Runge Kutta methods are ways of finding numerical solutions to the ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial value $(x_0, y_0)$. The simplest such method is Euler's method where to approximate the solution $y$ at some value $x$ we divide the interval $[x_0, x]$ up into intervals of equal size $h$ (called the step size) and we define the sequence $(x_n, y_n)$, by

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hf(x_n, y_n).$$

We hope that as $h \to 0$, the calculated approximation to $f(x)$ will tend to the actual value of $f(x)$.

Thus in between $x_n$ and $x_{n+1}$ we are approximating the solution by a straight line whose slope is $f(x_n, y_n)$. The corresponding change in $y$ is $hf(x_n, y_n)$.

In the special case where $f(x, y)$ is independent of $y$, call it $f(x)$, the solution to the differential equation is $y = \int f(x)dx$. Thus Euler's method would be a numerical integration of the function $f(x)$. The value $hf(x_n)$ is the area of a rectangle which would occur if we were using the "left end-point" integration. Thus Euler's method generalises the left end-point rule, a very crude method.

Runge Kutta methods use different values of $f(x, y)$ for $x$ in the interval $[x_n, x_{n+1}]$ to come up with an approximation of $y_{n+1}$. The most well known of these appears to be *RK4* also known as *the* Runge-Kutta method.

To define RK4 all we have to do is specify how $y_{n+1}$ is obtained from $y_n$. For this purpose one defines (with $x_{n+1} = x_n + h$)

$k_1 = f(x_n, y_n)$
$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$
$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$
$k_4 = f(x_{n+1}, y_n + hk_3)$

EXERCISE 1: Draw a diagram of a direction field for $x$ between $x_n$ and $x_{n+1} = x_n + h$ to illustrate the meaning of the slopes $k_1, k_2, k_3, k_4$.

Then $y_{n+1}$ is defined to be

$$y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

In the case where $f(x, y)$ is independent of $y$ we see that $k_2 = k_3$ and that $RK4$ is nothing but Simpson's rule!

EXERCISE 2: Use RK4 with step size 0.1 to approximate the solution to $\frac{dy}{dx} = x + y$ with $x_0 = 0, y_0 = 1$ for $x = 0, 0.1, 0.2, 0.3$. Check that $y = 2e^x - x - 1$ is the actual solution to the ODE and compare the values with those calculated by RK4 and Euler's method (values calculated in the book, page 576).