The Jones polynomial for dummies.

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February 12, 2014

[2] supported by NSF under Grant No. DMS-XYZ

Abstract

Acknowledgements.

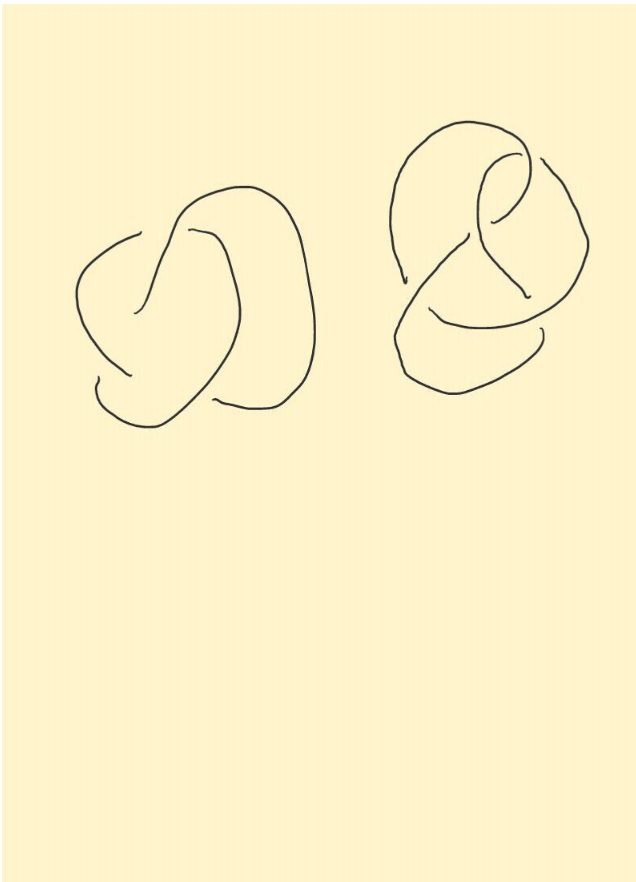
1 Introduction.

This is a series of 8 lectures designed to introduce someone with a certain amount of mathematical knowledge to the Jones polynomial of knots and links in 3 dimensions. The amount of mathematical knowledge required will increase from high school mathematics in the first two lectures to at least graduate student level in the last lecture, which will be a survey of developments of the Jones polynomial. A particular aim of the course will be to obtain the Jones polynomial for torus knots. This will require a little representation theory. In general proofs will be discussed rather than given as it is easy to find proofs with the help of google.

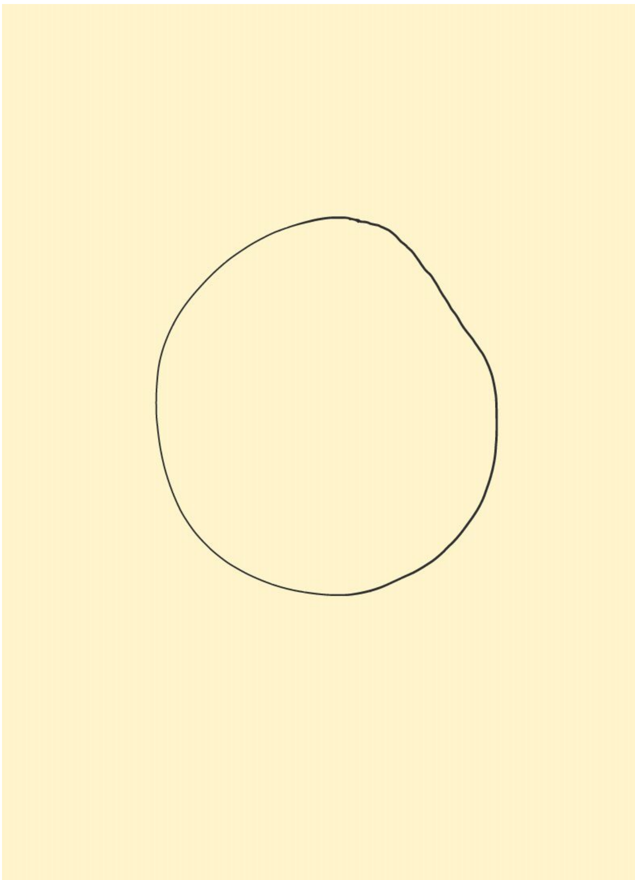
2 Knots, links braids and tangles.

A knot is a smooth closed curve in three dimensional space $\mathbb{R}^3$. As such it is an object of topology, two knots are "the same" if one can be obtained by the other by smooth deformations of $\mathbb{R}^3$. A link is a disjoint union of several knots.

Two fortunate features make the theory of knots particularly accessible and allow us to actually forget the analysis underlying the word "smooth". The first is that knots are perfectly adequately modeled by bits of string and the "smooth deformations" are modeled by manual manipulation of the strings (without ever cutting them). Thus the question "is this knot the same as that knot" can be approached quite experimentally by tying the knots and seeing if the first can be manipulated to look like the other. One immediately meets the necessity of mathematical proof - how can one be sure that a little more manipulation would not have turned the first knot (the "trefoil") below into the second (the "figure 8"):



or indeed that either of them can be converted into the "unknot":

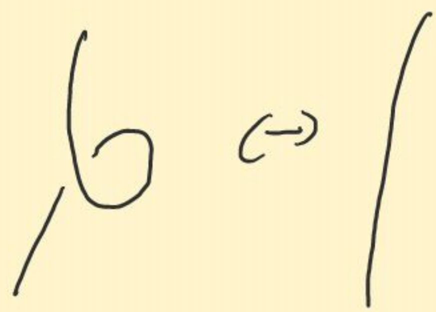


Whatever sense one may attach to the word "topology", knot theory fits into topology and one should search for ways of distinguishing knots from topology.

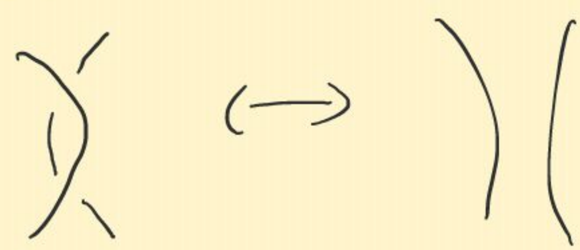
Moreover the second fortunate feature of knot theory is now visible: although knots are inherently 3 dimensional, they can be faithfully represented by two dimensional pictures such as the ones we have seen above, all the 3 dimensionality being reduced to whether the crossings in the picture are over or under. And the fundamental question of knot theory becomes: when do two pictures of knots represent the same knot?

This question was answered in the early twentieth century by Reidemeister. There are three "Reidemeister moves" which act locally on pictures, only changing that part included in the move. We draw the Reidemeister moves below:

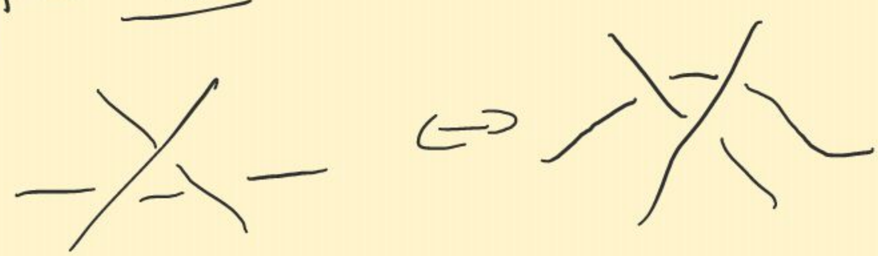
Type I



Type II



Type III

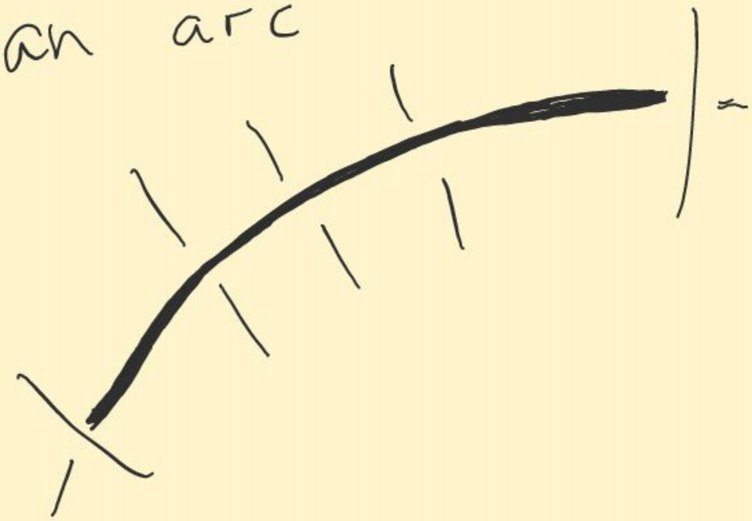


The theorem is that any two pictures of the same knot can be obtained from one another by two dimensional deformations (called isotopies) and Reidemeister moves. It is in fact not too hard to convince oneself of this but the proof is really a four dimensional thing-one considers a "movie" of pictures and one must arrange for the simplest possible things to happen when crossings meet one another.

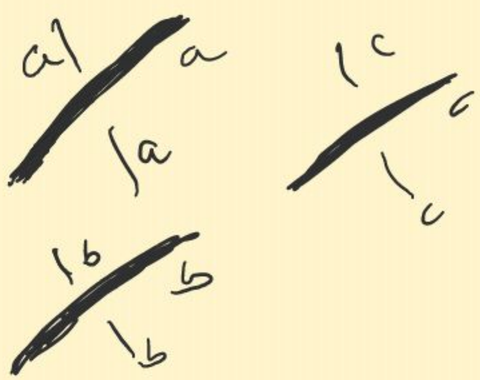
The Reidemester moves reduce knots to objects of planar combinatorics! This does not however necessarily simplify matters. But one can search for combinatorial formulae that don't change under the Reidemeister moves. For links one could give the rather trivial example of the number of components.....The simplest example which gives information about knots is the number of Fox 3-colourings explained in the picture below:

Fox 3-colouring

an arc



colours a, b, c associated to arcs - at a crossing

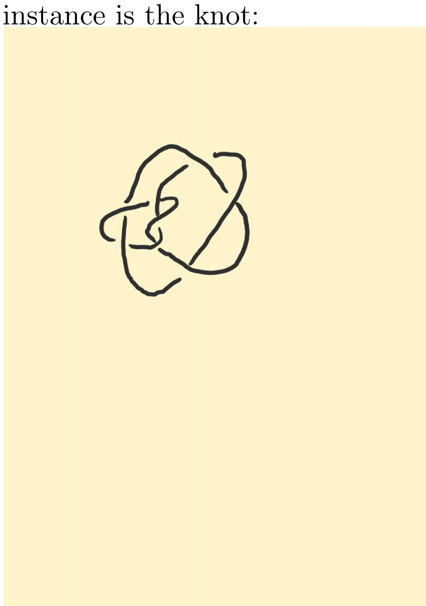


or all three colours used

One can check without much pain that the number of three colourings is invariant under the Reidemeister moves.

Knots and links may or may not be the same as their mirror images (=all crossings reversed). If they are they are called amphicheiral and sometimes called chiral if they are not. The trefoil is chiral, the figure 8 is amphicheiral. The first fact is not obvious, the second can be easily demonstrated.

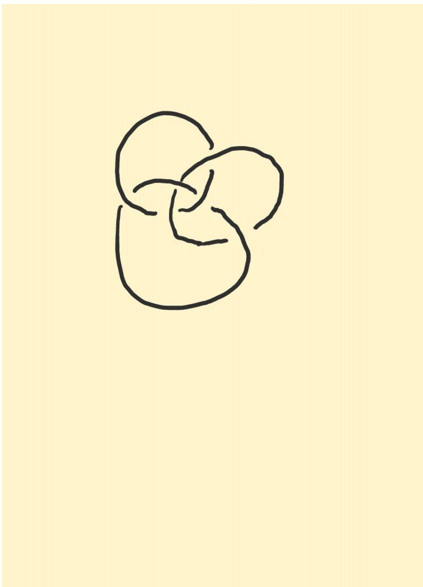
A simple measure of the complexity of a knot is the number of crossings required to draw a picture of it. This number is obviously an invariant but is not always easy to compute. Tables of knots are arranged by the crossing number. All knots have been classified by Morwen Thistlethwaite up to 16 crossings. Up to 10 crossings there are about 200 different ones (not counting chirality) and they were classified, almost entirely correctly, by Tait and Little in the 19th century. They are referred to by some standard numbering as in Rolfsen's book "Knots and Links" so that 8_4 for instance is the knot:



The knot atlas: katlas.math.toronto.edu/wiki/ is a mine of information.

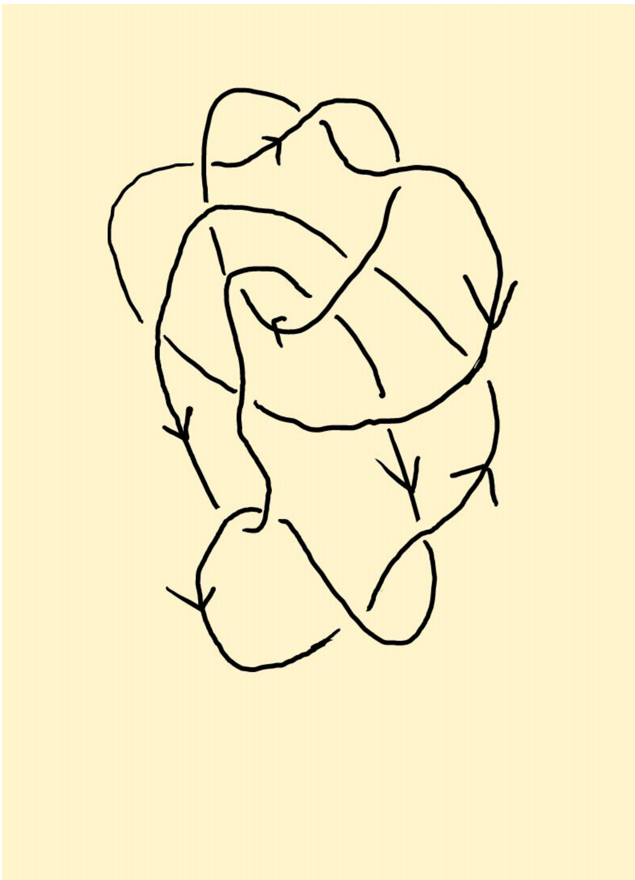
Gauss code:

Links present knottedness of a different kind from knots. The extreme is the "Borromean Rings" which consists of three unknotted circles which come apart if any of the three is removed:



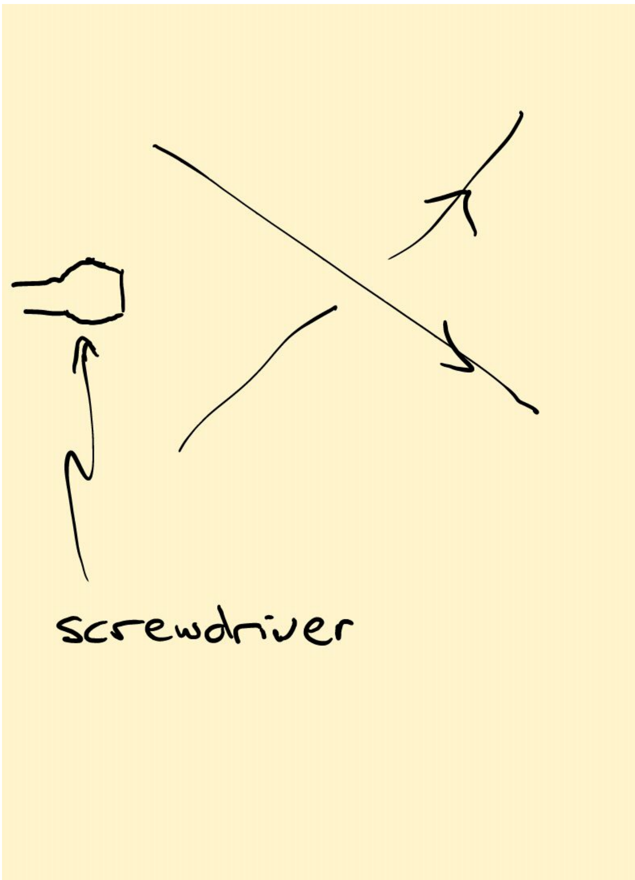
Links with this property are called "Brunnian"

Knots and links may be oriented. In pictures this is achieved simply by giving a direction to the components thus:



For oriented links one needs to consider the Reidemeister moves with all possible orientations. The "reverse" of a knot is the same knot with opposite orientation. A knot may or may not be the same as its reverse, if it is it is called reversible. It is somewhat surprising that small knots tend to be reversible. in fact the smallest knot that is not reversible has 8 crossings.

Once oriented, a crossing has a sign. By convention the following crossing is positive:



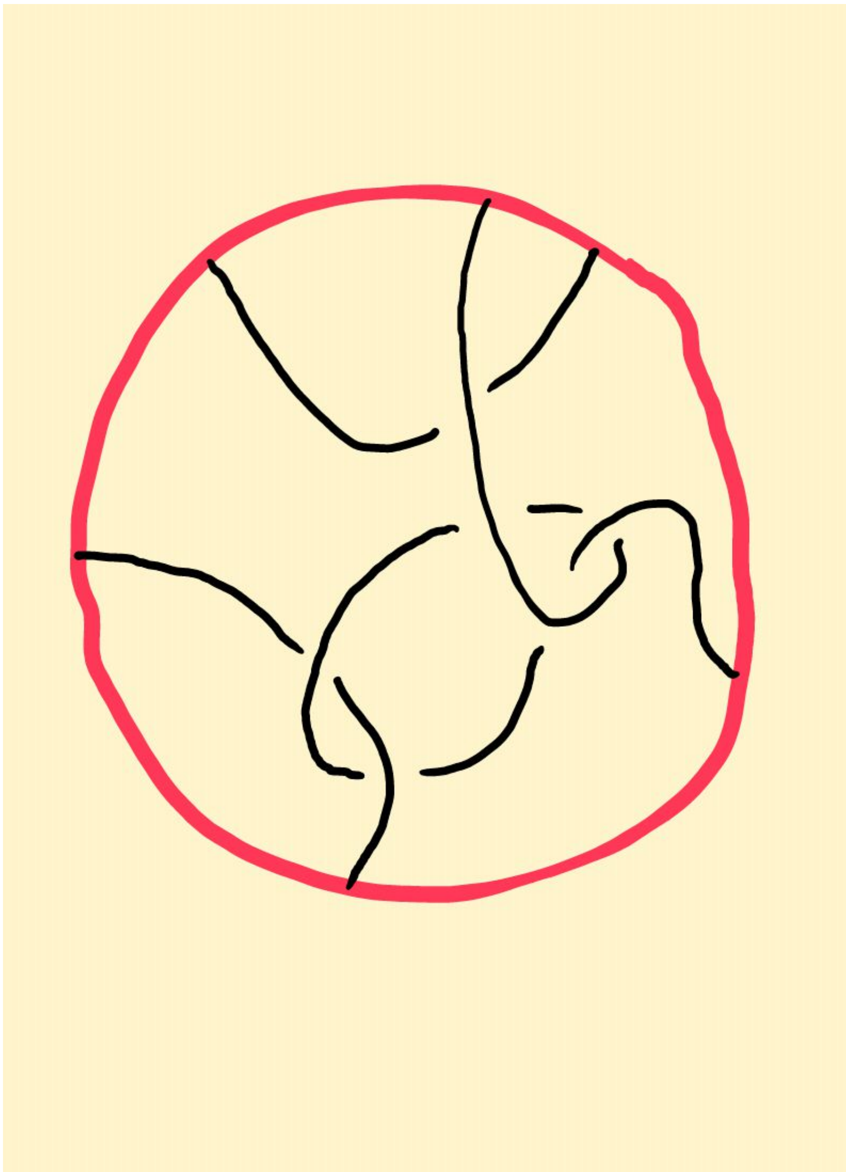
One may think of a screwdriver going "into" the crossing with its extremities on the strings and turning in the positive direction (tightening up the screw).

For links, the number of crossings counted with signs is invariant under the Reidemeister moves. It is called the linking number. Gauss obtained a beautiful integral formula for it given the knot as a subset of $\mathbb{R}^3$.

Braids are links with a rather organized boundary. Informally speaking a braid consists of two bars which may be assumed to be top and bottom, and strings which tie the top bar two the bottom bar. The rule about braids is that the strings never have a horizontal tangent vector so once they start going up they keep going up. Here is an example of a braid:

Like links, braids are considered to be the same if, after making sure the top and bottom points of contact between the bar and the strings are the same, they can be deformed to one another by smooth deformations happening entirely in the region between the bars. We will return to braids in some detail later.

Tangles are the full Monty-links with boundary, which may be considered to be a ball or a box, or ever something more exotic, with no restriction on what happens to the (smooth) strings inside. The only obvious restriction is that the number of boundary points must be even-but is not necessarily divided into two equal halves like a braid. Here is a picture of a tangle:



3 The Jones polynomial.

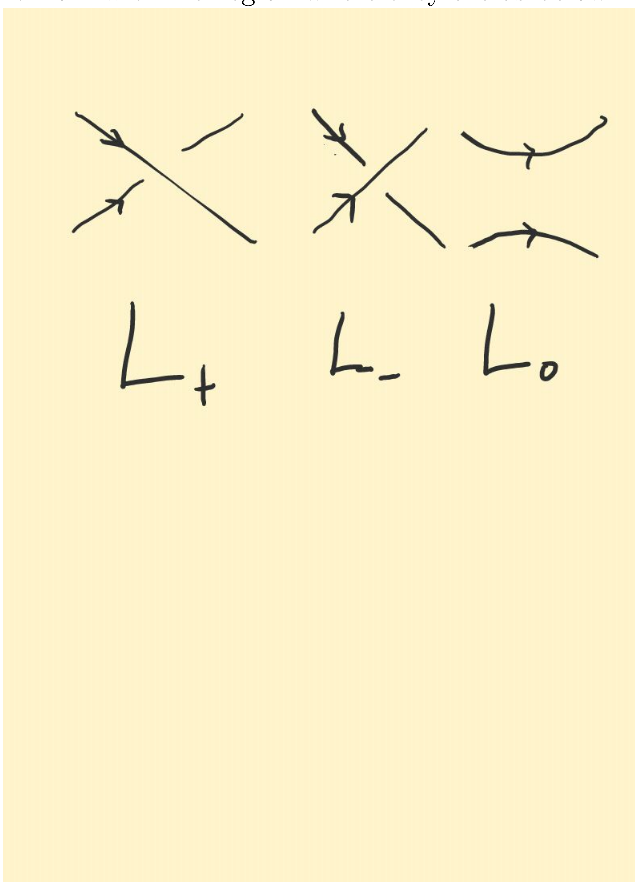
Second definition of the Jones polynomial. (The first will be last...)

The Jones polynomial is an assignment of Laurent polynomials $V_L(t)$ in the variable $\sqrt{t}$ to oriented links L subject to the following three axioms.

(a) Two equal links have the same polynomial.

(b) The polynomial of the unknot is equal to 1

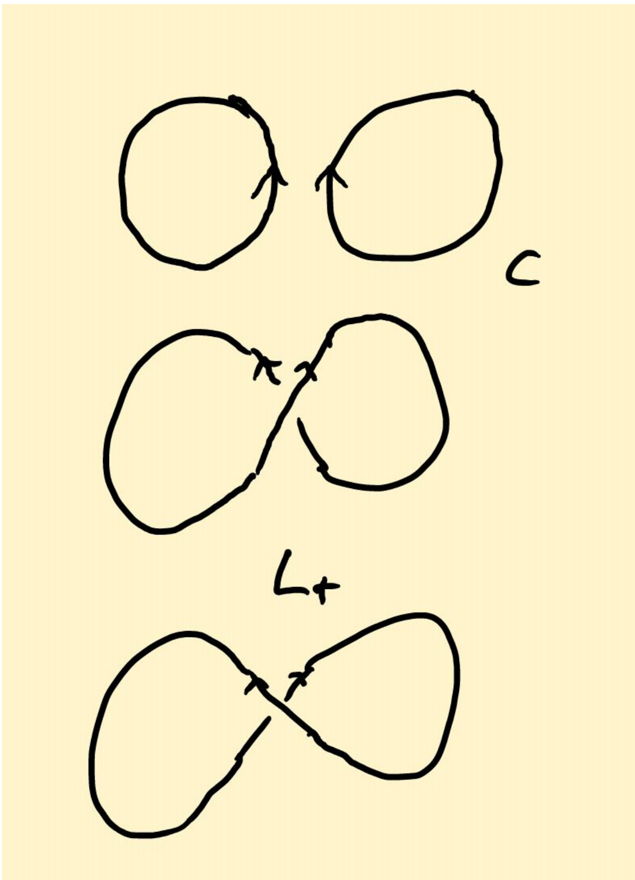
(c) (The skein relation) If three links L_+, L_- and L_0 have pictures which are identical apart from within a region where they are as below:



then

$$1/tV_{L_+} - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_{L_0}$$

This is less a definition than a calculational method. There is no guarantee at this stage that such an invariant exists. I want to convince you though, right away, that the "skein" formula suffices to calculate $V_L(t)$, inductively on all links. We begin with two unlinked circles. Call that link C. Then consider the following picture:



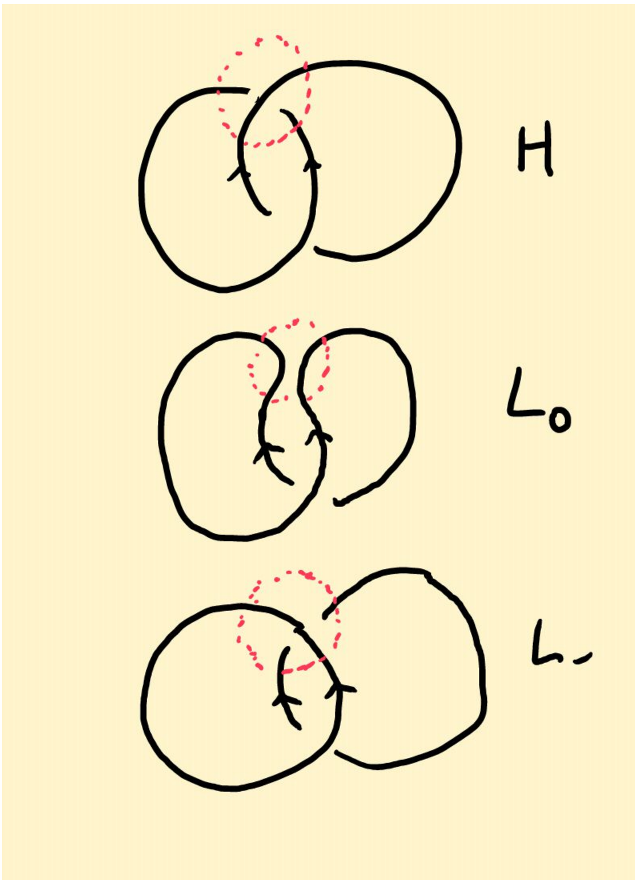
Clearly we may apply axiom (c) with both L_+ and L_- being the unknot and L_0 being C. So we have

$$1/tV_{L_+} - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_C$$

which gives

$$V_C = -(\sqrt{t} + 1/\sqrt{t})$$

Now consider the link called H in the following picture (sometimes called the Hopf link):



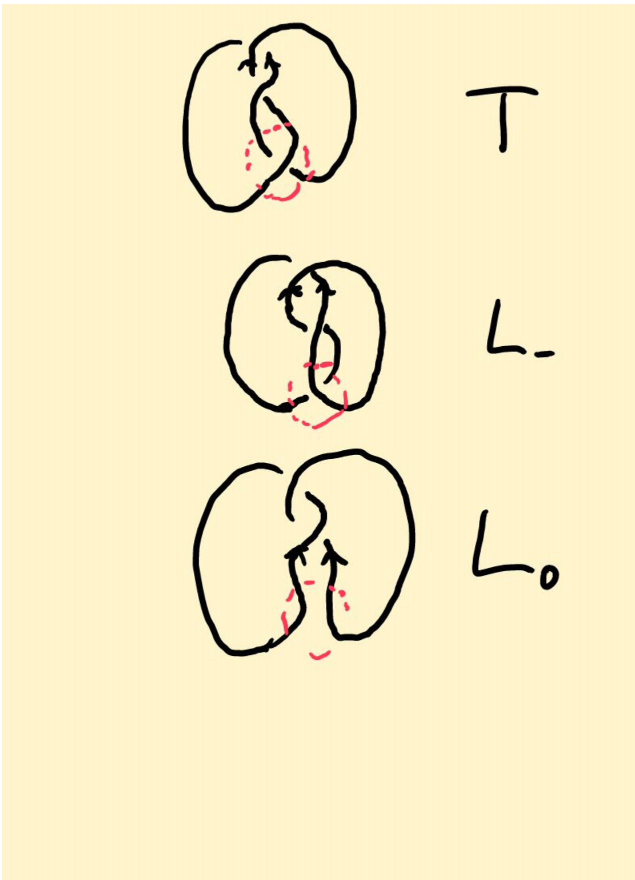
again by axiom (c) we have:

$$1/tV_H - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_{L_0}$$

which by our previous calculation gives

$$V_H = -\sqrt{t}(1 + t^2)$$

Now finally we can calculate the polynomial of the trefoil:



We have

$$1/tV_T - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_{L_0}$$

and since L_- is the unknot and L_0 is now H we get:

$$V_T = t + t^3 - t^4$$

We invite the reader to try his/her hand at the figure 8 knot, the answer is

$$V_8 = t^{-2} - t^{-1} + 1 - t + t^2$$

It should now be fairly obvious that the axioms for the Jones polynomial suffice to calculate it. The point is that any knot/link can certainly be untied by changing enough crossings. So there must be, somewhere in any picture, a crossing that "simplifies" the knot if it is changed. If one makes this crossing the L_+ part (or L_- as the case may be) then L_- is simpler and L_0 has one less crossing so is way simpler. This argument can be made to work by introducing the right notion of complexity for pictures of links.

Some observations are immediate:

(1)

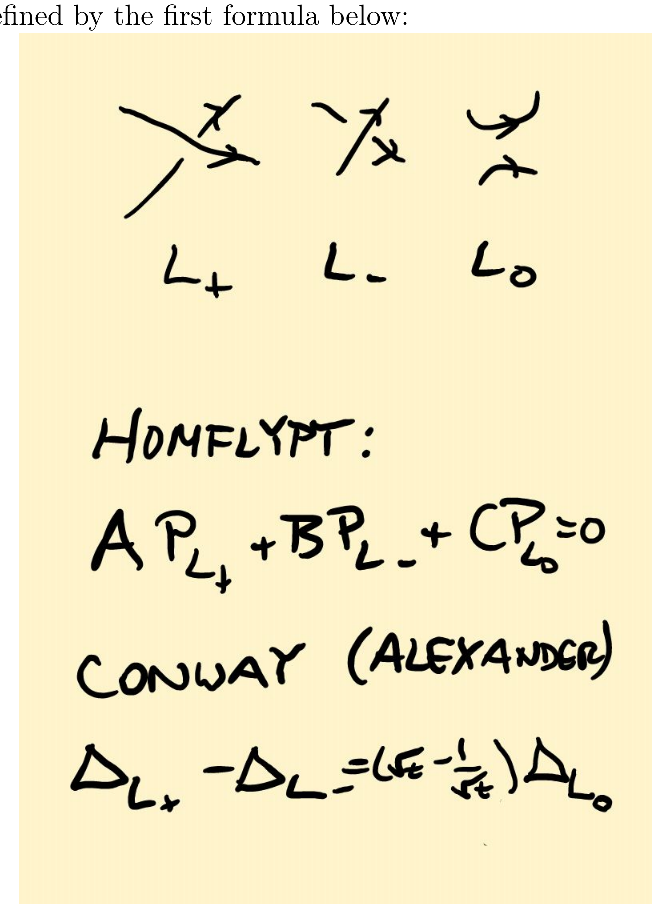
$$V_{L'}(t) = V_L(1/t)$$

if L and L' are mirror images of one another.

(2) So the mirror image of the trefoil is not the same as the trefoil.

(3) The polynomial of a link with an odd number of components is a (Laurent) polynomial in t and the polynomial of a link with an even number of components is $\sqrt{t}$ times a (Laurent) polynomial in t. (4) The polynomial of the "connected sum" of two two is the product of their polynomials.

One may wonder what is special about the choice of coefficients in the skein relation for V_L. The answer is absolutely nothing-there is a polynomial in 3 variables (only two really by an obvious normalization) called the HOMFLYPT polynomial defined by the first formula below:



The second formula is a skein relation noticed by Conway for the "classical" Alexander polynomial which came originally from algebraic topology.

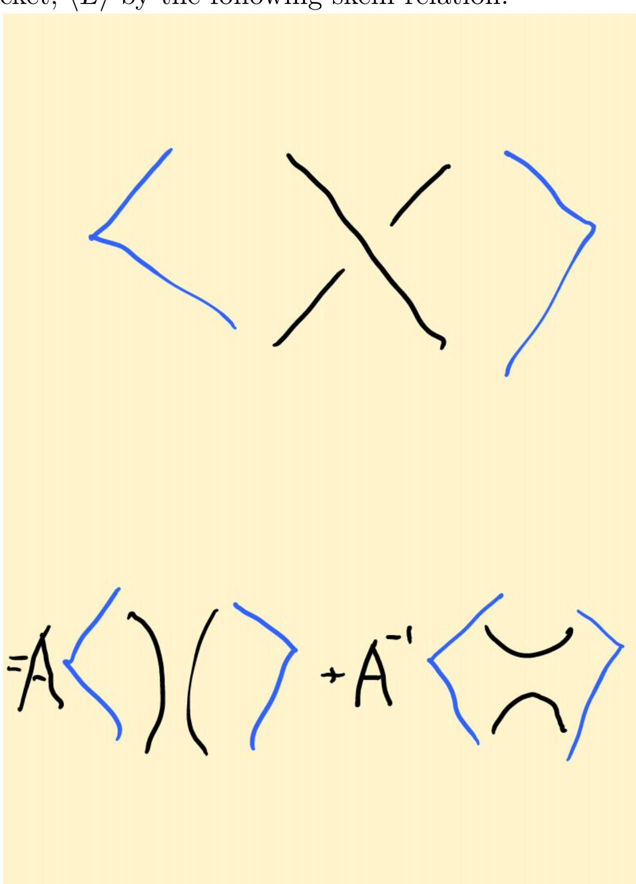
There are pairs of different knots with equal Jones, indeed HOMFLYPT, polynomials though they are not particularly easy to find. There are non-trivial knots with trivial Alexander polynomials and devices for producing such. But the existence of knots with trivial Jones polynomial remains an unsolved problem 30 years after the discovery of the polynomial. Remarkably, Thistelthwaite, or more correctly his computer, discovered a two-component link whose Jones polynomial is $-(\sqrt{t}+1/\sqrt{t})$ - the same as that of the two-component unlink. We have drawn this link above as our first example of an oriented link.

Third definition of the Jones polynomial (due to Kauffman).

It is a surprising property of the Jones polynomial, from the above definition, that it only changes by a power of t if one changes the orientation of any component of a link. This is certainly not true of the Alexander polynomial even. This was observed by myself using the Temperley-Lieb algebra below, and fully explained and exploited by Kauffman.

Definition 3.1. *Two unoriented link diagrams are said to be "regular isotopic" if they can be converted one to another by Reidemeister moves of types II and III.*

Kauffman defined an invariant of regular isotopy of a link L, the Kauffman bracket, $\langle L \rangle$ by the following skein relation:



The meaning of this elegant notation should be clear. Unlike the first skein relation for the Jones polynomial, this one is particularly simple since the number of crossings is strictly reduced when it is applied. This means that there is a completely explicit formula for the Kauffman bracket as a sum over all 2^n ways of "smoothing" all the n crossings of the product over all the crossings of a factor $A^{\pm 1}$ depending on how that particular crossing was smoothed, times a factor which accounts for the number of (disjoint) closed curves formed when all the crossings are smoothed. It follows easily from the bracket skein relation that a closed curve must count for a factor $\delta = -A^2 - A^{-2}$.

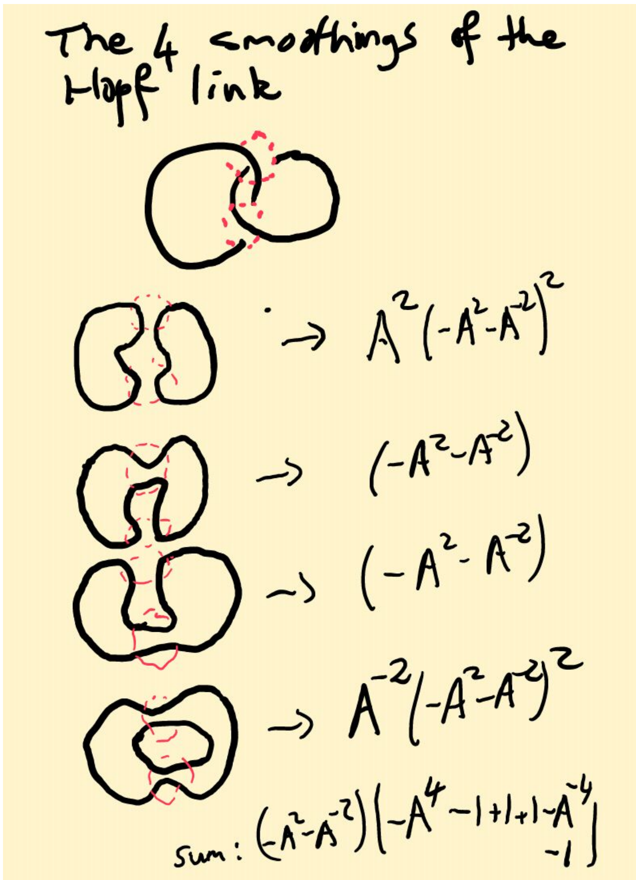
With this formula the Kauffman bracket is simply *defined* for any link diagram. It is a surprisingly easy exercise to check that the bracket thus defined is invariant under the type II and III Reidemeister moves!!

In fact regular isotopy is not very far from isotopy. The type I move is quite innocent and all the simple twists involved can be concentrated in one little region of the knot that does not see the II and III moves. To obtain the Jones polynomial from the Kauffman bracket one orients the link L to obtain $\vec{L}$ and multiplies the Kauffman bracket by a factor

$$A^{-3wr(\vec{L})}$$

where $wr(\vec{L})$ is the number of crossings, counted with their signs. It is not difficult to show that this is invariant under all the Reidemeister moves and satisfies the skein relation of definition 2 of $V_{\vec{L}}$, with $t = A^4$. One must also divide by a factor of δ to account for the normalization of the unknot.

Here is the Kauffman bracket calculation for the Hopf link:



which gives

$$\langle H \rangle = (A^2 + A^{-2})(A^4 + A^{-4})$$

If the link is oriented appropriately the writhe is 2 so after dividing by δ one obtains $-A^6(A^4 + A^{-}4) = -A^2(1 + A^8) = -\sqrt{t}(1 + t^2)$, agreeing with our previous calculation.

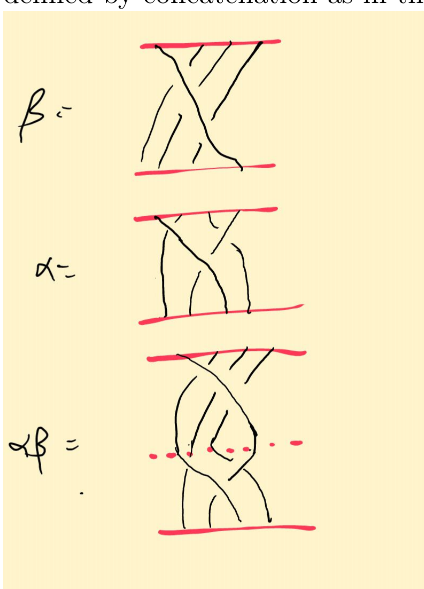
What is extremely useful about the Kauffman bracket is that one can under certain circumstances, locate the terms of highest and lowest degree. One obtains immediately that $deg(V_L(t))$, defined as the difference between the highest and lowest powers of t is $\leq$ the number of crossings! And things are particularly simple for "alternating" links where the crossings alternate over and under as one goes around the components of the link. In this case the term of highest degree comes from smoothing all the crossings the same way (A smoothing) and the lowest degree comes from smoothing them all with the A^{-1} smoothing. (One has to be a bit careful because one may form the connected sum of two links in an alternating way by adding a crossing where the two links are joined, but magically this problem takes care of itself.) Putting the two results together one obtains that, for an alternating n-crossing link (that does not have a crossing splitting it as above), $degV_L(t)$ is actually equal to the number of crossings! Thus L has no picture whatsoever with less than n crossings. This had been a conjecture for over a century when it was proved by Kauffman, Murasugi and Thistlethwaite in the mid 1980's.

By either the Kauffman bracket or the first skein definition the Jones polynomial is inherently exponential time to compute. It is known to be NP-hard but not known to be NP.

4 The braid group and links.

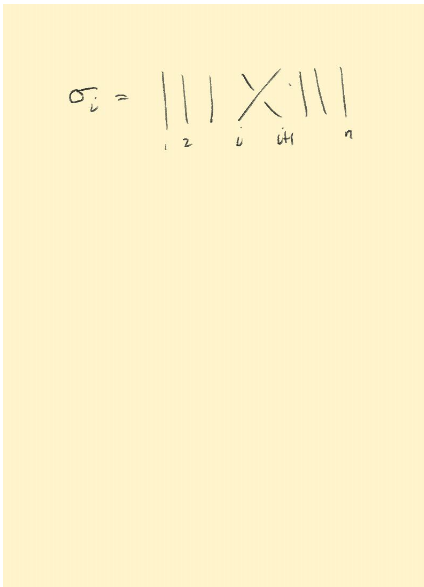
4.1 Definition of the braid groups.

The set of all braids on n strings forms a group B_n. The group multiplication is defined by concatenation as in the picture below:



The identity braid is the braid which consists of n vertical straight lines. The inverse of a braid can be seen by looking at it in a mirror, or bit by bit as below. Note that a braid defines a permutation of its end points. Thus there is a homomorphism from B_n onto the symmetric group S_n. The kernel of this homomorphism is called the *pure braid group* P_n.

With a little nudging, all the crossings in a braid can be assumed to occur on different horizontal levels. This means that any braid can be written as a word on the generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ depicted below:

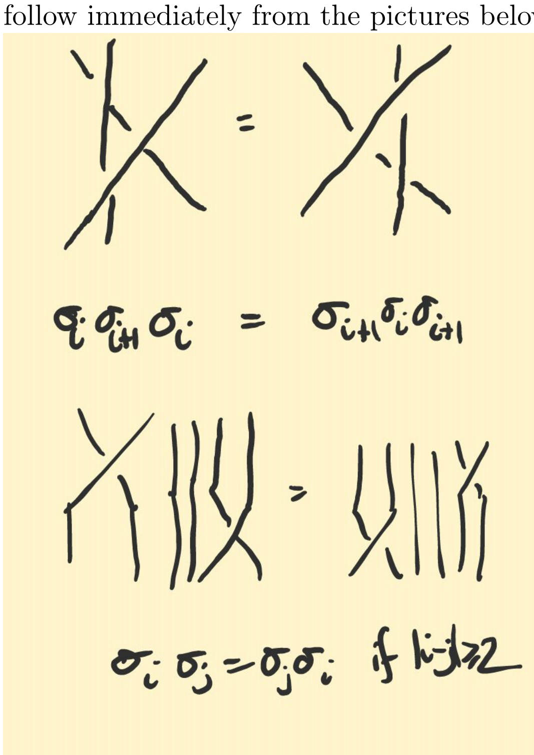


The relations

$$\sigma_1\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \quad \text{and}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

follow immediately from the pictures below:



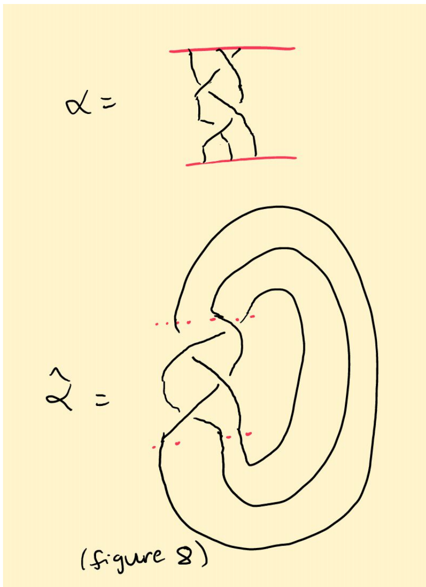
It is a result of E. Artin that B_n is actually presented on these generators and relations so B_n has a separate life as a finitely presented discrete group. This has great significance - for instance if one wants to find a representation of the braid group it suffices to find matrices satisfying the appropriate relations.

Note that if one adds $\sigma_i^2 = 1$ to the braid group relations one obtains a well known presentation of S_n. This corresponds to the action of the braid group on the end points of the braid.

Here is a potpourri of elementary facts about the braid group (all more or less true):
0) The braid group B_n is embedded as a subgroup of B_{n+1} by adding a vertical string to the right of the braid. 1) The braid group B_2 is isomorphic to $\mathbb{Z}$.
2) The element $\zeta = (\sigma_1\sigma_2\sigma_3...\sigma_{n-1})^n$ generates the centre of the braid group. (Geometrically it is a full twist so clearly in the centre.)
3) The quotient of B_3 by its centre is the modular group $PSL_2(\mathbb{Z})$.
4)B_3 itself is the inverse image of $SL_2(\mathbb{Z})$ in the universal covering group $S\tilde{L_2}(\mathbb{R})$ (under the covering map).
5) B_n acts on the free group F_n in an obvious way when F_n is realized as π_1 of $\mathbb{R}^2$ minus n points. This action is faithful.
6)The pure braid group P_{n+1} is the semi direct product of the free group F_n and P_n.
7)The braid groups are not amenable but they do not have Kazhdan's property T.
8) Any braid can be written as a power of ζ times a "negative" braid- word in the inverses of the σ_i. 9) The braid groups are known to be linear ($\cong$ a subgroup of $GL_m(\mathbb{R})$ via a representation that came out of the knot polynomials.

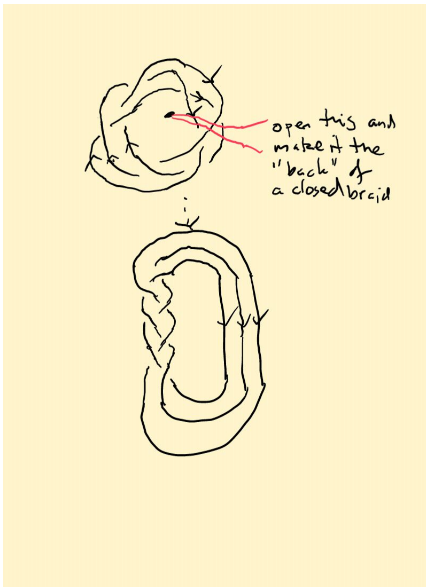
4.2 Closing a braid.

There is a powerful way to form a link $\hat{\alpha}$ from a braid α. It is given by the following picture:

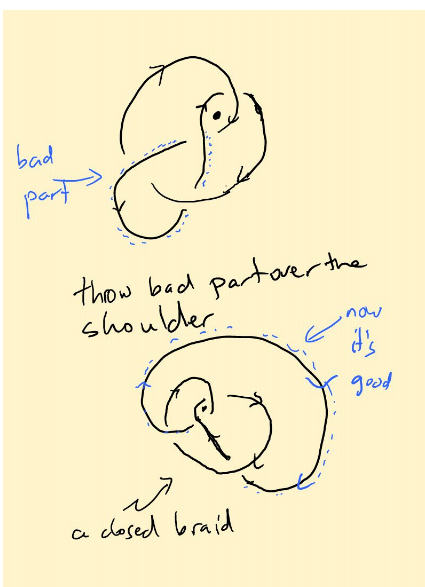


Note that the closure of a braid is a knot iff the element of S_n determined by the braid is an n-cycle. In general the number of components of the closure is the number of orbits of the permutation on the end points.

Closed braids are naturally oriented, say from the bottom to the top of the braid. It is a theorem of Alexander that any oriented link can be obtained as the closure of the braid. Several cunning algorithms have surfaced recently, e.g. one by Vogel, for turning a link into a closed braid, but the procedure is simple enough-one only has to find a point in the plane about which the strings of the link always turn in the same direction. By "opening up" as below one sees a closed braid:



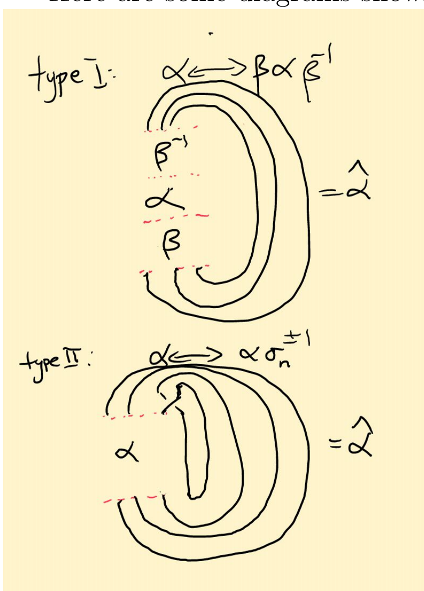
Of course there may not be a point about which the strings all travel clockwise or anticlockwise. But this can always be achieved by "throwing the bad parts over one's shoulder" as below:



Thus we have a surjective map from braids to links. This raises the question as to the "kernel" of this map, i.e. when do two braids represent the same link? The answer to this is provided by the Markov moves:
Type I: conjugation within B_n
Type II: If $\alpha \in B_n$ then replace α by $\alpha\sigma_n^{\pm 1} \in B_{n+1}$ (And inversely.) Markov's

theorem asserts that two braids represent the same link iff they can be transformed from one to another by a sequence of Markov moves.

Here are some diagrams showing the "if" direction of Markov's theorem:



Conjugation in a group is a rather natural operation which occurs everywhere in representation theory so Markov's theorem strongly suggests looking at representations of groups for invariants of links. Going between different braid groups means one will have to have coherent representations of all the braid groups at once to be able to use Markov's theorem.

4.3 The torus knots.

The torus link of type (p, q) is the closure of the braid $(\sigma_1\sigma_2...\sigma_{p-1})^q \in B_q$. If p and q are relatively prime this is a knot. Thus the trefoil is the $(2, 3)$ trefoil knot. Our goal in these lectures is to calculate the Jones polynomial of the (p, q) torus knot.

4.4 Representations.

Very few (linear) representations of the braid groups had been considered before 1984. There was only really the Burau representation which we describe from the combinatorial point of view. It suffices to construct matrices $S_1, S_2, ..., S_{n-1}$ satisfying the braid relations. Here is a family of such matrices:

$$S_1 = \begin{pmatrix} 1-t & t & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & & \\ 0 & 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 1 & \cdots & \\ 0 & 0 & 0 & \cdots & & 1 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1-t & t & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdot\ 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1-t & t & & \\ & & 1 & 0 & & \\ & 0 & & & 1 & \\ & & & & & \ddots \end{pmatrix}$$

$$S_{n-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ 0 & & & 1-t & t \\ & & & 1 & 0 \end{pmatrix}$$

The above matrices are row-stochastic (at least for $0 \leq t \leq 1$) so the representation is not irreducible. The quotient by the one-dimensional invariant subspace is called the "reduced Burau representation". It has a simple interpretation in terms of algebraic topology and is intimately related to the Alexander polynomial. In fact the determinant of $1-$Burau(a braid) is a multiple of the Alexander polynomial of that braid.

The Burau representation is faithful for $n = 3$ and not faithful for $n \geq 5$. Its faithfulness for $n = 4$ is a major open question.

Note how the Burau representation becomes a permutation representation when $t = 1$.

5 The Temperley Lieb algebra.

5.1 First definition of the Temperley-Lieb algebra.

Definition 5.1. *For $\tau \in \mathbb{C}$ $TL_{n+1}(\tau)$ is the associative unital (i.e. has an identity) *-algebra with generators $e_1, e_2, ..., e_n$ and relations*

(a) $e_i^2 = e_i = e_i^*$
(b) $e_i e_j = e_j e_i$ *for* $|i-j| \geq 2$
(c) $e_i e_{i\pm1} e_i = \tau e_i$.

This algebra was first encountered by Temperley and Lieb in work on statistical mechanics. It was rediscovered as a von Neumann algebra involving index for subfactors and its structure was given for all relevant values of τ.

Let us warm up by figuring out the structure of TL_1, TL_2 and TL_3. Let us assume they are *semisimple*. Since we are working over the complex numbers this just means they are direct sums of matrix algebras of various sizes. Semisimplicity is a "generic" condition- it is true whenever the "Killing form" is non-singular so for an algebra depending on a parameter like TL, it will be semisimple for all but a finite number of values of τ provided it is semisimple for a single value. We will later see why it is semisimple for $\tau = \frac{1}{4}$ for all n. One may suppose that $*$ is conjugate transpose.

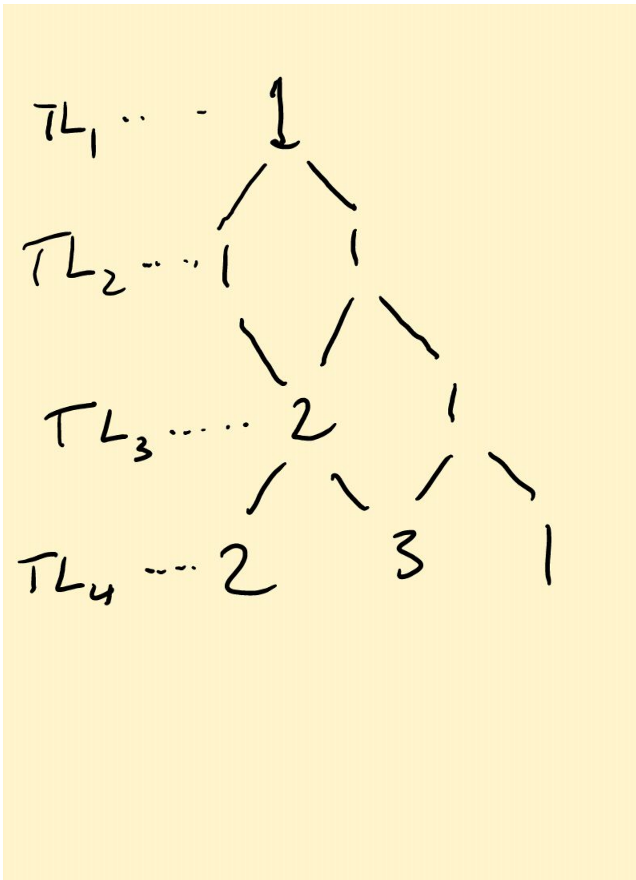
TL_1: No e_i's at all, just the identity so $\qquad \cong \mathbb{C}$

TL_2 : $1 - e_1$ is also idempotent and $e_1(1-e_1) = 0$ so $\qquad \cong \mathbb{C} \oplus \mathbb{C}$
TL_3: Spanned by $1, e_1, e_2, e_1e_2, e_2e_1$, and non abelian so $\qquad \cong M_2(\mathbb{C}) \oplus \mathbb{C}$

Counting words on e_1, e_2 and e_3 and a little wishful thinking gives $dim TL_4 = 14$ and the observation that $e_1 = e_3$ gives a representation of TL_4 onto TL_3 more or less forces:

$TL_4 \qquad \cong M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C}$

We can deduce that the obvious "inclusion" maps between TL_1, TL_2, TL_3 and TL_4 are actually injective and thus genuine inclusions. And we can schematically begin the structure of all TL_n with the diagram:



Note that there is a problem with TL_3 if $\tau = 1$ (or zero). For in the 2x2 matrix part, e_1 and e_2 will be rank one projections and $e_1e_2e_1 = e_1$ forces $e_1 = e_2$ and $e_1e_2e_1 = 0$ forces e_1 and e_2 to commute! For these values of τ the algebra is not semisimple. The problem gets worse for $n = 4$ where the value $\frac{1}{2}$ causes problems. This can actually be seen geometrically. Once again in the $M_2(\mathbb{C})$ part, e_1, e_2 and e_3 are rank one projections, and we might as well be working over $\mathbb{R}$. The relation $e_1e_2e_1 = \tau e_1$ determines the *angle* between the subspaces onto which the e_i's project. For $\tau = \frac{1}{2}$ this angle is $\frac{\pi}{4}$ for e_1, e_2 and e_2, e_3 and by $e_1e_3 = e_3e_1$, it is $\frac{\pi}{2}$ for e_1 and e_3. Thus all three subspaces live in a plane, so projections onto them cannot generate a 3x3 matrix algebra!

This nuts and bolts approach would start to become difficult for larger n.

5.2 Some easy facts.

1) Any reduced word (after applying the relations in the definition) on $e_1, e_2, \ldots e_n$ contains e_n at most once.

Proof. Write a word as $w_1e_nw_2e_nw_3$. If we can get rid of one of the e_n's we are done.

First we may suppose that w_2 does not contain e_n by taking a nearest pair of e_n's. By induction we may write $w_2 = xe_{n-1}y$ or $w_2 = x$ where x and y do not contain e_n or e_{n-1}. In the first case use relations (b) and (c) of TL and in the second case relations (b) and (a) to get rid of one of the two e_n's. □

2) Corollary: $dimTL_n < \infty$.
3) Corollary: $dimTL_n \leq \frac{1}{n+1}\binom{2n}{n}$

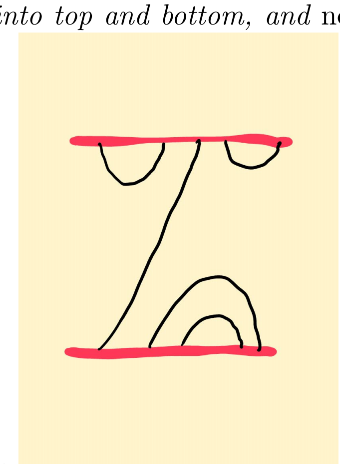
Proof. By pushing e_{max} to the right as much as possible any word on $e_1, e_2, ..., e_n$ is proportional to a product of strings of e_i's with indexes decreasing by one, with the beginning and end of each string strictly increasing, such as

$$(e_5e_4e_3)(e_6e_5)(e_8e_7e_6)$$

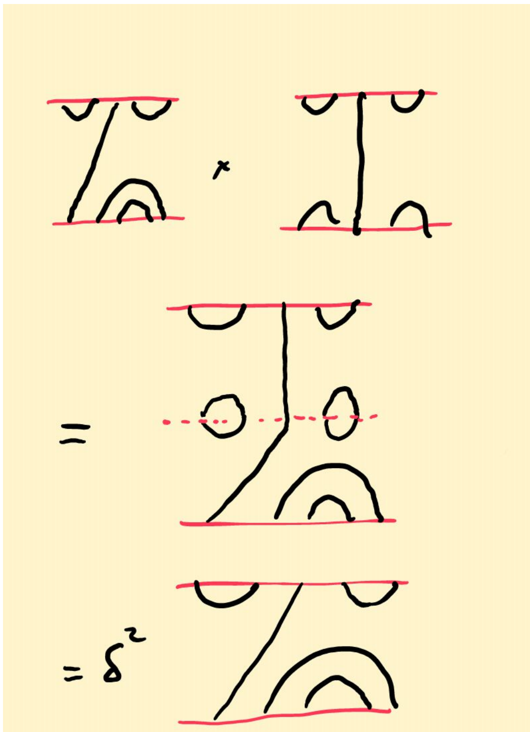
It is an exercise to show that such words are counted by the famous Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. □

5.3 Second definition of Temperley Lieb, due to Kauffman.

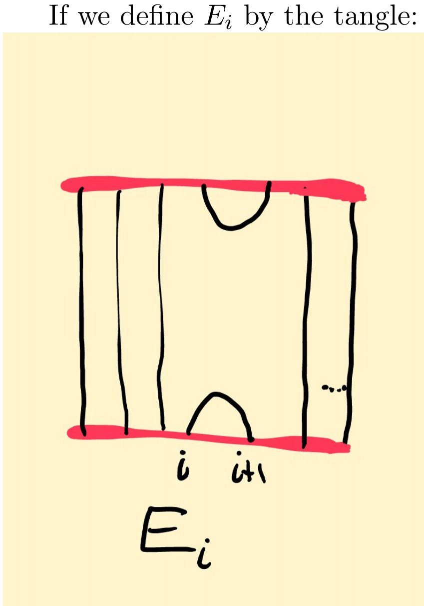
Definition 5.2. *For $\delta \in \mathbb{C}$ $tl_n(\delta)$ is the algebra whose basis is the set of (planar isotopy classes of) tangles without crossings (called planar tangles) with $2n$ boundary points, divided into top and bottom, and* no closed loops*. Thus an example of a tl_5 basis element is:*



Basis elements are multiplied by concatenation, like braids, with the (inevitable) closed loops being removed, each one giving a multiplicative factor of δ. Thus:



If we define E_i by the tangle:



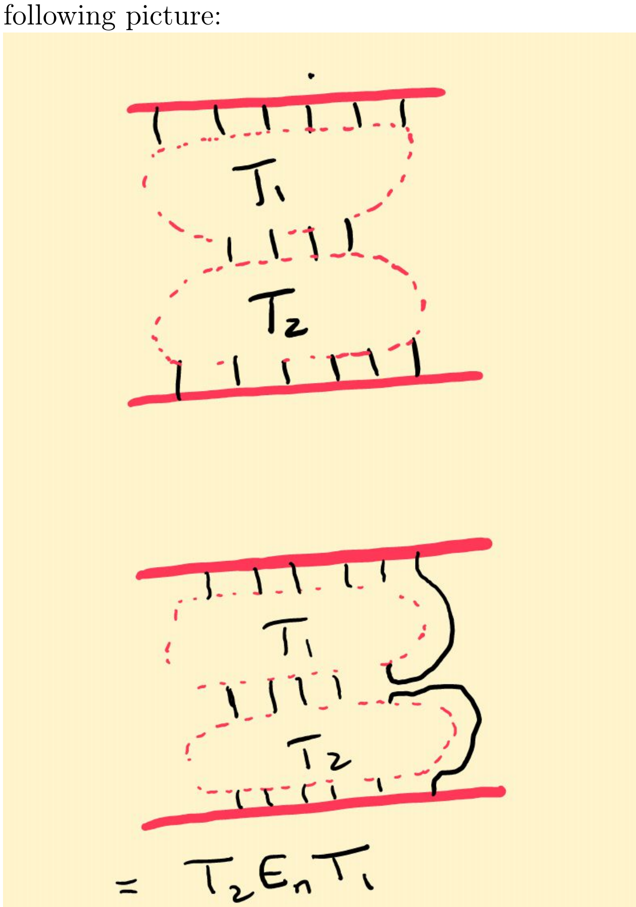
then the following relations are shown by simple diagrams:
(a)$E_i^2 = \delta E_i$
(b)$E_iE_j = E_jE_i$ if $|i-j| \geq 2$

(c)$E_i E_{i\pm 1} E_i = E_i$.

Thus if $\tau = \delta^{-2}$ there is an algebra (*-algebra if obvious * structure on tl) homomorphism $\iota : TL_n \to tl_n$ given by $\iota(e_i) = \frac{1}{\delta} E_i$.

Proposition 5.1. *ι is an isomorphim of *-algebras.*

Proof. It is well known that the dimension of tl_n is $\frac{1}{n+1}\binom{2n}{n}$ (e.g. by generating functions). So it clearly suffices by fact (3) concerning TL to show that ι is surjective, i.e. any tl basis element is 1 or a product of e_i's. This is not hard. If One takes a tl_{n+1} diagram and moves all local minima close to the top and all local maxima close to the bottom one is left with a picture that decomposes into a top part, a bottom part and $n-p$ "through strings" in between. If the element is not the identity there have to be local maxima and minima. This is illustrated in the top part of the following picture:



The second diagram above is the same picture after an isotopy which exhibits the original element as $T_2 E_n T_1$ with T_1 and T_2 both in tl_n (embedded in tl_{n+1} via a vertical string on the right). By induction we are through. $\square$

As corollaries we have that the inclusion maps between the TL_n are indeed all inclusions and that the dimension of TL_n is $\frac{1}{n+1}\binom{2n}{n}$ (independent of semi simplicity but provided $\tau \neq 0$).

5.4 Representations and restricition.

The decomposition above of an arbitrary *tl* element into a top and a bottom suggests one should focus on the top and bottom parts separately.

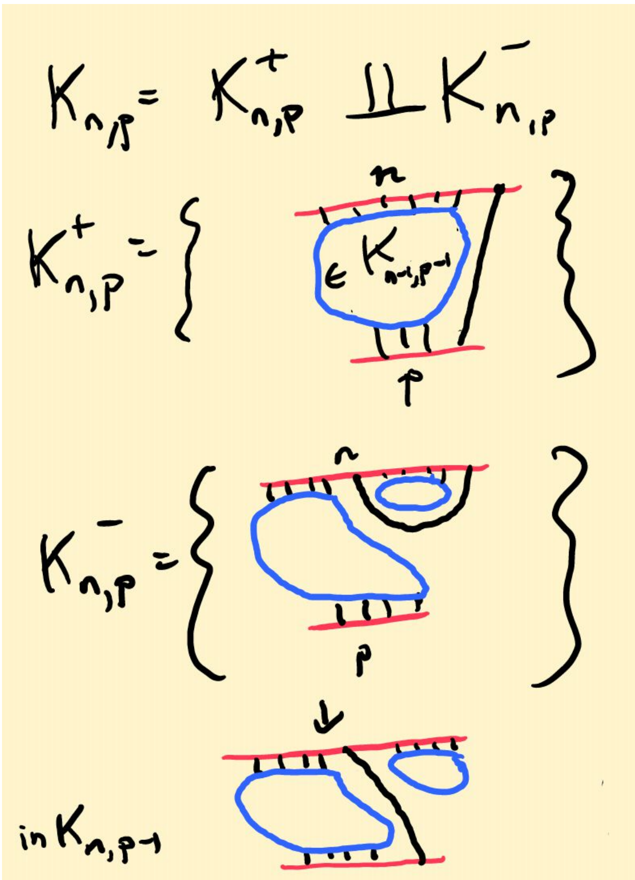
Definition 5.3. *The set $K_{n,p}$, for $n \equiv p \mod 2$, is the set of all planar tangles (no crossings) with n strings on the top and p strings on the bottom which are all connected to the top (and no closed loops). $\mathcal{K}_{n,p}$ is the vector space having $K_{n,p}$ as a basis.*

Definition 5.4. *The representation $\pi_{n,p}$ of TL_n (identified with tl_n) on $\mathcal{K}_{n,p}$ is defined by concatenation and removing closed loops as in the Kauffman definition of TL, but the answer is zero if there are less than p through strings in the concatenated tangle.*

This representation can actually be deduced from a filtration of tl_n according to the number of through strings in a natural way. We leave the details to the reader.

The following observation highly illuminates the structure of the Temperley-Lieb algebra.

Let $K^+_{n,p}$ be the set of all $K_{n,p}$ elements such that the top right string is a through string and $K^-_{n,p}$ be the complement. $K^+_{n,p}$ is in obvious bijection with $K_{n-1,p-1}$. Moreover moving the string attached to the top right point down to the bottom gives a bijection from $K^-_{n,p}$ to $K_{n-1,p-1}$, as illustrated in the picture below:



Linearising we see that

$$K_{n,p} \cong K_{n-1,p-1} \oplus K_{n-1,p+1}$$

It is a simple observation that this map intertwines the actions of TL_{n-1}, on $K_{n,p}$ by the inclusion of TL_{n-1} in TL_n and on $K_{n-1,p-1} \oplus K_{n-1,p+1}$ as the direct sum of modules.

Corollary 5.1. *$dim K_{n,n-2r} = \binom{n}{r} - \binom{n}{r-1}$*

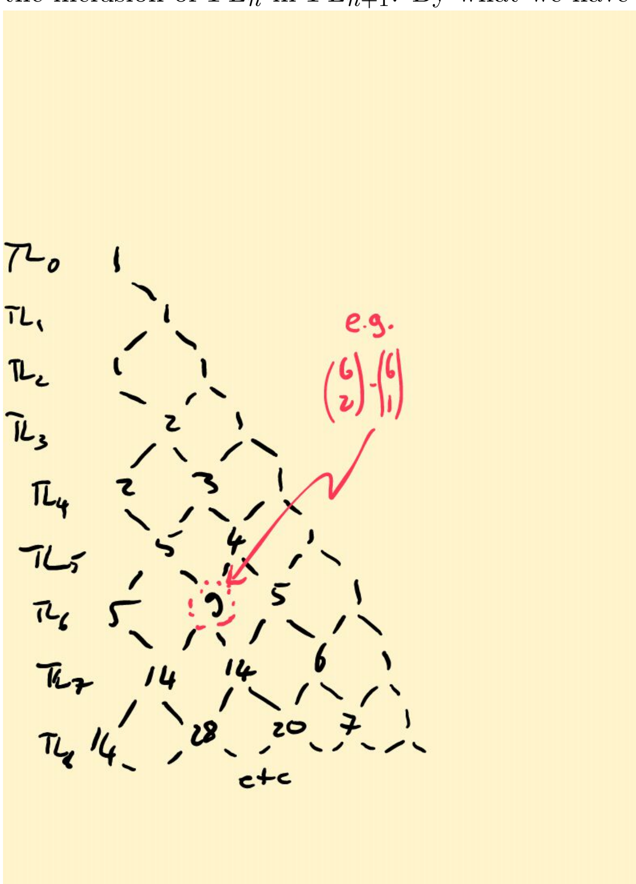
Proof. By the above, both the left and right hand sides satisfy the same recurrence relation and boundary conditions. □

Assume for the moment that all these representations are irreducible. We will see that this happens for generic τ and the numerics are compelling:

$$\frac{1}{n+1}\binom{2n}{n} = \sum_{r=0}^{\infty}(\binom{n}{r} - \binom{n}{r-1})^2$$

is an easy binomial identity (in Feller's book for instance).

We can now complete the Bratteli diagram which we gave above, interpreting it in terms of the irreducible representations of TL_n for all n and how they restrict via the inclusion of TL_n in TL_{n+1}. By what we have shown, the answer is:



The numbers appearing on each row are the dimensions of the $K_{n,n-2r}$, numbered with $r = 0$ on the right, increasing to the left. The diagonal lines indicate the restriction of each representation to the previous TL. For semisimple (complex) algebras is entirely equivalent to giving the structure of the TL_n as sums of matrix algebras and defining the inclusions between them.

One could describe the Bratteli diagram as Pascal's triangle truncated to the right of a vertical line, with the numbers adjusted so that the familiar addition rule of Pascal's triangle still holds.

So if TL is semisimple, we are done, we know its structure completely. We now need to return to the thorny question of its semisimplicity which we can actually address using the $K_{n,n-2r}$. For the structure of the algebra is remarkably simple by decomposing into a top and bottom with through strings in between. In fact we see immediately that, as vector spaces, independent of any semisimplicity,

$$tl_n \cong \bigoplus_{r=0}^{[\frac{n}{2}]} K_{n,r} \otimes K_{n,r}$$

To see the algebra structure we remind the reader of a very simple fact from linear algebra:

If V is a finite dimensional vector space and $\langle , \rangle$ is a nondegenerate bilinear form, then the action of $V \otimes V$ on V given by

$$v \otimes w(u) = \langle w, u \rangle v$$

establishes a linear isomorphism of $V \otimes V$ with $End(V)$, the algebra of all linear transformations of V.

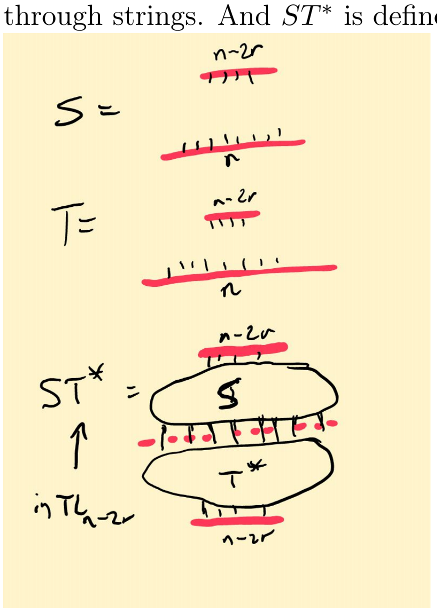
Moreover, the algebra strucure on $End(V)$, translated back to $V \otimes V$ is

$$(v_1 \otimes v_2)(w_1 \otimes w_2) = \langle v_2, w_1 \rangle v_1 \otimes w_2$$

But now consider the bilinear $\langle , \rangle$ form on $K_{n,r}$ defined by:

$$ST^* = \langle S, T \rangle id$$

where id is the identity in TL_{n-2r} mod the ideal spanned by diagrams with $< n-2r$ through strings. And ST^* is defined as follows:



with T^* being T reflected in a horizontal mirror.

It is clear that basis diagrams in TL_n with exactly $n-2r$ through strings, span the quotient $Q_{n,r}$ by the ideal spanned by diagrams with less than $n-2r$ though strings and that they multiply exactly as the multiplication in the isomorphism above with $K_{n,n-2r} \otimes K_{n,n-2r}$ with our bilinear form $\langle , \rangle$. The conclusion is that $Q_{n,r} \cong End(K_{n,r})$ as soon as $\langle , \rangle$ is non-degenerate. Which means that TL will be semisimple as soon as all the $\langle , \rangle$ (as r varies) are non-degenerate. But consider the matrix of $\langle , \rangle$. It consists of powers of δ and zeros. A moments thought suffices to convince oneself that the highest power of δ is δ^r and that this occurs only on the diagonal.

"Diagonal dominance" implies that the determinant of this matrix is non-zero as soon as δ is sufficiently large. We conclude that all the $\pi_{n,r}$ are irreducible, and $tl_n(\delta)$ is semisimple as soon as δ is large enough. (The exact determination of the rank of $\langle , \rangle$ for all δ is much more difficult but we will not need it.)

6 The Markov trace on the Temperley Lieb algebra.

6.1 The Markov trace, uniqueness and existence.

In [] the main object of study was in fact a trace

$$: \bigcup_{n=1}^{\infty} TL_n(\tau)$$

defined by the three properties:
(a)$tr(ab) = tr(ba)$
(b)$tr(1) = 1$
(c) $tr(we_n) = \tau tr(w)$ for $w \in TL_n(\tau)$

It is not hard to show, given what we did in the last section, that such a trace exists and is uniquely defined by these properties. In fact existence is even easier via the isomorphism with tl. If T is a basis diagram just define $tr(T)$ to be δ^{-n+k} where k is the number of closed loops formed when T is closed just as we closed braids above. It is a simple exercise to prove properties (a),(b) and (c).

Now suppose $TL_(n)$ is semisimple so its structure is given by our demi-Pascal Bratteli diagram. A trace on a direct sum of matrix algebras is a weighted sum of the traces in the individual matrix algebras, the weight being the trace of a rank one projection in that algebra.

6.2 The weights of the Markov trace

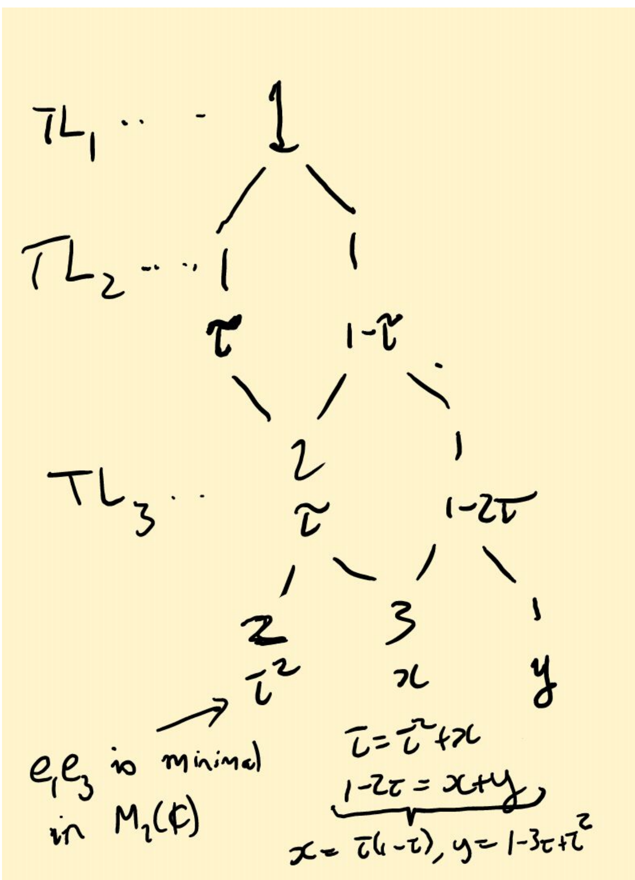
We will assume throughout this section that TL is semisimple, e.g. for large δ so that our previous analysis of its structure holds. In particular we have an explicit isomorphism $Tl_n \cong \bigoplus_{r=0}^{[\frac{n}{2}]} End(K_{n,n-2r})$. So we can choose minimal projections $p_{k,r}$ in $End(K_{n,n-2r})$ and view them as elements of TL_n. As elements of TL_n they are characterized by the property that they are minimal projections on $K_{n,n-2r}$ and zero on Kn, ℓ for $\ell \neq n - 2r$.

Definition 6.1. *The weights of the Markov trace are*

$$w_{n,r} = tr(p_{k,r})$$

Let us begin gently and calculate the weights of the trace for TL_2, TL_3 and TL_4.

We know that TL_2 is $\mathbb{C} \oplus \mathbb{C}$, with the first summand being multiples of e_1 and the second being multiples of $1 - e_1$. Thus the weights are trivially τ and $1 - \tau$. For TL_3 e_1 is a rank one projection inside $M_2(\mathbb{C})$ so the weight coresponding to the representation on $K_{3,1}$ is τ. Since the trace of the identity is 1 we conclude that the weight of the trace for the representation $K_{3,0}$ is $1 - 2\tau$. The argument for TL_4 is given in the next picture which also summarizes the results:

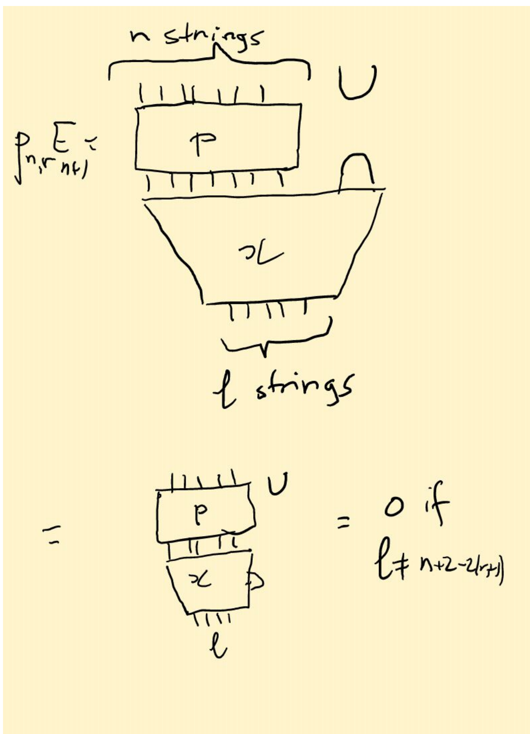


Definition 6.2.

We could get a bit further with this ad hoc approach but for an elegant derivation of the weights of the trace we need two more ingredients, the first is a simple observation:

Proposition 6.1. $w_{n,r} = \tau w_{n+2,r+1}$

Proof. This will follow immediately from property (c) of the Markov trace and the fact that $e_{n+1}p_{n,r}$ is a minimal projection in $End(K_{n+2,r+1})$. This follows from the corresponding property of $p_{n,r}$ and the following picture:



□

Looking at the Bratteli diagram this shows that $w_{n,r}$ is determined from $wn-2, r-1$ so inductively all we need to know is wn, n. So we need to know about $p_{n,n}$. Note that $dimK_{n,n} = 1$ so there is a unique choice of minimal projection $p_{n,n}$.

Definition 6.3. *$p_{n,n} \in TL_n$ is called the nth. Jones-Wenzl idempotent, written f_n for short.*

Let P_n be the polynomials in τ defined by $P_0 = 0, P_1 = 1$ and $P_{n+1} = P_n - \tau P_{n-1}$.

Lemma 6.1. *(Wenzl) Suppose τ is such that $P_n(\tau) \neq 0$ for all $n \geq 1$, then*

$$f_{n+1} = f_n - \frac{P_n \tau}{P_{n+1}(\tau)} f_n e_n f_n$$

and

$$tr(f_n) = P_{n+1}(\tau)$$

Proof. We have to show that $e_i f_n = 0 = f_n e_i$ for $1 \le i \le n-1$, which identifies f_n as a multiple of $p_{n,n}$, that f_n is non-zero and that $f_n^2 = f_n$. These follow easily by induction- the formula

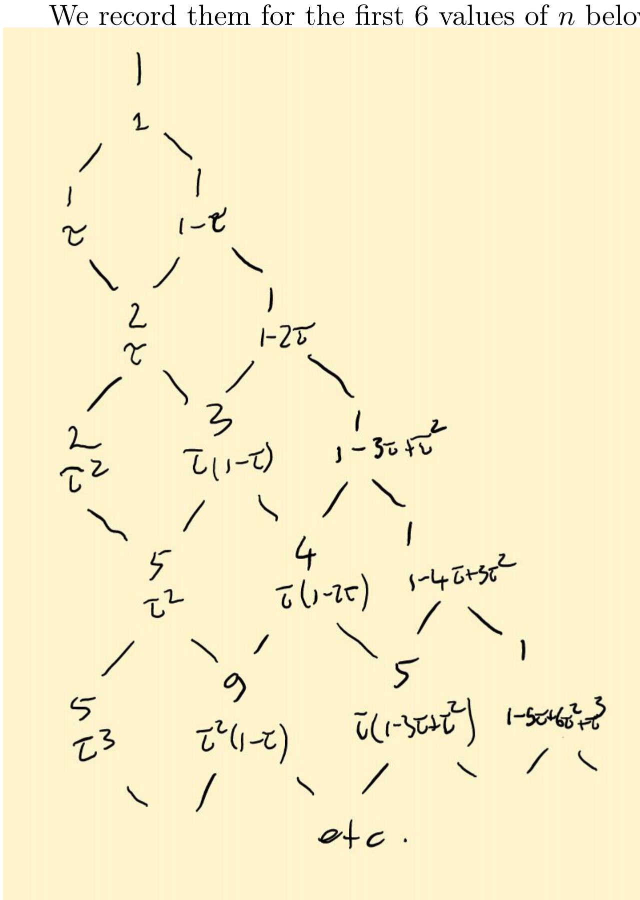
$$e_n f_n e_n = \frac{P_{n+1}}{P_n} f_{n-1} e_n$$

is useful to carry along. $\square$

We have thus completely calculated the $w_{n,n-2r}$. They are

$$w_{n,n-2r} = \tau^r P_{n+1-2r}(\tau)$$

We record them for the first 6 values of n below:



Solving the difference equation for P one readily gets

$$P_n(\tau) = \frac{s^n - s^{-n}}{(s - s^{-1})(s + s^{-1})^{n-1}}$$

where $\delta = (s + s^{-1})$ and $s^2 = t$ (so $\tau = \frac{1}{(s+s^{-1})^2}$).

7 First definition of the Jones polynomial, The Jones polynomial of torus knots.

7.1 Representations of the braid group into the TL algebra.

A similarity between the Temperley-Lieb presentation and that of the braid group will not have escaped the reader's notice. One can use this as follows:

Proposition 7.1. *If t is a complex number then if we define*

$$\psi_i = -te_i + (1 - e_i) \in TL_n(\tau)$$

for $i = 1, 2, ..., n-1$ then

$$\psi_i \psi_{i+1} \psi_i = \psi_{i+1} \psi_i \psi_{i+1} \quad \textit{iff} \quad \tau_{-1} = 2 + t + t^{-1}$$

.

We leave this easy computation to the reader.

Definition 7.1. *For t and ψ as above we define $\beta_t : B_n \to TL_n(\tau)$ by $\beta_t(\sigma_i) = \psi_i$.*

Now all the representations we have defined of TL become representations of the braid groups, with restriction rules exactly as those of the TL representations. We call them $\beta_{n,p}(= \pi_{n,p} \circ \beta_n$

The 1-dimensional representations $\beta_{n,n}$ send all the σ_i to 1.

The $n-1$ dimensional representations $\beta_{n,n-2}$ are the reduced Burau representations.

β_4 is faithful iff the Burau representation of B_4 is faithful.

7.2 First definition of the Jones polynomial.

Observe that for a braid $\alpha \in B_n$ the Markov trace of $\beta_n(\alpha)$ is invariant under the type I Markov move. It is obvious that it only changes by a simple factor under the second Markov move. Thus we have:

Definition 7.2. *If L is an oriented link realized as $\hat{\alpha}$ for $\alpha \in B_n$, the Jones polynomial of L is*

$$(-1)^{\textit{number of components}-1} \quad t^{\frac{wr(\hat{\alpha})}{2}} (\sqrt{t} + \frac{1}{\sqrt{t}})^{n-1} tr(\beta_n(\alpha))$$

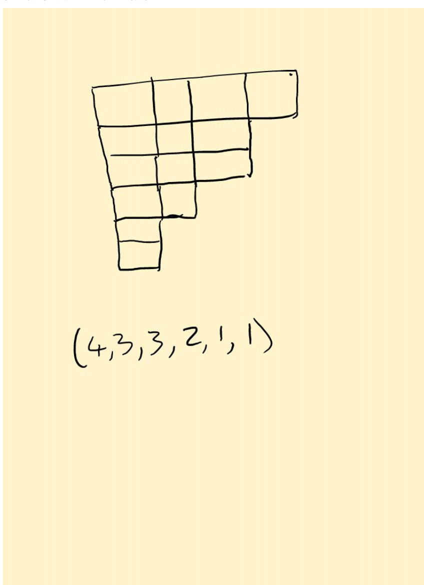
We need to see that the normalization of the trace is invariant under the type II Markov move. For this it suffices to calculate $tr(\beta_n(\sigma_i^{\pm 1}))$. We have $\beta_n(\sigma_1) = 1 - (1+t)e_1$ so $tr(\beta_n(\sigma_i)) = \frac{1}{1+t} = \frac{t^{-\frac{1}{2}}}{\sqrt{t}+\frac{1}{\sqrt{t}}}$ and $tr(\beta_n(\sigma_i^{-1}) = \frac{t^{\frac{1}{2}}}{\sqrt{t}+\frac{1}{\sqrt{t}}}$. So the normalization factor exactly cancels the change under the type II move and we have a link invariant.

To check that it is the same as the second definition, one has simply to check that the first definition satisfies the skein relation. This is true by focusing on a single crossing and turning the link into a braid without changing that crossing. Then the three members of the skein relation become closed braids and the skein relation is simple algebra since $t^{-1/2}\beta_n(\sigma_i) - t^{1/2}\beta(\sigma_i) = 1$ (or something).

7.3 Representations of S_n

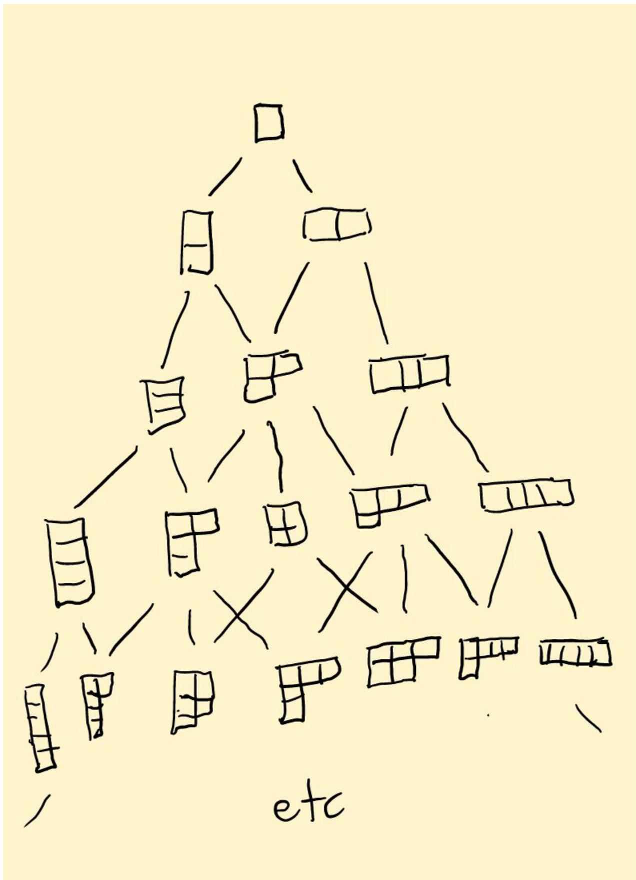
The group algebra of a finite group (over $\mathbb{C}$) is semisimple so any finite dimensional representation is a direct sum of irreducible ones.

The irreducible representations of S_n are indexed by Young diagrams-partitions of n written as rows of boxes aligned on the left, with non-increasing length as one goes down thus:



is the Young diagram for the partition $\{4, 3, 3, 2, 1, 1\}$ of 14.

The restriction to S_{n-1} of an irrep π_Y given by a Young diagram Y is the direct sum of all irreps $\pi_{Y'}$ for all Young diagrams Y' that can be formed from Y by removing a single box. Thus the Bratteli diagram for these representations (alternatively that of the group algebras of the S_n is the "Young's lattice":



In particular the dimension of the representation π_Y is the number of descending paths on the Young lattice beginning with the one box and ending with Y. These paths are called Young tableaux.

Conjugacy classes in S_n are also given by partitions and there is a rule, called the Murnaghan-Nakayama rule for calculating the character of π_Y in terms of ways of displaying the partition of a conjugacy class inside the Young diagram. There is also an elegant formula of Frobenius in terms of symmetric polynomials with as many variables as there are rows in the Young diagram.

The only result we will need, which is a very simple example of these character computations, is the following:

Proposition 7.2. *(a) If Y is the Young diagram with two rows, the first with $n-1$ boxes and the second with 1 box, and $g \in S_n$ is an n-cycle, then the character $trace(\pi_Y(g)) = -1$.*
(b) If Y is any other 2-row Young diagram then $trace(\pi_Y(g)) = 0$.

7.4 $\tau = \frac{1}{4}$

Our calculation of the Jones polynomial of torus knots involves identifying β_t when $t = 1$. We know that for this value the braid group representations factor through the symmetric group and since the e_i's can be recovered from the braid generators we know that TL is semisimple (since a group algebra always is) so that our analysis of its representations holds (small justification required here) so that the symmetric group representations obtained from the $\beta|n, p$ are irreducible and satisfy the same restriction rules as the irreps of TL. There are only two systems of Young diagrams that satisfy these restriction rules-ones with two rows and ones with two columns. This is a matter of convention so we may assume that *our S_n representations coming from TL at $\tau = \frac{1}{4}$ are the π_Y for Y a Young diagram with two rows.* Note that the dimensions necessarily agree.

(There is another way to see this via an action of TL_n on $\otimes^n \mathbb{C}^2$ whose commutant for $\tau = \frac{1}{4}$ is the algebra generated by the tensor product action of $SU(2)$, a special case of what is called "Schur-Weyl" duality.)

7.5 The Jones polynomial of a torus knot.

All the ingredients bar one are now in place for the calculation.

The main observation is that $\zeta = (\sigma_1\sigma_2\sigma_3...\sigma_{n-1})^n$ is in the centre of the braid group, hence in the centre of TL_n hence equal to a scalar in any irreducible representation such as $\beta_{n,p}$. We want to know what scalar it is and a determinant argument will give that to us. To get $det(\zeta)$ we need to know $det(\beta_{n,p}(\sigma_i)$. Since $\beta(\sigma) = te - (1 - e)$ it suffices to know the rank of the e_i on $K_{n,p}$. But if we return to the argument that gave minimal projections in $K_{n,p}$ from ones on $K_{n-2,p}$, we see that in fact the rank of e_n on Kn, p is exactly $dimK_{n-2,p}$ so equal to $\binom{n-2}{r-1} - \binom{n-2}{r-2}$.

Whatever it is, we have that $\beta_{n,p}(\zeta) = xid$. So taking an nth root of x we find a scalar so that $(y\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1}))^n = id$. Now if we put $\tau = \frac{1}{4}$ we have that the trace of $\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1})^b = 0$ for any relatively prime to n, except when $p = n - 2$ or $p = n$. Now the trace of the bth power of an n-cycle depends only on the dimensions of the eigenspaces of the various nth roots of unity and these vary continuously with τ. We conclude that:

The trace of $\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1})^b$ is zero unless $p = n$ or $p = n - 2$.

Thus to calculate the Markov trace of $\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1})^b$ we only need to figure out the trace in the two representations $\beta_{n,n}$ and $\beta_{n,n-2}$ and add them up with their weights! For $\beta_{n,n}$ this is trivial since $\beta_{n,n}(\sigma_i) = 1$ and the weight is $\frac{s^{n+1} - s^{-n-1}}{(s-s^{-1})(s+s^{-1})^n}$.

For $\beta_{n,n-2}$ the rank of the $\beta(e_i)$ is one so

$$det(\beta_{n,n-2}(\zeta) = t^{n(n-1)}$$

Looking explicitly at the representation (Burau) we see that $\beta_{n,n-2}(\zeta)$ must be a sign time t^n. But putting $t = 1$ that sign must be 1. We conclude that $t^{-1}\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1})$

is a root of unity varying continuously with t. By the deformation argument it has the same trace of as what it does when $t = 1$ so the trace of $(t^{-1}\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1}))^b$ is -1 and the trace of $\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1})^b$ is $-t^b$. The weight of the trace is $\tau P_{n-2}(\tau) = \frac{s^{n-1}-s^{-n+1}}{(s-s^{-1})(s+s^{-1})^n}$.

So altogether we get the Markov trace of $\beta_{n,p}(\sigma_1\sigma_2\sigma_3...\sigma_{n-1})^b$ to be

$$-s^{2b}\frac{s^{n-1}-s^{-n+1}}{(s-s^{-1})(s+s^{-1})^n} + \frac{s^{n+1}-s^{-n-1}}{(s-s^{-1})(s+s^{-1})^n}$$

The normalization factor to get the Jones polynomial is $-s^{b(n-1)}(s+s^{-1})^{n-1}$ (the minus sign coming from changing $s^2 - s^{-2}$) so altogether we get $V(s^2) = -\dfrac{s^{(n-1)(b-1)+n+1}}{1-s^4}(-s^{n-1+2b} + s^{-n+1+2b} + s^{n+1} - s^{-n-1})$ and, if p and q are relatively prime, the Jones polynomial of the (p, q) torus knot is

$$\frac{t^{\frac{(p-1)(q-1)}{2}}}{1-t^2}(1 - t^{p+1} - t^{q+1} + t^{p+q})$$

8 Other polynomials and their algebras.

9 More recent developments and approaches.

References