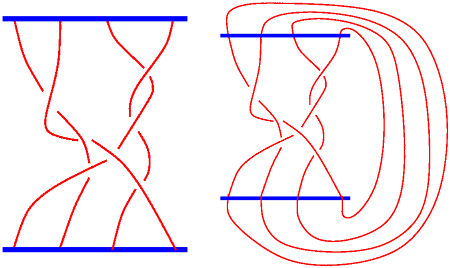

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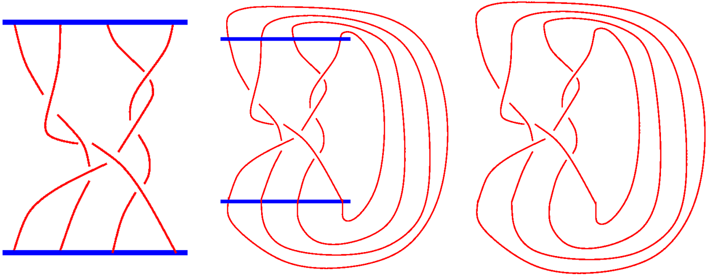

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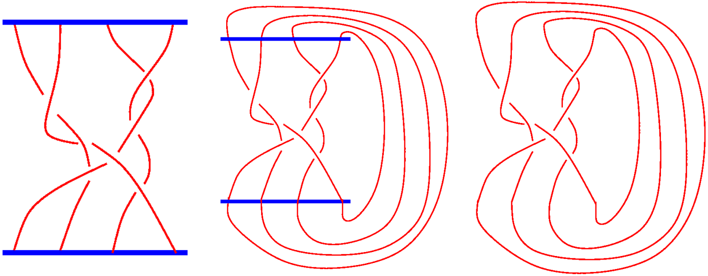

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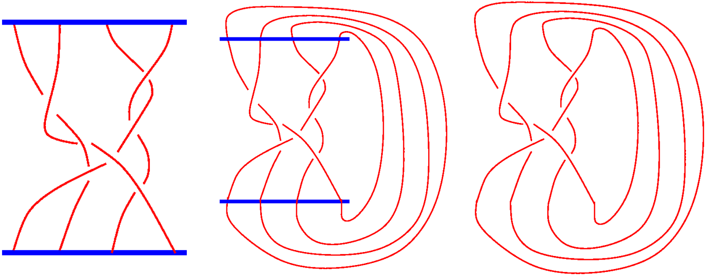

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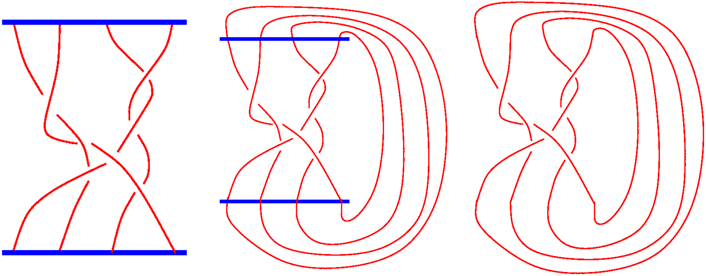

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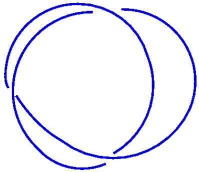
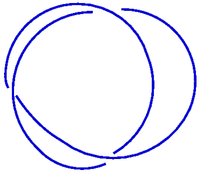
For the trefoil knot $K =$ 

For the trefoil knot $K =$ 

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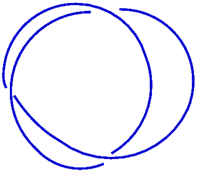
For the trefoil knot $K =$ 

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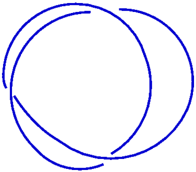
For the trefoil knot $K =$ 

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2)Several applications including braid index estimates and answers to some questions about Conway's "skein theory".
3) Observation from stat mechanical models that $V_L(t)$ is essentially independent of the orientation of L. Followed within a few weeks by Kauffman's great diagrammatic insights:
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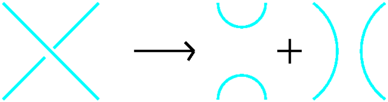

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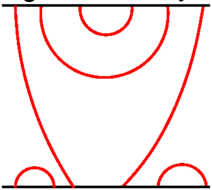

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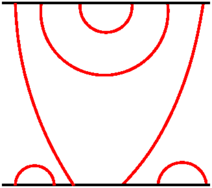

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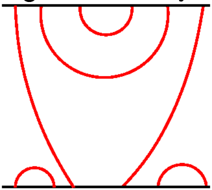

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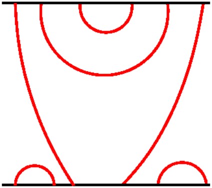

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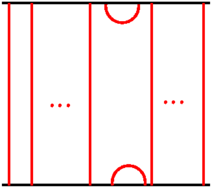

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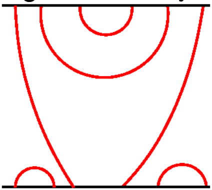

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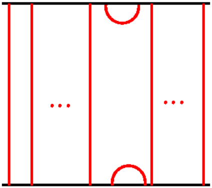

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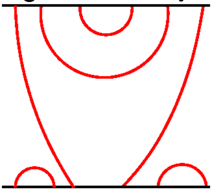

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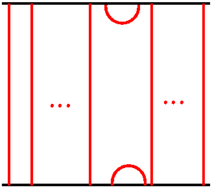

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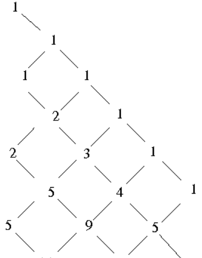

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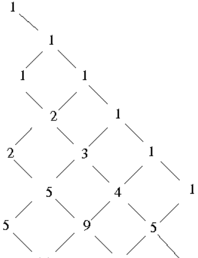

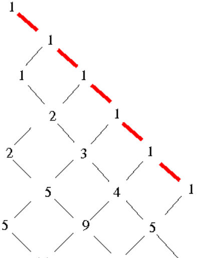

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ii) A topological interpretation was needed for $V_L(t)$ and the other polynomials.

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Organisational period. Jimbo,Woronowicz, Drinfeld - quantum group.
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Roots of unity:
Tricky in quantum group world but here the subfactor picture reigns supreme (Wenzl, Feng Xu) as all the non semi-simple clutter is pared away by positivity.

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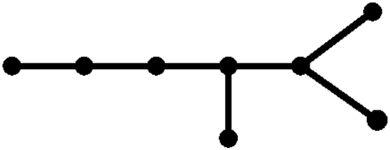

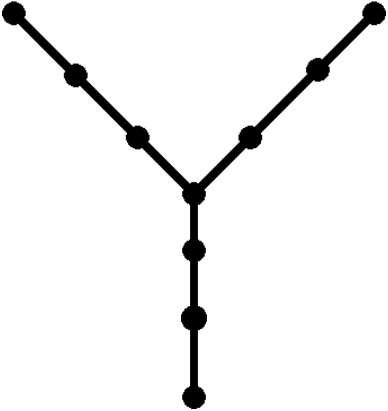

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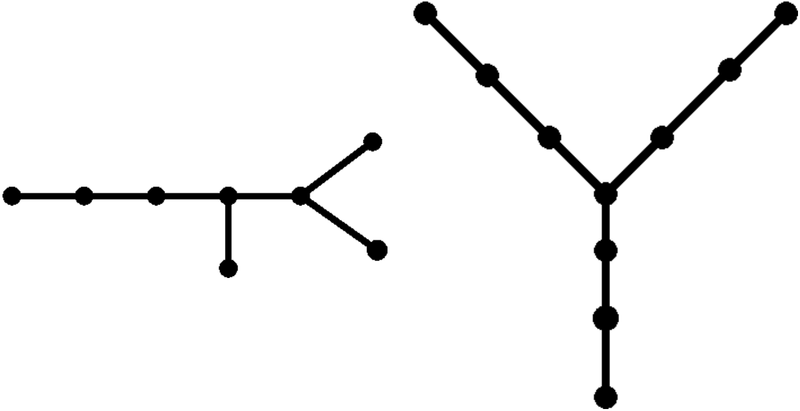

These objects remain "exotic" creatures in the zoo.

The 600 pound gorilla: CONFORMAL FIELD THEORY
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Even more murky, Fredenhagen, Rehren, Schroer, and Longo, suggesting that braid groups and subfactors should crop up in general low dimensional QFT in the Haag-Kastler algebraic QFT framework following Dopplicher, Haag, Roberts.

1988: Witten- TQFT (Atiyah)
An interpretation of $V_L(t)$, at $t = e^{2\pi i/k}$, as a functional integral in a 2+1 dimensional gauge theory, gauge group SU(2) (as expected), with Chern -Simons action, the knot invariant being the expected value of the trace of the monodromy along the components of the link which now became known as "Wilson loops".

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EXPLICIT formula for links in an arbitrary 3-manifold via surgery on links or Heegard splitting!
Complete verificaiton by Reshetikhin-Turaev.
Turaev-Viro Ocneanu-3 manifold invariants from general categorical data as provided for instance by a subfactor.

MSRI meeting January 1989. Lots of key actors present, Atiyah, Bott, Witten, Jimbo, Miwa, Kauffman, Faddeev, Wassermann.... Greatly clarified the situation. Finally got over the feeling that the Russians and the Japanese knew something that we didn't.

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Subfactor outcome: Wassermann's construction of subfactors from loop group representations:

take positive energy projective representations of $LSU(2)$, parametrized by a central extension (level) and an irrep of $SU(2)$, then if I is an interval in the circle with I^c the complementary interval,

$$L_I SU(2)'' \subseteq L_{I^c} SU(2)'$$

is a subfactor realising the indices $4\cos^2 \pi/(level)$ (more generally $\frac{\sin^2(k\pi/\ell)}{sin^2 \pi/\ell}$).

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And in fact this is an instance of the Fredenhagen-Rehren-Schroer theory, with braiding as predicted!
The main technical and conceptual tool used by Wassermann is the Connes tensor product of bimodules over a von Neumann algebra, which in turn relies on the Tomita-Takesaki theory.

Turning again to topology, a new ingredient entered the game in the early 1990's - Vassiliev theory.
Thinking of knots as smooth functions from S^1 to $\mathbb{R}^3$, Vassiliev began the study of the algebraic topology of this space and obtained new knot invariants. Birman and Lin furthered their study and the concept of "finite type invariant" emerged. The idea is to extend any additive knot invariant *Inv* to an invariant of immersed curves with at worst double point singularities by locally setting

Inv  $= Inv$  $- Inv$ 

Then *Inv* is of finite type n if it vanishes on immersed curves with n double points.
The knot polynomials are a source of finite type invariants but it was shown by Vogel that there are others. Kontsevich discovered a beautiful general integral formula for finite type invariants, thinking of them more as a perturbative expansion of a functional integral.

Bar-Natan put everything together in a stunningly comprehensible paper which made the whole theory broadly accessible and showed that one could construct knot invariants by studying "chord diagrams" modulo certain linear relations.
There emerged a basic relationship with (not necessarily simple) Lie algebras and understanding of how these invariants fit together, even for the simplest knots, gives information about Lie algebras. (Bar-Natan, Le, Thurston-Alexeev.) Vogel showed that not all finite type invariants come from Lie algebras.
All this has to a certain extent been extended to knots and links in arbitrary 3-manifolds though I believe that, beyond homology 3-spheres, things remain a little murky.
Another major problem in this area is to understand how the Witten-Reshetikhin-Turaev invariant depends on the root of unity. In the best of all worlds there would be a holomorphic function of q of which it is the boundary value but this is appears to be too naive. Recent progress by Garoufalidis.

There are various conjectures concerning the asymptotic behaviour of the WRT invariants and how they relate to other geometric or topological invariants. Let me mention Kashaev's volume conjecture (just for knots). If K is a knot with hyperbolic complement, one looks at

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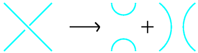
Then as $k \to \infty$, the asymptotic growth of this number is controlled by the hyperbolic volume.

Result is known for some small knots by explicit computations. The experts are not entirely in agreement as to whether it should be true.

Other asymptotic expansions proved recently by J. Anderson.

Marcos Marino, Aganagic, Klemm, Vafa and others made connections with Calabi-Yau three-folds via matrix models.

A more recent development has been Khovanov homology. This uses the basic Kauffman resolution  of a crossing but "categorifies" it in that the basic invariant associated to a link is a graded complex and the relation between the complexes at a crossing is an exact sequence between the three complexes. The invariant ends up, at least in the hands of Bar-Natan, being a complex up to homotopy equivalence. The Euler characteristic of the complex is $V_L(t)$. Bar-Natan's version is highly calculable by a "planar algebra" approach.

Khovanov and others have extended categorification to many other knot invariants and beyond.

Khovanov homology has yielded proofs of topological results (Rasmussen), it is strictly more powerful than the corresponding Euler Characteristic polynomials, and it looks like it is the arena for connections with gauge theory, e.g. Floer homology.

What about subfactors?
The 90's saw deep classification results by Sorin Popa who showed that in "amenable" cases (both the factor, the subfactor and the principal graph), the subfactor is completely classified by what is known as the "standard invariant" (an enrichment of the principal graphs containing not just the bimodules and their tensor powers but intertwiners between them). These results extend Connes' breakthrough results from the 70's classifying automorphisms and group actions.

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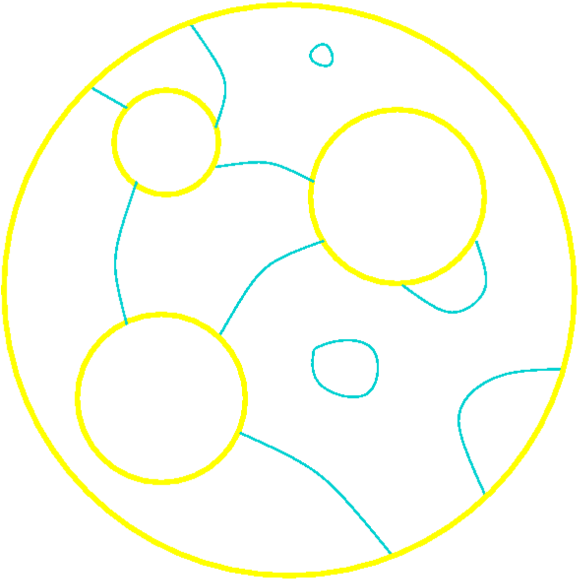
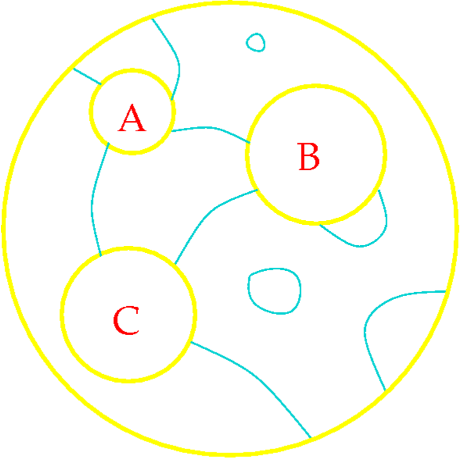
An attempt to use subfactors to provide information on a huge variety of combinatorial problems. Difficulty: Compute the principal graph.

May eventually prove fruitful though no doubt less universal than hoped, but one thing it did lead to was the development of *planar algebras* which provided a very convenient axiomatisation of the standard invariant of a subfactor and a useful technique for analysing their structure. Similar in spirit to Conway's "skein theory".

A tangle:



Given A, B, C in the standard invariant,



$= D$

Gives D also in the standard invariant. Recent interaction with random matrices and free probability (Voiculescu). (with Guionnet and Shlyakhtenko.)

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The idea is to spread out the quantum system in some 2-dimensional quantum fluid.
The qbits are then replaced by quasiparticles. According to FRS, if one carries out a motion in such a fluid, returning n points to their original position, the Hilbert space should evolve according to a braid group representation. The ones described at the beginning of this talk are precisely those one would expect for such a system. One could then encode a calculation as a braid.

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Freedman's Navier Stokes on Mars analogy.