Beginning Quantization

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1 classical mechanics and an easy quantization

This is not the time or place to trace the historical development of the theory of "quantization", nor even to give a complete picture of quantum mechanics. See [2] for a modern mathematical treatment of quantum mechanics, and [4] for a version better-suited for physics undergraduates. Rather, we will tell a highly simplified and anachronistic story in an attempt to motivate quantization for mathematicians. Nevertheless, the author is a mathematical physicist, and so we will begin our story with classical mechanics.

The theory of a classical point-like particle moving in one dimension is well-understood. Following Newton, let us suppose that the laws of physics comprise a second-order differential equation for the position of the particle. If the equations of motion are known, then the full evolution of the particle is determined by giving two numbers: its *position* x and its *momentum* p at time $t = 0$ — that the equations of motion are second-order means precisely that these two numbers describe the state. Let us then graph the evolution of the particle's state as motion on the x, p-plane.

We have not described yet how the coordinates x and p relate; their relationship is part of the equations of motion. Indeed, it was Hamilton who introduced the following formalism: We consider a function H on the x, p-plane, called the *Hamiltonian*, and study the motion determined by the coupled first-order differential equations $\dot{x} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial x$, where $\dot{x}, \dot{p}$ are the time-derivatives of x and p. When $H = \frac{1}{2m}p^2 + V(x)$, these equations of motion reduce to Newton's laws: we have $\dot{x} = p/m$, relating velocity with momentum, and $\ddot{x} = F(x)/m$, where $F(x) = -\partial V/\partial x$ is the (necessarily conservative) force field defined by the potential energy function V. Then to solve the differential equation is straightforward: one checks that the quantity H is conserved, and so motion is confined to level sets of H, which are generically one-dimensional curves in the x, p-plane, and for each such curve the second-order differential equation reduces to a first-order separable differential equation in one variable.

Let us describe the *Hamiltonian vector field* v_H given in coordinates by $v_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p}$ more invariantly (we follow the standard convention of writing the basic vector fields as partial differential operators; in this way we have identified vector fields and their associated derivations). It is clear that v_H depends linearly on the differential $dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial p}dp$. Indeed, $v_H = \pi \cdot dH$, where π is the *Poisson bivector* $\pi : dx \mapsto -\frac{\partial}{\partial p}$ and $dp \mapsto \frac{\partial}{\partial x}$. In terms of the canonical pairing $\frac{\partial}{\partial a} \cdot db = \delta_{ab}$ (the Kronecker delta), we see that $\pi = \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial x} - \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial p} = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x}$. This is the inverse bivector to the symplectic form $dp \wedge dx$.

The algebraic geometers say that we should consider the geometric x, p-plane as an algebraic space, associating to it its algebra of functions $\mathbb{K}[x, y]$, the ring of polynomials in two (commuting) variables. Then the Poisson bivector π becomes the *Poisson bracket* $\{,\} : \mathbb{K}[x,p] \otimes \mathbb{K}[x,p] \to \mathbb{K}[x,p]$, given by $\{f, g\} \stackrel{\text{def}}{=} \pi \cdot (df \otimes dg)$. Since π is antisymmetric, so is $\{,\}$: i.e. $\{f, g\} = -\{g, f\}$. Moreover, $\{f, -\}$ is the Hamiltonian vector field v_f, and so is a derivation: $\{f, gh\} = g\{f, h\} + \{f, g\}h$. One can also check that the bracket of functions acts as the bracket of vector fields:

$$\{\{f, g\}, -\} = [v_f, v_g] = v_f \circ v_g - v_g \circ v_f$$

which follows from the fact that $dp \wedge dx = \pi^{-1}$ is a closed two-form. Therefore $\{,\}$ satisfies the Jacobi identity.

Generalizing, we say that a *Poisson algebra* is an (associative, unital) algebra A along with a map $\{,\} : A \otimes A \to A$ satisfying:

antisymmetry $\{f, g\} = -\{g, f\}$

Jacobi $\{f, \{g, h\}\} = \{g, \{f, h\}\} + \{\{f, g\}, h\}$

Leibniz $\{f, gh\} = g\{f, h\} + \{f, g\}h$

The first two conditions make $A, \{,\}$ into a *Lie algebra*. We remark that the axioms for a Poisson structure are homogenous: if $\alpha \in \mathbb{K}$ and $\{,\}$ is a Poisson structure on A and algebra over $\mathbb{K}$, then $\alpha\{,\}$ is also a Poisson structure.

In our example above, we have $A = \mathbb{K}[x, p]$. By the Leibniz rule, it suffices to determine the bracket on generators: $\{p, x\} = 1$. The physics, then, is determined by the Poisson structure and a choice of Hamiltonian H: the observables x and p evolve by $\dot{a} = \{H, a\}$, and the Leibniz rules assure that this formula is true for any function in x and p.

Let A be any (unital, associative) algebra, and consider the bracket $[f, g] \stackrel{\text{def}}{=} fg - gf$. (In any algebra, we will reserve the bracket $[,]$ for this commutator.) One can check directly that $[,]$ defines on A the structure of a Poisson algebra. Of course, not every Poisson algebra arises in this way — the example above shows this. The process of "quantization" is an attempt to turn a generic Poisson algebra into one of this form, at least approximately.

Consider the associative unital algebra A generated by two elements x, p subject to the relation $[p, x] = 1$. Assuming that we are working over a field $\mathbb{K}$ of characteristic 0, let us also define the symmetric "multiplication" $(,) : A \otimes A \to A$ given by $(a, b) \stackrel{\text{def}}{=} \frac{1}{2}(ab + ba) = \frac{1}{4}\left((a+b)^2 - (a-b)^2\right)$; as a "multiplication", it is commutative but not necessarily associative. More generally, we define the symmetric n-linear multiplication

$$(a_1, \dots, a_n) \stackrel{\text{def}}{=} \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} \dots a_{\sigma(n)} = \frac{1}{n!\, 2^n} \sum_{\vec{\epsilon} \in \{\pm 1\}^n} \epsilon \left(\sum_{i=1}^n \epsilon_i a_i \right)^n$$

where $\epsilon = \prod_{i=1}^n \epsilon_i$, and S_n is the symmetric group of permutations of the numbers $1, \dots, n$. In particular, we define $() = 1$ and $(a) = a$, and $(a_1, \dots, a_n)$ is fixed under permutations. Then one can prove directly that A has a basis consisting of all symmetric products: the multiplication respects the filtration of A given by $A_{\leq 0} = \mathbb{K}$, $A_{\leq 1} = \mathbb{K} \oplus \mathbb{K}x \oplus \mathbb{K}p$, …, and the associated graded algebra $\operatorname{gr} A$ is precisely the polynomial ring $\mathbb{K}[x,p]$.

We can construct $\mathbb{K}[x,p]$ from the algebra A in the previous paragraph another way. Let $\hbar \in \mathbb{K} \setminus \{0\}$ and consider the (invertible, linear) map $A \to A$ given by $x \mapsto X = \hbar x$ and $p \mapsto P = \hbar p$, and more generally by $(a_1, \dots, a_n) \mapsto \hbar^n(a_1, \dots, a_n) = (\hbar a_1, \dots, \hbar a_n)$. This map is not an algebra homomorphism from A to itself unless $\hbar = 1$.

In the new basis, the algebra A is generated by X, P with defining relations $[P, X] = \hbar^2$.[1] Let us write $A_{\hbar^2}$ for the algebra generated by two elements X, P subject to $[P, X] = \hbar^2$. Then for $\hbar \neq 0$, we have $A_{\hbar^2} \cong A_1 = A$, but $A_0 \cong \mathbb{K}[X, P]$ is the commutative algebra. It is in this sense that $\mathbb{K}[X, P]$ is the *classical limit* of the algebra A.

Moreover, we can continue to pick up the Poisson structure on $\mathbb{K}[X, P]$ described in the opening discussion. Indeed, for $\hbar \neq 0$, consider $\{f, g\}_{\hbar^2} \overset{\text{def}}{=} \frac{1}{\hbar^2}[f, g]$. The axioms for a Poisson structure are homogeneous, and so $\{,\}_{\hbar^2}$ is a Poisson structure on $A_{\hbar^2}$, since $[,]$ is. But $\{X, P\}_\hbar = 1$, and so in the limit $\hbar \to 0$, we see by the Leibniz rule that the Poisson structure $\{,\}_0$ on $A_0 \cong \mathbb{K}[X, P]$ is precisely the one defined in the discussion of physics.

Let us investigate the algebra $A_{\hbar^2}$ more deeply. Rather than letting $\hbar$ represent a fixed scalar, let us consider it as a variable, so that $A_{\hbar^2}$ is generated by the three elements X, P, and $\hbar$. In fact, since $[P, X] = \hbar^2$, it's clear that $A_{\hbar^2}$ includes a subalgebra generated by X, P, and $H = \hbar^2$. This subalgebra can be presented by the relations $[P, X] = H$ and $[H, X] = 0 = [H, P]$; it is thus precisely the universal enveloping algebra $\mathcal{U}\mathfrak{h}$, where $\mathfrak{h}$ is the three-dimensional *Heisenberg* Lie algebra. In $\mathcal{U}\mathfrak{h}$, H is central, and so for any element $\hbar \in \mathbb{K}$ we can quotient by the ideal generated by $H - \hbar$. Doing so yields back our algebra $A_\hbar$. We have replaced $\hbar^2 \mapsto \hbar$ in the algebra $A_{\hbar^2} \mapsto A_\hbar$. The Poisson bracket is $\{,\}_\hbar = \frac{1}{\hbar}[,]$ and $\{,\}_0 = \lim_{\hbar \to 0}\{,\}_\hbar$.

The rescaling trick we used to define $A_\hbar$ we can use for any universal enveloping algebra. Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[,]$, and for $\hbar \in \mathbb{K}$ consider the algebra $\mathfrak{g}_\hbar$ with Lie bracket $[,]_\hbar = \hbar[,]$. If $\hbar \neq 0$, then $\mathfrak{g}_\hbar \cong \mathfrak{g}$ by $x \mapsto x/\hbar$. In particular, $\mathcal{U}\mathfrak{g} \cong \mathcal{U}\mathfrak{g}_\hbar$ for $\hbar \neq 0$. But as $\hbar \to 0$, we have $\mathcal{U}\mathfrak{g}_\hbar \to \mathcal{S}\mathfrak{g}$, where $\mathcal{S}\mathfrak{g}$ is the symmetric algebra over the vector space $\mathfrak{g}$, i.e. the (commutative) polynomial algebra in a basis of $\mathfrak{g}$. The bracket $\{,\} = \frac{1}{\hbar}[,]_\hbar$ continues to a Poisson bracket on $\mathcal{S}\mathfrak{g}$ as $\hbar \to 0$; it is the same as the extension of the bracket $\{,\} = [,]$ on $\mathfrak{g}$ to all of $\mathcal{S}\mathfrak{g}$ by the Leibniz rule. This Poisson structure goes by the name *Lie-Kirollov-Kostant*. The symmetric algebra $\mathcal{S}\mathfrak{g}$ is the algebra of (polynomial) functions on the dual vector space $\mathfrak{g}^*$, and the Poisson bracket $\{,\}$ can be defined in terms of a (non-constant) bivector field on $\mathfrak{g}^*$.

Conversely, we can consider $\mathfrak{g}_\hbar$ as a Lie algebra over $\mathbb{K}[\hbar]$, so that $\hbar$ is a formal variable, and then $\mathcal{U}\mathfrak{g}_\hbar$ is an associative algebra over $\mathbb{K}[\hbar]$. Then the 0th-order part of $\mathcal{U}\mathfrak{g}_\hbar$ is precisely the symmetric algebra $\mathcal{S}\mathfrak{g}$, and is commutative to 0th order. The first-order part — namely $\{x, y\} = \frac{1}{\hbar}[x, y] \bmod \hbar$ — is just the Poisson bracket $\{,\}_\mathfrak{g}$ on $\mathcal{S}\mathfrak{g}$ induced by the Lie bracket $[,]$ on $\mathfrak{g}$. In this way, $\mathcal{U}\mathfrak{g}_\hbar \cong \mathcal{U}\mathfrak{g}[\hbar]$ is a *deformation* of $\mathcal{S}\mathfrak{g}$ in the direction of the Poisson bracket $\{,\}_\mathfrak{g}$. For any vector space V over $\mathbb{K}$, we will write $V[\hbar]$ for the vector space (free module) $V \otimes_\mathbb{K} \mathbb{K}[\hbar]$ over $\mathbb{K}[\hbar]$, and will will similarly write $V[[\hbar]]$ for $V \otimes \mathbb{K}[[\hbar]]$, where $\mathbb{K}[[\hbar]]$ is the ring of formal power series in $\hbar$. The

[1] We could have considered a slightly different map, which sense $x \mapsto x$ and $p \mapsto \hbar p$, extended to the whole algebra by the symmetric multiplication. This would have yielded the more physically significant formula $[P, X] = \hbar$. Of course, when $\mathbb{K} = \mathbb{C}$, we could just as easily use $p \mapsto i\hbar p$, which gives the standard relations $[P, X] = i\hbar$.

algebra isomorphism $\mathcal{U}\mathfrak{g}_\hbar \cong \mathcal{U}\mathfrak{g}[\hbar]$ is given by $x \mapsto \hbar x$ for $x \in \mathfrak{g}$.

The remainder of these notes will discuss quantization of Poisson Lie groups and Lie bialgebras, rather than physics. We return to the physics discussion to mention that in any Poisson algebra $A, \{,\}$, a choice of Hamiltonian element $H \in A$ gives a notion of physics by $\dot{a} = \{H, a\}$. In quantum mechanics of a point particle moving in n dimensions, we have $\mathbb{K} = \mathbb{C}$, and the Poisson algebra normally taken is generated by x^a, p_a, $a = 1, \dots, n$ with $[p_a, x^b] = \delta_a^b i\hbar$ and $[p_a, p_b] = 0 = [x^a, x^b]$, and then completed; the Poisson structure is $\{,\} = \frac{1}{i\hbar}[,]$, where $\hbar$ is a fixed constant. This algebra is naturally a subalgebra of the algebra of unbounded operators on the Hilbert space $\mathscr{L}^2(\mathbb{R}^n)$, where x^a acts by multiplication by the coordinate function $x^a \in \mathscr{L}^2(\mathbb{R}^n)$ and p_a acts by the partial derivative $\frac{\partial}{\partial x^a}$. In particular, the commutative but non-associative subalgebra consisting of all self-adjoint (unbounded) operators (with multiplication given by the symmetric product $(,)$ described earlier) is closed under the Poisson bracket, and comprises the algebra of *observables*.

2 groups

We turn our attention to a very important geometrical object, the Lie group.

In the previous section, we explained how the Poisson bivector field on the plane gave a Poisson structure on the algebra of functions on the plane. Let us take this as the definition of a Poisson bivector field: a manifold M is *Poisson* if it comes equipped with a bivector field π so that the map $\{f, g\} \stackrel{\text{def}}{=} \pi \cdot (df \otimes dg)$ is a Poisson structure on the algebra of functions on the manifold. A smooth map of manifolds is *Poisson* if the induced map on the algebras of functions is in fact a homomorphism of Poisson algebras. We were motivated to study Poisson structures because they provide the necessary structure with which to define physics.

The *product of Poisson manifolds* is easy: $T_{m_1,m_2}(M_1 \times M_2) = T_{m_1}M_1 \oplus T_{m_2}M_2$, so $\bigwedge^2 T_{m_1,m_2}(M_1 \times M_2) = \bigwedge^2 T_{m_1}M_1 \oplus \bigwedge^2 T_{m_2}M_2 \oplus (T_{m_1}M_1 \otimes T_{m_2}M_2)$, and we take the bivector on the product to be the sum of the bivectors living in the first two components. On functions, this corresponds to the *tensor product of Poisson algebras*: on the algebra $A_1 \otimes A_2$ we take the Poisson structure $\{a_1 \otimes a_2, b_1 \otimes b_2\} = a_1 b_1 \otimes \{a_2, b_2\} + \{a_1, b_1\} \otimes a_2 b_2$.

Recall that a group object in a category with products is an object G along with maps $m : G \times G \to G$, $i : G \to G$, and $e : \{\text{pt}\} \to G$ such that some diagrams commute. We spell this out in the category of Poisson manifolds, assuming that we already understand Lie groups.

The tangent bundle to a Lie group is trivializable by right translations, and under this trivialization a section π of $\bigwedge^2 TG$ becomes a map $\pi : G \to \bigwedge^2 \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. That the multiplication map, then, is Poisson is equivalent to the condition that $\pi(gh) = \pi(g) + \text{Ad}_g^{\otimes 2} \pi(h)$, where $\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the adjoint action of $g \in G$. Then $\pi(e \cdot e) = \pi(e) + \pi(e)$, and so $\pi(e) = 0$. In particular, $d\pi : \mathfrak{g} \to T \bigwedge^2 \mathfrak{g} = \bigwedge^2 \mathfrak{g}$ is an antisymmetric map. The Jacobi identity for $\{,\}$ induces the Jacobi identity for $d\pi$, and so $d\pi$ is a Lie cobracket on $\mathfrak{g}$. Moreover, by expanding $\pi(ghg^{-1}) = \text{Ad}_{gh}^{\otimes 2} \pi(g^{-1}) + \text{Ad}_g^{\otimes 2} \pi(h) + \pi(g)$ and differentiating at the identity, we get the cocycle condition. Thus $d\pi$ is necessarily a Lie bialgebra structure for $\mathfrak{g}$.

We give another argument, which also proves a converse statement. Let G by a Lie group with Lie algebra $\mathfrak{g}$, and consider the group $\bigwedge^2 \mathfrak{g} \rtimes G$, with the multiplication given by $(x, g) \cdot (y, h) = (x + \text{Ad}_g^{\otimes 2} y, gh)$. Then a map $(\pi, \text{id}) : G \to \bigwedge^2 \mathfrak{g} \rtimes G$ that is the identity in the second component is a group homomorphism precisely if π satisfies $\pi(gh) = \pi(g) + \mathrm{Ad}_g^{\otimes 2}\,\pi(h)$. Let us call a Lie group G with a map $\pi : G \to \bigwedge^2 \mathfrak{g}$ satisfying $\pi(gh) = \pi(g) + \mathrm{Ad}_g^{\otimes 2}\,\pi(h)$ is a *quasiPoisson Lie group*. What is the Lie algebra $\mathrm{Lie}\big(\bigwedge^2 \mathfrak{g} \rtimes G\big)$? It is $\mathrm{Lie}\big(\bigwedge^2 \mathfrak{g}\big) \rtimes \mathfrak{g}$, where $\mathrm{Lie}\big(\bigwedge^2 \mathfrak{g}\big)$ is the trivial Lie algebra with dimension $\dim \bigwedge^2 \mathfrak{g}$, and the action gives the bracket $\big[(x,a),(y,b)\big] = \big([a\otimes 1 + 1\otimes a, y] + [x, b\otimes 1 + 1\otimes b], [a,b]\big)$. This proves that if π is a quasiPoisson Lie structure on G then $d\pi$ satisfies the cocycle condition and also that any Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Lie}\big(\bigwedge^2 \mathfrak{g} \rtimes G\big)$ that is the identity in the second component satisfies the cocycle condition. But any such homomorphism lifts to a group homomorphism from the simply connected cover of G. This proves that for a connected simply connected Lie group the notions of quasiPoisson structure on the group and Lie quasibialgebra structure on the Lie algebra are equivalent. Moreover, let $[,]_{\mathfrak{g}^*} : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ be the dual to the map $\delta = d\pi$, and let $\{,\} : C(G) \otimes C(G) \to C(G)$ be $(f_1, f_2) \mapsto \pi \cdot (df_1 \otimes df_2)$. Then one can check directly that $[df_1, df_2]_{\mathfrak{g}^*} = d\{f_1, f_2\}$, and so the quasibialgebra structure $d\pi$ is in fact a Lie bialgebra structure, i.e. it satisfies the coJacobi identity, if and only if π is an honest Poisson Lie structure on the Lie group G.

We have seen that the notions of connected simply-connected Poisson Lie group and of finite-dimensional Lie bialgebra are equivalent, just as the notions of connected simply-connected Lie group and of finite-dimensional Lie algebra are equivalent.

Recall the algebraic geometers' dictionary: to a space (affine algebraic) we associate its algebra of functions (finitely-generated over $\mathbb{C}$). Then to a Lie group we associate a commutative Hopf algebra. Remember that the tensor product of commutative Poisson algebras is given by the Poisson structure $\{f_1 \otimes f_2, g_1 \otimes g_2\} = \{f_1, g_1\} \otimes f_2 g_2 + f_1 g_1 \otimes \{f_2, g_2\}$.[2] Then it is natural to define a Poisson Hopf algebra as being the algebraic structure so that a commutative Poisson Hopf algebra is precisely the algebra of functions of a Poisson Lie (algebraic) group. Namely, we demand that the antipode, comultiplication, and counit maps be homomorphisms of Poisson algebras — to the ground field we assign the trivial Poisson structure.

Let G be an algebraic group with Lie group $\mathfrak{g}$; then we have a commutative Hopf algebra $\mathcal{C}(G)$ of the polynomial functions on G and a cocommutative Hopf algebra $\mathcal{U}\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$. Recall that $\mathcal{U}\mathfrak{g} \curvearrowright \mathcal{C}(G)$ by left-invariant differential operators, and the action and then evaluation $\mathrm{ev}_e : \mathcal{C}(G) \to \mathbb{K}$ defines a Hopf pairing $\cdot : \mathcal{U}\mathfrak{g} \otimes \mathcal{C}(G) \to \mathbb{K}$; it defines an embedding $\mathcal{U}\mathfrak{g} \hookrightarrow \big(\mathcal{C}(G)\big)^*$, and the embedding the other way depends on whether G is connected ($\mathcal{C}(U) \hookrightarrow (\mathcal{U}\mathfrak{g})^*$ where U is a small neighborhood of the identity). That it is a Hopf follows from the Leibniz rule — $x \cdot fg = \Delta x \cdot (f \otimes g)$ for $x \in \mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$ and $f, g \in \mathcal{C}(G)$ — and the invariance. If G moreover is Poisson, then we can try to define a coPoisson ρ structure on $\mathcal{U}\mathfrak{g}$ such that for $x \in \mathcal{U}\mathfrak{g}$,

[2] This is not a great definition for noncommutative Poisson algebras. If A_1, A_2 are noncommutative with Poisson structures $\{,\} = [,]$, then the Poisson structure on $A_1 \otimes A_2$ is not $[a_1 \otimes a_2, b_1 \otimes b_2] = a_1 b_2 \otimes [a_2, b_2] + [a_1, b_1] \otimes b_2 a_2$ — notice the switch in the last line. Indeed, the tensor product of noncommutative Poisson algebras is not a Poisson algebra with the proposed tensor structure — the Leibniz rule is not satisfied. Thus an amended tensor product is necessary for noncommutative spaces.

Indeed, the proposed tensor product of noncommutative Poisson algebras is not even antisymmetric. A better tensor product is:

$$\{a_1 \otimes a_2, b_1 \otimes b_2\} \stackrel{\text{def}}{=} \frac{a_1 b_1 + b_1 a_1}{2} \otimes \{a_2, b_2\} + \{a_1, b_1\} \otimes \frac{a_2 b_2 + b_2 a_2}{2}$$

This still fails to satisfy the Leibniz rule, though.

$\rho x \cdot (f \otimes g) = x \cdot \{f, g\}$.

In fact, we can define a coPoisson structure on $\mathcal{U}\mathfrak{g}$ out of a Lie bialgebra structure on $\mathfrak{g}$. The condition of a (commutative) Poisson Hopf algebra is that $\Delta\{a, b\} = \{\Delta a, \Delta b\}$, where remember that we define $\{a_1 \otimes a_2, b_1 \otimes b_2\} = \{a_1, b_1\} \otimes a_2 b_2 + a_1 b_1 \otimes \{a_2, b_2\}$. In fact, this is a "cycle" condition in some Hochschild homology theory. Then the dual condition on a coPoisson structure $\rho : A \to A \otimes A$ is that $\rho(ab) = \Delta(a)\rho(b) + \rho(a)\Delta(b)$, where the multiplication as always is $(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$. This is precisely the condition that ρ be a 1-cocycle in the Hochschild cohomology complex of A with coefficients in $A \otimes A$, where $A \curvearrowright A \otimes A$ by the comultiplication: $a \cdot (b_1 \otimes b_2) = \Delta(a)\,(b_1 \otimes b_2)$. We call a (cocommutative) bialgebra A with such a 1-cocycle a *quasi-coPoisson bialgebra*; it is a *coPoisson bialgebra* if ρ also satisfies the Jacobi identity, and a (quasi-)*coPoisson Hopf algebra* if ρ respects the antipode. We should also demand some sort of *coLeibniz rule*, namely that $\Delta_{12}\rho = \rho_{23}\Delta + \rho_{13}\Delta$.

Then given a Lie bialgebra structure $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, we define a coPoisson structure on $\mathcal{U}\mathfrak{g}$ by $\rho(1) = 0$, $\rho(x) = \delta(x)$ for $x \in \mathfrak{g}$ and inductively by the cocycle rule $\rho(ab) = \Delta(a)\rho(b) + \rho(a)\Delta(b)$. It is antisymmetric by induction, using the cocommutativity. We check that this is well-defined: $\rho(xy - yx) = \rho x \Delta y + \Delta x \rho y - \rho y \Delta x - \Delta y \rho x = [\rho x, \Delta y] + [\Delta x, \rho y]$, and if $x, y \in \mathfrak{g}$, this is $[\delta x, \Delta y] + [\Delta x, \delta y] = \delta[x, y]$ by the cocycle condition for the Lie bialgebra structure on δ. This proves that ρ is well-defined on $\mathcal{U}\mathfrak{g}_{\leq 2}$, and we continue by induction. Thus, a Lie quasibialgebra structure on $\mathfrak{g}$ defines a unique Hopf quasicoPoisson structure on $\mathcal{U}\mathfrak{g}$ that restricts correctly. If δ satisfies coJacobi, then of course ρ does on $\mathcal{U}\mathfrak{g}_{\leq 1} = \mathbb{K} \oplus \mathfrak{g}$; higher degrees follow by induction.

So we have seen that a Lie bialgebra structure on $\mathfrak{g}$ induces on $\mathcal{U}\mathfrak{g}$ the structure of a (cocommutative) coPoisson Hopf algebra and if G is connected, simply connected, then $\mathcal{C}(G)$ is a (commutative) Poisson Hopf algebra. By construction the coPoisson and Poisson structures pair correctly for $x \in \mathfrak{g}$ — $\rho x \cdot (f \otimes g) = x \cdot \{f, g\}$ — and by the multiplicativity formulas and induction, this extends to higher degrees. Thus $\mathcal{U}\mathfrak{g}$ and $\mathcal{C}(G)$ are very much a dual pair. If we are hoping to quantize $\mathcal{C}(G)$, we may instead decide to quantize $\mathcal{U}\mathfrak{g}$ — this time by deforming the comultiplication.

Before we sketch this program of quantization, we mention a few more coincidences. Any Lie bialgebra $\mathfrak{g}$ has a dual $\mathfrak{g}^*$, and this has a simply-connected cover G^*. In any Poisson manifold M, the Poisson bivector π defines a map $\Omega^1(M) \to \mathrm{Vect}(M)$, and one can pull the canonical bracket of vector fields back along this map to get a bracket on Ω^1; the bracket is given by $[\omega, \omega'] \stackrel{\text{def}}{=} d\pi(\omega \wedge \omega') + \pi\omega(d\omega') - \pi\omega'(d\omega)$, and its existence is a theorem of Koszul. Then if M is a Poisson Lie group G, then the natural mapping $\mathfrak{g}^* \to \Omega^1(G)$ into the left-invariant vector fields is in fact a Lie algebra isomorphism; this is a theorem of Weinstein's [5]. Combining the map $\mathfrak{g}^* \to \Omega^1(G)$ with $\pi : \Omega^1(G) \to \mathrm{Vect}(G)$ gives a Lie algebra map $\mathfrak{g} \to \mathrm{Vect}(G)$ and thus a local action $G^* \curvearrowright G$. In a similar way we have an action $G \curvearrowright G^*$.

In any case, from a Lie bialgebra $\mathfrak{g}$ we can construct two commutative Poisson Hopf algebras $\mathcal{C}(G)$ and $\mathcal{C}(G^*)$, and two cocommutative coPoisson Hopf algebra $\mathcal{U}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}^*$. Then $\mathcal{C}(G)$ and $\mathcal{U}\mathfrak{g}$ are a dual pair of (co)Poisson Hopf algebras, as are $\mathcal{C}(G^*)$ and $\mathcal{U}\mathfrak{g}^*$. We will quantize each, preserving these pairings. If we are lucky, we achieve $\mathcal{C}_q(G^*) \cong \mathcal{U}_q\mathfrak{g}$ and $\mathcal{C}_q(G) \cong \mathcal{U}_g\mathfrak{g}^*$, where for now the q just denotes "quantized". In fact, we will be lucky, at least algebraically — the different Hopf algebras will have different topologies.

3 example: quantized borel

Rather than explaining this here, we work an example. We begin by classifying the two-dimensional Lie bialgebras.

Recall that up to a change of basis, a two-dimensional Lie algebra $\mathfrak{g}$ is either the commutative Lie algebra $\langle x, y : [x, y] = 0\rangle$ or the non-commutative $\langle x, y : [x, y] = y\rangle$. The proof is straightforward: in d dimensions, the bracket assigns a d-dimensional vector to each of the $\binom{d}{2}$ pairs of basis vectors, so in $d = 2$ dimensions the bracket consists of two numbers; the Jacobi identity is trivial if any of the three inputs agree, so it consists of $\binom{d}{3}$ vector equations, but when $d = 2$, $\binom{d}{3} = 0$. Thus in $d = 2$ either the algebra $\mathfrak{g}$ is abelian or the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is one-dimensional. We then pick our basis to include one element $y \in \mathfrak{g}'$ and normalize the other element so that $[x, y] = y$. We remark that this presentation does not completely determine the basis: any rescaling $y \to \alpha y$ for $\alpha \in \mathbb{K}$ is an automorphism of $\mathfrak{g}$.

If $\mathfrak{g}$ is the abelian two-dimensional Lie algebra, then any Lie coalgebra structure on $\mathfrak{g}$ is a Lie bialgebra structure — the cocycle identity is trivially satisfied. We are still free to pick the basis in this case — we did not have to fix a basis to present the abelian algebra as $\langle x, y : [x, y] = 0\rangle$ — and so we can do so now: if $\mathfrak{g}$ is not cocommutative, then we take the coalgebra to be $\delta x = 0$, $\delta y = x \otimes y - y \otimes x = x \wedge y$. Then $\mathcal{U}\mathfrak{g} \cong \mathcal{S}\mathfrak{g}$ is the symmetric algebra, and $\mathcal{U}\mathfrak{g}^* = \langle x^*, y^* : [x^*, y^*] = y^*\rangle$, where we have taken $\{x^*, y^*\}$ to be the basis of $\mathfrak{g}^*$ dual to the basis $\{x, y\}$ of $\mathfrak{g}$. The pairing $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^*$ given by $x^n y^m \otimes (x^*)^{n'} (y^*)^{m'} \mapsto \delta_{n,n'} \delta_{m,m'} n! m!$ defines on $\mathcal{U}\mathfrak{g}$ the structure of a commutative bialgebra and on $\mathcal{U}\mathfrak{g}^*$ the structure of a cocommutative bialgbera — on $\mathcal{U}\mathfrak{g}^*$ this structure is the normal Hopf structure. Of course, there was nothing special about two dimensions here: in any bialgebra with one side commutative, one of the two universal enveloping algebras $\mathcal{U}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}^*$ is commutative and is precisely the polynomial algebra $\mathbb{K}[x_1, \dots, x_d]$ where $d = \dim \mathfrak{g}$, and the pairing $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^*$ endows the noncommutative algebra with (the standard) cocommutative Hopf structure.

Let us then classify the nonabelian noncoabelian Lie bialgebras in two dimensions. We have a bracket on $\mathfrak{g}$ given by $[x, y] = y$, and we let $\{x^*, y^*\}$ be the basis of $\mathfrak{g}^*$ dual to $\{x, y\}$. Then any choice $[x^*, y^*] = \alpha x^* + \beta y^*$ defines a bracket on $\mathfrak{g}^*$, the Jacobi identity being trivial in this case. One can check by hand that the cocycle identity is trivially satisfied. We still have two degrees of freedom with which to pick our basis: any map of the form $x \mapsto x + ay$ and $y \mapsto by$ is an automorpihsm of $\mathfrak{g}$. A matrix in $GL(\mathfrak{g})$ acts by its transpose inverse on $GL(\mathfrak{g}^*)$: the change of basis $\{x \mapsto x + ay, y \mapsto by\}$ corresponds to $\{x^* \mapsto x^*; y^* \mapsto -ab^{-1}x^* + b^{-1}y^*\}$. Thus, if $\beta \neq 0$, we can find a change of basis $x^* \mapsto x^*$, $y^* \mapsto \alpha x^* + \beta y^*$, and then $\mathfrak{g}^*$ is $[x^*, y^*] = \beta y^*$. If $\beta = 0$, we cannot get rid of α, but we can set it equal to 1. We have shown that up to a change of basis there is a one-parameter family of Lie bialgebras given by $[x, y] = y$, $[x^*, y^*] = \beta y^*$, and an extra one given by $[x, y] = y$, $[x^*, y^*] = x^*$.

We turn now to the universal enveloping algebras. We now rewrite the defining relations: $\mathcal{U}\mathfrak{g} = \langle x, y : yx = (x - 1)y\rangle$ and either $\mathcal{U}\mathfrak{g}^* = \langle x^*, y^* : y^*x^* = x^*(y^* - 1)\rangle$ or $\mathcal{U}\mathfrak{g}^* = \langle x^*, y^* : y^*x^* = (x^* - \beta)y^*\rangle$. Let us pick a PBW basis for each, so that the basis of $\mathcal{U}\mathfrak{g}$ is $\{x^n y^m\}$ and that of $\mathcal{U}\mathfrak{g}^*$ is $\{x^{*n} y^{*m}\}$. We will try to construct a pairing $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^* \to \mathbb{K}$ so that the the multiplication on $\mathcal{U}\mathfrak{g}^*$ pulls back to a (topological) comultiplication on $\mathcal{U}\mathfrak{g}$ that makes the pair into a bialgebra, and that restricts to the pairing $\mathfrak{g} \otimes \mathfrak{g}^* \to \mathbb{K}$ on $\mathfrak{g}, \mathfrak{g}^* \hookrightarrow \mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}^*$. We also demand that $\mathbb{K} \hookrightarrow \mathcal{U}\mathfrak{g}$ pairs only with $\mathbb{K} \hookrightarrow \mathcal{U}\mathfrak{g}^*$. We define the matrix coefficients of the multiplication

μ on $\mathcal{U}\mathfrak{g}$ to be given by $x^n y^m x^{n'} y^{n'} = \sum_{N,M} \mu_{N,M}^{n,m;n',m'} x^N y^M$; then one immediately computes $\mu_{N,M}^{n,m;n',m'} = \delta_M^{m+m'} \binom{n'}{N-n} (-m)^{n'+n-N}$. We list the values for some small N, M, since we want to guess the adjoint matrix:

$$\begin{aligned}
\mu_{N,M}^{n,m;n',m'} &= \delta_M^{m+m'} \binom{n'}{N-n} (-m)^{n'+n-N} \\
\mu_{N,0}^{n,m;n',m'} &= \delta_0^m \delta_0^{m'} \delta_N^{n'+n} \\
\mu_{0,M}^{n,m;n',m'} &= \delta_M^{m+m'} \delta_0^n (-m)^{n'} \\
\mu_{1,0}^{n,m;n',m'} &= \delta_0^m \delta_0^{m'} \left(\delta_0^n \delta_1^{n'} + \delta_1^n \delta_0^{n'} \right) \\
\mu_{0,1}^{n,m;n',m'} &= \delta_0^m \delta_0^n \delta_0^{n'} \delta_1^{m'} + \delta_1^m \delta_0^n \delta_0^{m'} (-1)^{n'}
\end{aligned}$$

In particular, let's demand that $x \in \mathcal{U}\mathfrak{g}$ pair only with $x^* \in \mathcal{U}\mathfrak{g}^*$ and that $y \in \mathcal{U}\mathfrak{g}$ pair only with $y^* \in \mathcal{U}\mathfrak{g}^*$. Then μ pulls back to a matrix $\mu^* : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ with

$$\begin{aligned}
\mu^*(x^*) &= 1 \otimes x^* + x^* \otimes 1 \\
\mu^*(y^*) &= 1 \otimes y^* + y^* \otimes \sum_{n'=0}^{\infty} (-1)^{n'} (x^n)^*
\end{aligned}$$

where $(x^n)^*$ is whatever is the dual basis element of $\mathcal{U}\mathfrak{g}^*$ dual to $x^n \in \mathcal{U}\mathfrak{g}$. More generally,

$$\begin{aligned}
\mu^*\big((x^N)^*\big) &= \sum_{n=0}^{N} \left[(x^n)^* \otimes (x^{N-n})^* \right] \\
\mu^*\big((y^M)^*\big) &= \sum_{m=0}^{M} \left[(y^m)^* \otimes \left(\sum_{n'=0}^{\infty} (-m)^{n'} (x^{n'})^* \right) (y^{m'})^* \right]
\end{aligned}$$

Let us guess that x^N pairs only with $(x^*)^N$, but perhaps not to unity — if one of $\mathfrak{g}$ or $\mathfrak{g}^*$ were abelian, the pairing would be $N!$. So we suppose that $(x^N)^* = (x^*)^N / [N]_x!$, where $[N]_x! \in \mathbb{K}$ is the pairing of x^N with $(x^*)^N$, and we do not currently read more meaning into the single symbol $[N]_x!$. Let us make a similar guess that $(y^M)^* = (y^*)^M / [M]_y!$. If we demand that μ^* be multiplicative, then we see that

$$\begin{aligned}
\sum_{n=0}^{N} \frac{N!}{n!\,(N-n)!} (x^*)^n \otimes (x^*)^{N-n} &= \big(\mu^*(x^*)\big)^N = [N]_x! \mu^*\big((x^N)^*\big) \\
&= [N]_x! \sum_{n=0}^{N} (x^n)^* \otimes (x^{N-n})^* = \sum_{n=0}^{N} \frac{[N]_x!}{[n]_x!\,[N-n]_x!} (x^*)^n \otimes (x^*)^{N-n}
\end{aligned}$$

Then $[0]_x! = 1 = [1]_x!$, and we proceed by induction to conclude that $[n]_x! = n!$. We see that by choosing $(x^N)^* = (x^*)^N / [N]_x!$ we force almost everything, and in particular

$$\begin{aligned}
\mu^*(y^*) &= 1 \otimes y^* + y^* \otimes e^{-x^*} \\
\mu^*\big((y^M)^*\big) &= \sum_{m=0}^{M} \left[(y^m)^* \otimes e^{-mx^*} (y^{m'})^* \right]
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}\sum_{m=0}^{M}\frac{[M]_y!}{[m]_y!\,[M-m]_y!}(y^*)^m\otimes e^{-mx^*}(y^*)^{M-m} &= [M]_y!\sum_{m=0}^{M}(y^m)^*\otimes e^{-mx^*}(y^{M-m})^* \\ &= [M_y]!\mu^*\big((y^M)^*\big) = \mu^*\big((y^*)^M\big) \\ &= \big(\mu^*(y^*)\big)^M = \big(1\otimes y^* + y^*\otimes e^{-x^*}\big)^M \\ &= \sum_{m=0}^{M}(y^*)^m\otimes \sum_{\substack{\text{subsets } S\subseteq \mathbf{M}\\ \text{s.t. } |S|=m}}(e^{-x^*})^S(y^*)^{\mathbf{M}\smallsetminus S}\end{aligned}$$

We have introduced the notation $\mathbf{M}=\{1,\dots,M\}$ and $a^S b^{M\smallsetminus S}=\prod_{i=1}^M s_i$, where $s_i=a$ if $i\in S$ and $s_i=b$ if $i\in\mathbf{M}\smallsetminus S$. Equivalently,

$$\sum_{S\subseteq\mathbf{M},|S|=m}(e^{-x^*})^S(y^*)^{\mathbf{M}\smallsetminus S} = \binom{M}{m}(\underbrace{e^{-x^*},\dots,e^{-x^*}}_{m},\underbrace{y^*,\dots,y^*}_{M-m})$$

is the symmetric product of an earlier section.

Everything we've done so far did not depend on the bialgebra structure. We now focus on the case when $[x^*,y^*]=x^*$. Then $[e^{-x^*},y^*]=-x^*e^{-x^*}$ by the chain rule, using the fact that $[-,y^*]$ is a derivation. In particular, $\sum_{S\subseteq\mathbf{M},|S|=m}(e^{-x^*})^S(y^*)^{\mathbf{M}\smallsetminus S}$ is not a scalar multiple of $e^{-mx^*}(y^*)^{M-m}$. And so the whole program is hopeless. Well, at least our original guess that $(x^n)^*$ should be proportional to $(x^*)^n$ and that $(y^m)^*$ should be proportional to $(y^*)^m$ — that guess is hopeless.[3]

On the other hand, if $[x^*,y^*]=\beta y^*$, then we are in business. We have $y^*x^*=(x^*-\beta)y^*$, and so $y^*e^{-x^*}=e^{-x^*+\beta}y^*$, as can be seen by writing out the formal power series. But $e^{-x^*+\beta}y^*=e^{\beta}e^{-x^*}y$ picks up just a scalar, and in particular when a y^* moves past k-many e^{-x^*}s we pick up the scalar $e^{k\beta}$. In particular, it is a straightforward combinatorial exercise that:

$$\sum_{S\subseteq\mathbf{M},|S|=m}(e^{-x^*})^S(y^*)^{\mathbf{M}\smallsetminus S} = \begin{bmatrix}M\\ m\end{bmatrix}_q e^{-mx^*}(y^*)^{M-m}$$

Here we define $\begin{bmatrix}a\\ b\end{bmatrix}_q=\frac{[a]!}{[b]!\,[a-b]!}$, where $[a]!=[a][a-1]\dots[2][1]$ and $[a]=q^{a-1}+q^{a-2}+\dots+q+1=\frac{q^a-1}{a-1}$, where $q=e^\beta$. One proof is to interpret the rule that $y^*e^{-x^*}=qe^{-x^*}y^*$ in terms of paths descending from the top of Pascal's triangle down to lower terms: a path is weighted by q^A where A is the area enclosed by the figure with right-hand-side that path and left-hand-side the left-most path with the given endpoints. Then the proof follows from writing down recurrence relations from this "q-Pascal triangle". Thus we take $[m]_y!=[m]!$ to be the q-factorial, which we will now write as

[3] My preferred basis is not the PBW basis used here, but rather the symmetric basis. It seems unlikely that the goal of pairing $\langle x,y:[x,y]=y\rangle$ with $\langle x^*,y^*:[x^*,y^*]=x^*\rangle$ will work in this basis either — the rule should be that the pairing of a symmetric polynomial in x,y with a symmetric polynomial in x^*,y^* should be nonzero only if the degrees are the same — but not impossible, since the change-of-basis matrix from the PBW basis to the symmetric basis is sufficiently complicated (it is upper-triangular with ones on the diagonal, but fairly densely filled in general). However, I have not yet computed the multiplication in this basis.

$[m]_q!$, dropping the y from the notation. Again, we have $q = e^\beta$, where β is the non-zero eigenvalue of $[x^*, -] : \mathfrak{g}^* \to \mathfrak{g}^*$.

Thus we almost completely understand the two algebras $\mathcal{U}\mathfrak{g} = \langle x, y : [x, y] = y\rangle$ and $\mathcal{U}\mathfrak{g}^* = \langle x^*, y^* : [x^*, y^*] = \beta y^*\rangle$ and the pairing between them. We can repeat the argument with the roles of $\mathfrak{g}$ and $\mathfrak{g}^*$ reversed. We find out that the comultiplications (now writing Δ for μ^*) are given by

$$\Delta x = 1 \otimes x + x \otimes 1$$
$$\Delta x^* = 1 \otimes x^* + x^* \otimes 1$$
$$\Delta y = 1 \otimes y + y \otimes e^{-\beta x}$$
$$\Delta y^* = 1 \otimes y^* + y^* \otimes e^{-x^*}$$

The pairing is $x^n y^m \otimes (x^*)^{n'}(y^*)^{m'} \mapsto \delta^{n,n'}\delta^{m,m'} n!\,[m]_q!$, where $q = e^\beta$. We almost, but did not quite, prove that these maps define $\mathcal{U}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}^*$ to be a dual pair of (topological) Hopf algebras; the proof requires more carefully computing $\Delta(x^n y^m)$, etc. It does not seem difficult.

The next thing that we ought to do is to recognize $\mathfrak{g}$ as the Lie algebra $\mathfrak{b}_+ \subseteq \mathfrak{sl}(2)$ and $\mathfrak{g}^*$ as the Lie algebra $\mathfrak{b}_-$; we have $x = H/2$, $y = E$, $x^* = -\beta H/2$ and $y^* = F$. Then we should describe the Double construction that turns a dual pair of Hopf algebras into a Hopf algebra: for now we say that the end result is a Hopf algebra $\mathcal{U}\mathfrak{g} \bowtie \mathcal{U}\mathfrak{g}^*$ generated as an algebra by x, y, x^*, y^*, with the brackets $[x, y^*]$ and $[x^*, y]$ determined by the condition that the pairing between $\mathcal{U}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g}^*$ extends to an invariant pairing between $\mathcal{U}\mathfrak{g} \otimes 1 \hookrightarrow \mathcal{U}\mathfrak{g} \bowtie \mathcal{U}\mathfrak{g}^*$ and $1 \otimes \mathcal{U}\mathfrak{g}^* \hookrightarrow \mathcal{U}\mathfrak{g} \bowtie \mathcal{U}\mathfrak{g}^*$. Then it turns out that the algebra ideal generated by $x^* + \beta x$ is a Hopf ideal, and that quotienting by this ideal gives a Hopf algebra whose algebra part is just $\mathcal{U}\mathfrak{sl}(2)$.

But rather than do all this now, we will say a few philosophical words about the preceding construction, and compare it with the quantization we discussed earlier.

Recall that for a Lie algebra $\mathfrak{g}$, we understood $\mathcal{U}\mathfrak{g}$ as a deformation of $\mathcal{S}\mathfrak{g}$ by considering the algebra $\mathfrak{g}_\hbar$ with $[,]_\hbar = \hbar[,]$; then as $\hbar \to 0$, $\mathcal{U}\mathfrak{g} \to \mathcal{S}\mathfrak{g}$, and $\{,\}_\hbar \overset{\text{def}}{=} \frac{1}{\hbar}[,]_\hbar$ tends to the Lie-Kirollov-Kostant bracket $\{,\}_0$. Let us take this approach here: namely, we consider not the fixed Lie algebra $\mathfrak{g}$ but the Lie algebra $\mathfrak{g}_\alpha$ with $[x, y] = \alpha y$, and we let $\mathfrak{g}^* = \mathfrak{g}^*_\beta$ with $[x^*, y^*] = \beta y^*$. Then for $\alpha \neq 0$, the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ are isomorphic. However, we now impose the pairings $(x, x^*) = 1 = (y, y^*)$ and $(x, y^*) = 0 = (y, x^*)$, making $(\mathfrak{g}_\alpha, \mathfrak{g}^*_\beta)$ into a Lie bialgebra. Then rescaling just α or β yields non-isomorphic Lie bialgebras: the isomorphism class of the bialgebra is determined by the product $\alpha\beta$ provided that this product is not 0. When $\alpha\beta = 0$, there are three isomorphism classes: the commutative but noncocommutative one, the cocommutative but noncommutative one, and the commutative cocommutative one. (Of course, there is another isomorophism class of two-dimensional Lie bialgebras all together, as we mentioned above.) But we choose not to think about isomorphism classes, but rather keep a bit more data: we parameterize the space of two-dimensional Lie bialgebras, or at least the part of the space we are interested in, by points (α, β) in the plane.

Then by the construction above we end up with the following dual pair of associative bialgebras[4]:

$$\begin{array}{ll} \underline{\mathcal{U}\mathfrak{g}} & \underline{\mathcal{U}\mathfrak{g}^*} \\ [x,y] = \alpha y & [x^*,y^*] = \beta y^* \\ \Delta x = 1\otimes x + x\otimes 1 & \Delta x^* = 1\otimes x^* + x^*\otimes 1 \\ \Delta y = 1\otimes y + y\otimes e^{-\beta x} & \Delta y^* = 1\otimes y^* + y^*\otimes e^{-\alpha x^*} \end{array}$$

$$\underline{\text{Pairing:}}\quad \left(x^n y^m, (x^*)^{n'}(y^*)^{m'}\right) = \delta^{n,n'}\,\delta^{m,m'}\,n!\,[m]_q!,\ \text{where } q = e^{\alpha\beta}$$

It's then reasonable to set Plank's constant to $\hbar = \alpha\beta$. When α, β are small, we think of this pair of Hopf algebras as deforming the pairing of commutative cocommutative Hopf algebras $\mathcal{S}\mathfrak{g}\otimes\mathcal{S}\mathfrak{g}^* \to \mathbb{K}$ given by $\left(a^n, (a^*)^m\right) = \delta^{n,m}\,n!\,(a,a^*)^n$ for any $a \in \mathfrak{g}, a^* \in \mathfrak{g}^*$.

References

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[2] L.D. Faddeev and O.A. Yakubovskiĭ. *Lectures on Quantum Mechanics for Mathematics Students*. Translated by H. McFaden. Student Mathematical Library; 47. American Mathematics Society, Providence, RI, 2009.

[3] N. Reshetikhin. *Quantum Groups*, graduate seminar, UC Berkeley, Spring 2009. Lecture notes by T. Johnson-Freyd available at `http://math.berkeley.edu/~theojf/QuantumGroups.pdf`.

[4] J.J. Sakurai. *Modern Quantum Mechanics*. Revised edition. Addison-Wesley Publishing Company, Reading, MA, 1994.

[4] We'd like to write "dual pair of Hopf algebras", but we have not yet defined the antipode. The map $x \mapsto -x, y \mapsto -y$ defines an algebra antiautomorphism, but a coalgebra automorphism rather than antiautomorphism. The problem seems to be that the pairing is not preserved under switching the orders:

$$\begin{aligned} \left(y^m x^n, (y^*)^{m'}(x^*)^{n'}\right) &= \left((x-m\alpha)^n y^m, (x^*-m'\beta)^{n'}(y^*)^{m'}\right) \\ &= \left(\sum_{k=0}^{n}\binom{n}{k}(-m\alpha)^k x^{n-k}y^m, \sum_{k'=0}^{n'}\binom{n'}{k'}(-m'\alpha)^{k'}(x^*)^{n'-k'}(y^*)^{m'}\right) \\ &= \sum_{k=0}^{n}\sum_{k'=0}^{n'}\binom{n}{k}\binom{n'}{k'}(-m\alpha)^k(-m'\beta)^{k'}\left(x^{n-k}y^m, (x^*)^{n'-k'}(y^*)^{m'}\right) \\ &= \sum_{k=0}^{n}\sum_{k'=0}^{n'}\binom{n}{k}\binom{n'}{k'}(-m\alpha)^k(-m'\beta)^{k'}\delta^{n-k,n'-k'}\delta^{m,m'}(n-k)![m]! \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{n'}{n-k}(-m)^{2k+n'-n}\alpha^k\beta^{n'-n+k}(n-k)!\,\delta^{m,m'}[m]! \end{aligned}$$

In particular, it's clear that this number is not proportional to $\delta^{n,n'}$.

[5] A. Weinstein. Some remarks on dressing transformations. *J. Fac. Sci. Univ. Tokyo.* Sect. 1A, Math. 35 (1988), 163-167.