Hopf algebras

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June 2, 2009

These notes will not record everything there is to know about Hopf algebras. Even books on quantum groups, e.g. [2, 4], do not contain everything. These notes were prepared as a study aid: the author took a class on quantum groups, for which unedited notes are available [7], and then put the topic on his qual. These notes would probably be better written in the graphical language, used, e.g., in [4] and in [5]. In particular, most of these notes will parallel [5]. But laying out such graphical notation on the computer is slow. We invite the reader to rewrite our story in that better notation.

1 Basic definitions

An *algebra* is an object A in a symmetric monoidal category (usually the category of $\mathbb{K}$-vector spaces, for $\mathbb{K}$ a field, or the category of (topologically) free K-modules, for K a (topological) commutative ring) along with a map $i : 1 \to A$ (1 is the unit object in the category) and a map $m : A \otimes A \to A$ such that:

assoc $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$ as maps $A \otimes A \otimes A \to A$

unit $m \circ (i \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes i)$ as maps $A \otimes 1 = A = 1 \otimes A \to A$

Of course, almost all the structure is in m: if there is a map i_L satisfying the first equation of **unit** and a map i_R satisfying the second, then the two maps are equal; i is thus uniquely defined.

We introduce two common notation:

- Primarily used in mathematics:

  In a tensor product $A^{\otimes n}$, we label the multiplicands by $i = 1, \dots, n$. Then the multiplication in the $i, i+1$ spots, for example, is the map $m^i_{i,i+1} : A^{\otimes n} \to A^{\otimes(n-1)}$. We do not write identity maps unless necessary. The above axioms read:

  assoc $m^1_{12} \circ m^1_{12} = m^1_{12} \circ m^2_{23}$
  unit $m^1_{12} \circ i^1 = \mathrm{id}^1_1 = m^1_{12} \circ i^2$

  Of course, one can vary the indices in any equation.

- Primarily used in physics:

  For any choice of basis $\{a_i\}$ of A, we let $\{a^j\}$ be the "dual basis" linear functionals $A \to 1$, so that $a^j \circ a_i = \delta_i^j$; then we define, for example, the numbers $m_{ij}^k = a^k \circ m \circ (a_i \otimes a_j)$, and this collection of numbers completely determines the map m. Composition of linear maps is achieved by "contraction of indices", and we adopt the Einstein convention of summing over repeated indices: we only repeat an index if it is raised once and lowered once. The notation, then, is "covariant". If $a_i \mapsto a_i'$ is any change of basis matrix, then it acts on *tensors* like m_{ij}^k in its usual representation on raised indices and in its adjoint representation on lowered indices. In particular, the change of basis cancels on contracted indices. Thus we can define symbols like m_{ij}^k invariantly: it is the function $m : A_i \otimes A_j \to A_k$, where we have labeled the copies of the algebra A by letters, rather than numbers as above.

  The axioms read:

  assoc $m_{jm}^i m_{kl}^j = m_{kj}^i m_{lm}^j$
  unit $m_{kj}^i i^k = \delta_j^i = m_{jk}^i i^k$

We remark that an *algebra* is precisely a monoidal functor from the category $\mathfrak{Alg}$: the objects of this category are nonnegative integers, the monoidal structure is $+$, and the monoidal generator is terminal. This is the same category whose non-monoidal functors define simplicial objects. This functorial language makes the question of what is an *algebra homomorphism* obvious: it is a natural transformation of functors. We spell this out: if A and B are algebras, a homomorphism is a map $\phi : A \to B$ such that $\phi \circ i_A = i_B$ and $\phi \circ m_A = m_B \circ \phi^{\otimes 2}$. Also the tensor product of algebras is natural. If A and B are algebras, we define $(m_{A\otimes B})_{12,34}^{12} = (m_A)_{1,3}^1 \otimes (m_B)_{2,4}^2$ and $i_{A\otimes B} = i_A \otimes i_B$. Thus the notion of the tensor product requires a "flip" map in the ambient category; this is why we required our objects to live in a *symmetric* monoidal category. The category of algebras is symmetric monoidal, since there is a natural isomorphism $A \otimes B \cong B \otimes A$, stemming from the fact that in our construction of the tensor product, the elements of A commute with the elements of B.

A *coalgebra* is a monoidal functor from the category $\mathfrak{Alg}^{\mathrm{op}}$. We write the comultiplication as Δ and the counit at ϵ. By the usual categorical yoga, a coalgebra in the category of algebras is equivalent to an algebra in the category of coalgebras, because both are precisely monoidal functors from the category $\mathfrak{Alg} \times \mathfrak{Alg}^{\mathrm{op}}$. Such a functor is a *bialgebra*. We spell this out: it is an object A with maps $m : A \otimes A \to A$, $i : 1 \to A$, $d : A \to A \otimes A$, and $e : A \to 1$ such that:

assoc $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$

unit $m \circ (i \otimes \mathrm{id}) = \mathrm{id} = m \circ (\mathrm{id} \otimes i)$

coassoc $(d \otimes \mathrm{id}) \circ d = (\mathrm{id} \otimes d) \circ d$

counit $(e \otimes \mathrm{id}) \circ d = \mathrm{id} = (\mathrm{id} \otimes e) \circ d$

compat $d \circ m = (m \otimes m) \circ (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (d \otimes d)$

We have introduced the map flip : $A \otimes A \to A \otimes A$. If A is an algebra, we write A_{op} for the algebra on A with multiplication $m_{\mathrm{op}} = m \circ \mathrm{flip}$. If A is a coalgebra, we write A^{op} for the coalgebra A with comultiplication $\delta^{\mathrm{op}} = \mathrm{flip} \circ \delta$. If A is a bialgebra, $A_{\mathrm{op}}^{\mathrm{op}}$ is the obvious bialgebra. A is *commutative* if $A = A_{\mathrm{op}}$ and *cocommutative* if $A = A^{\mathrm{op}}$.

A bialgebra A is *Hopf* if in addition to the maps above, it has an invertible map $s : A \to A$ called the *antipode* satisfying:

anti $s \circ m = m_{\mathrm{op}} \circ (s \otimes s)$ and $\delta \circ s = (s \otimes s) \circ \delta^{\mathrm{op}}$; thus s is a bialgebra isomorphism $A \to A_{\mathrm{op}}^{\mathrm{op}}$

Hopf $m \circ (\mathrm{id} \otimes s) \circ \delta = i \circ e = m \circ (s \otimes \mathrm{id}) \circ \delta$

As is traditional, we shorten the phrase "Hopf bialgebra" to "Hopf algebra". The map s, if it exists, is uniquely determined by the axiom **Hopf** and the bialgebra axioms. Indeed, if s and s' are two antipodes, then:

$$s = m \circ (\mathrm{id} \otimes i) \circ s \circ (\mathrm{id} \otimes e) \circ \delta = m^{(3)} \circ (s \otimes \mathrm{id} \otimes s') \circ \delta^{(3)} = m \circ (i \otimes \delta) \circ s' \circ (e \otimes \mathrm{id}) \circ \delta = s'$$

We have introduced the notation $m^{(3)} = m \circ (\mathrm{id} \otimes m) = m \circ (m \otimes \mathrm{id})$ and $\delta^{(3)} = (\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \mathrm{id}$, and we similarly define $m^{(n)} : A^{\otimes n} \to A$ and $\delta^{(n)} : A \to A^{\otimes n}$. In particular, we use the notation $m^{(2)} = m$, $m^{(1)} = \mathrm{id}$, and $m^{(0)} = i$, and similarly for $\delta^{(2)}, \delta^{(1)}, \delta^{(0)}$.

It is an easy exercise that if a Hopf algebra A is commutative or cocommutative, then $s^2 = \mathrm{id}$. This equation does not hold in general. We will have more to say about the isomorphism $s^2 : A \to A$ later.

It often is the case that the objects we are interested in are infinite-dimensional, and then it is often the case that they are controlled by some topology. If A is a complete topological vector space, we say that $m : A \otimes A \to A$ is *topological* if it is continuous; then it extends to a map $A \hat{\otimes} A \to A$, where $\hat{\otimes}$ is the completed tensor product, provided that A is also complete. The algebraic tensor product is designed to be mapped out of, not into; where the word "topological" becomes important is for coalgebras. A *topological coalgebra* is a complete vector space A along with a continuous map $\delta : A \to A \hat{\otimes} A$; thus topological coalgebras need not be (algebraic) coalgebras. The phrases *topological bialgebra* and *topological Hopf algebra* refer to bi- and Hopf algebras where the comultiplication is only topological; to make sense of **compat** and **Hopf**, we must impose that the multiplication is also topological, which is an extra condition, not a weakening. Equivalently, a topological Hopf (bi-) algebra is a Hopf (bi-) algebra in the category of complete vector spaces, whose morphisms are continuous linear maps and whose tensor product is the completed tensor product. Some results of coalgebra theory — every coalgebra is the sum of its commutative subalgebras, for example — depend on the coalgebra being an honest algebraic coalgebra, but we will not need those results here. In particular, we will often elide the word "topological".

We conclude this section with two more notions. A *topological basis* of a topological vector space is a linearly-independent set whose algebraic span is dense in the vector space. A subset of a topological algebra *topologically generates* the algebra if it algebraically generates a dense subalgebra. For example, the ring of formal power series in one variable $\mathbb{K}[[x]]$ is a complete topological algebra with the product topology (equivalently with the x-adic topology). It is topologically generated by the element x and the set $\{x^n\}_{n=0}^{\infty}$ is a topological basis. The index notation m_{ij}^k introduced above makes sense with a topological basis rather than a basis. We will always demand of our topological vector spaces that any one-dimensional subspace have the same topology as the ground field $\mathbb{K}$; we will usually give $\mathbb{K}$ the discrete topology here. The (topological) *dual* V^* to a topological vector space V over $\mathbb{K}$ is the space of all continuous linear maps $V \to \mathbb{K}$; it has a topology on it given by saying that a sequence $\alpha_n : V \to \mathbb{K}$ converges to $\alpha : V \to \mathbb{K}$ if for any $v \in V$, $\alpha_n(v) \in \mathbb{K}$ converges to $\alpha(v)$ in $\mathbb{K}$. Thus, this is the weakest topology so that the pairing $V \otimes V^* \to \mathbb{K}$ is continuous.[1] For example, the topological dual to the vector space $\mathbb{K}[x]$ with the discrete topology is isomorphic to the vector space $\mathbb{K}[[x]]$ with the x-adic topology.

We conclude by listing the standard examples of Hopf algebras (for bialgebras, replace the word "group" throughout by *monoid*, an algebra object in the category SET):

- Let G be a (discrete) group. The *group algebra* of G is the algebra $\mathbb{K}[G]$ with basis G and multiplication given on the basis by the multiplication in G. It is a Hopf algebra with $\mathscr{s}(g) = g^{-1}$ and $\mathscr{d}(g) = g \otimes g$ for $g \in G$. In general, if A is a Hopf algebra and $a \in A$ satisfies $\mathscr{d}(a) = a \otimes a$, then a is called *grouplike*. The grouplike elements of any Hopf algebra A form a group $\mathscr{G}(A)$, and we have $\mathscr{G}(\mathbb{K}[G]) = G$.

  If G is a topological group, we can extend $\mathbb{K}[G]$ to the algebra $\mathscr{C}(G)^*$ of *measures* on G, i.e. continuous linear maps $\mathscr{C}(G) \to \mathbb{K}$, where $\mathcal{C}(G)$ is the algebra of continuous functions on G with topology given by pointwise convergence. ($\mathbb{K}[G]$ only embeds in $\mathscr{C}(G)^*$ if G is Hausdorff.) The product is the *convolution product*: if μ, ν are measures on G, then $\mu \times \nu$ is a measure on $G \times G$, and its pushforward along the multiplication map $G \times G \to G$ gives the measure $\mu\nu$. The group G is still the group of grouplike elements of $\mathscr{C}(G)^*$. The comultiplication is only topological: it is given by $\mathscr{d}(\mu)(f, g) = \mu(fg)$, where we have used $\mathscr{C}(G)^* \hat{\otimes} \mathscr{C}(G)^* = (\mathscr{C}(G) \otimes \mathscr{C}(G))^*$ and fg is the pointwise multiplication $\mathscr{C}(G) \otimes \mathscr{C}(G) \to \mathscr{C}(G)$.

  The Hopf algebras $\mathbb{K}[G]$ and $\mathscr{C}(G)^*$ are cocommutative.

- Let G be a (topological) group. Then $\mathscr{C}(G)$ is a commutative (topological) Hopf algebra with pointwise multiplication. The comultiplication is only topological if G is infinite: it is given by the pullback of the multiplication $G \times G \to G$ to $\mathscr{C}(G) \to \mathscr{C}(G \times G) = \mathscr{C}(G) \hat{\otimes} \mathscr{C}(G)$. If G is a real (complex) Lie group, we can replace $\mathscr{C}(G)$ by the algebra of infinitely-differentiable or analytic (or holomorphic) functions. If G is an affine algebraic group, we can replace $\mathscr{C}(G)$ be the algebra of polynomial functions on G.

  The Hopf algebra $\mathscr{C}(G)$ is commutative. We can recover G form it as the set of algebra homomorphisms $\mathscr{C}(G) \to \mathbb{K}$, at least when G is Hausdorff. For consistency, we call an algebra homomorphism $A \to \mathbb{K}$ from an arbitrary Hopf algebra A *cogrouplike*. The cogrouplike functionals on any Hopf algebra form a group, with multiplication given $fg = (f \otimes g) \circ \mathscr{d}$.

- Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{U}\mathfrak{g}$ its universal enveloping algebra. Then $\mathcal{U}\mathfrak{g}$ is a cocommutative Hopf algebra with comultiplication given by $\mathscr{d}(g) = g \otimes 1 + 1 \otimes g$ for $g \in \mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie (or algebraic) group G, then $\mathcal{U}\mathfrak{g}$ embeds in the Hopf algebra $\mathscr{C}(G)^*$, where $\mathscr{C}(G)$ is the algebra of smooth (infinitely-differentiable, analytic, holomorphic, or polynomial)

[1] There is something missing here. In particular, we really ought to have a notion whereby V^* is complete, and if V is complete, then $(V^*)^* = V$. One problem is that the word "complete" requires more than a topology, but actually a metric. Oh, well. We would like the following probably false result: let α_n be a sequence of continuous maps so that for every $v \in V$, $\alpha_n(v)$ converges, and define the linear map α by $\alpha(v) = \lim \alpha_n(v)$. Then α is continuous. Unfortunately, we are not very good at topology.

functions on G, where $g \in \mathfrak{g} = T_e(G)$ acts on a function $f \in \mathscr{C}(G)$ as a point-derivation at the identity $e \in G$. In any Hopf algebra A, and element $a \in A$ such that $\Delta(a) = a \otimes 1 + 1 \otimes a$ is called *primitive*. The primitive elements of any Hopf algebra A form a Lie algebra $\mathfrak{a}$, and generate a Hopf subalgebra of A that is a Hopf quotient of $\mathcal{U}\mathfrak{a}$.

If $\mathfrak{g}$ is a Lie algebra over a field $\mathbb{K}$ of characteristic 0, we can also consider the Baker-Campbell-Hausdorff formal power series $\beta(x, y) = \log(\exp x \exp y) \in \mathbb{K}[[\mathfrak{g}^*]]$, where $\mathbb{K}[[\mathfrak{g}^*]]$ is the algebra of formal power series in a basis of $\mathfrak{g}^*$. (If $\mathbb{K}$ is not of characteristic 0, we can perform a similar construction with $\mathbb{K}[[\mathfrak{g}^*]]$ replaced by the algebra of "divided formal power series".) Then β defines on $\mathbb{K}[[\mathfrak{g}^*]]$ a topological comultiplication given by $\Delta : f \mapsto f \circ \beta$, where we have used $\mathbb{K}[[\mathfrak{g}^*]] \hat{\otimes} \mathbb{K}[[\mathfrak{g}^*]] = \mathbb{K}[[\mathfrak{g}^* \times \mathfrak{g}^*]]$. Then $\mathbb{K}[[\mathfrak{g}^*]]$ is a commutative Hopf algebra. Any topological Hopf algebra isomorphic to one from this construction a *formal group*, and we will write $\mathcal{F}\mathfrak{g}$ for this Hopf algebra. If $\mathfrak{g}$ is nilpotent (or if the Lie subalgebra generated by any two elements is nilpotent), then the construction works with $\mathbb{K}[[\mathfrak{g}^*]]$ replaced by $\mathbb{K}[\mathfrak{g}^*]$, the algebra of polynomial functions on $\mathfrak{g}$.

- Let $\mathfrak{b}$ be the two-dimensional nonabelian Lie algebra $\langle x, y \text{ s.t. } [x, y] = x \rangle$ over a field $\mathbb{K}$ of characteristic 0, and let $\mathcal{U}\mathfrak{b}$ be its universal enveloping algebra. We define the algebra $\mathcal{U}\mathfrak{b}[[h]] = \mathcal{U}\mathfrak{b} \hat{\otimes} \mathbb{K}[[h]]$. Then $\mathcal{U}_h\mathfrak{b}$ is this algebra with the coalgebra structure given on generators by $\Delta(y) = y \otimes 1 + 1 \otimes y$ and $\Delta(x) = x \otimes 1 + e^{hy} \otimes x$. One can check that this is a Hopf algebra, neither commutative nor cocommutative. As a coalgebra, it is isomorphic to the formal group $\mathcal{F}\mathfrak{b}_h$, where $\mathfrak{b}_h$ is the Lie algebra over $\mathbb{K}[h]$ given by $[x, y] = hx$, or possibly it is isomorphic to the opposite coalgebra. In particular, the conventions about algebras and their opposites are not particularly coherent in the literature; someone else might use the name $\mathcal{U}_h\mathfrak{b}$ for what we would call $(\mathcal{U}_h\mathfrak{b})^{\text{op}}$ or perhaps for $\mathcal{U}_{-h}\mathfrak{b}$, etc. In fact, $\mathcal{U}_h\mathfrak{b}^{\text{op}}$ and $\mathcal{U}_{-h}\mathfrak{b}$ are isomorphic, but nontrivially.

Lastly, we mention that in a Cartesian category (all finite products exist, and the monoidal structure is given by a choice of product), then any object is a coalgebra in a unique way. Then a Hopf algebra in a Cartesian category is precisely a group. This is the case, for example, in SET, TOP, and MAN.

2 The semidirect and knit products

We recall that a left *module* of an algebra A is a vector space V with a map $\lambda_V : A \otimes V \to V$ such that $\lambda_V \circ (\text{id} \otimes \lambda_V) = \lambda_V \circ (m \otimes \text{id})$ as maps $A \otimes A \otimes V \to V$, and also such that $\lambda_V \circ (i \otimes \text{id}) = \text{id}$. The category of left modules of A is abelian. If A is a bialgebra, then the category of left A-modules is monoidal: if $\lambda_V : A \otimes V \to V$ and $\lambda_W : A \otimes W \to W$ are two A-modules, then we define and action of A on $V \otimes W$ by $(\lambda_V \otimes \lambda_W) \circ (\text{id} \otimes \text{flip} \otimes \text{id}) \circ (\Delta \otimes \text{id} \otimes \text{id})$. The counit $\epsilon : A \to \mathbb{K}$ defines the *trivial representation* of A.

Recall that if V is a left A-module, then its vector-space dual V^* is a right A-module by defining $\langle \nu.a, v \rangle = \langle \nu, a.v \rangle$ for $\nu \in V^*, a \in A, v \in V$, where $\langle , \rangle$ is the pairing $V^* \otimes V \to \mathbb{K}$ and . is the action $a.v = \lambda_V(a, v)$. We can make V^* into a *left* A-module if A is Hopf, by defining $a.\nu = \nu.s(a)$. We could also use $\nu.s^{-1}(a)$ when $s^2 \neq \text{id}$; which is better depends on whether we want the canonical map $V^* \otimes V \to \mathbb{K}$ or $V \otimes V^* \to \mathbb{K}$ to be a morphism of A-modules.

- If $A = \mathbb{K}[G]$ is the group algebra of a group G with its cocommutative Hopf structure, then a representation of G is precisely an A-module, and the above construction defines the usual dual- and tensor product on the category of G-representations.

Let G be a group acting by automorphisms on an algebra B. We rewrite this in terms of the Hopf algebra $\mathbb{K}[G]$: the requirement that $g.m_B(b_1, b_2) = m_B(g.b_1, g.b_2)$ becomes the requirement that $m_B : B \otimes B \to B$ is a $\mathbb{K}[G]$-module map, where $B \otimes B$ has the $\mathbb{K}[G]$-module structure given in the previous paragraph.

We recall and generalize the semidirect product of groups. If G acts on H by automorphisms (from the left), we define $H \rtimes G$ to be the group structure on the set $H \times G$ with the group multiplication given by $(h_1, g_1)(h_2, g_2) = (h_1 \cdot g_1.h_2, g_1g_2)$, where $\cdot$ is the multiplication in H and . is the action of G on H. Equivalently, the semidirect product of groups G and H is the group so with subgroups G and H, so that for $g \in G$ and $h \in H$, we have $ghg^{-1} = g.h$. Conversely, G acts on itself by *inner automorphisms*, given by $g.h = ghg^{-1}$ for $g, h \in G$. So the semidirect product construction turns actions into inner actions.

In the semidirect product construction, there was nothing special about H being a group; it would work just as well if H were a monoid or an algebra. We now replace G by its group algebra $\mathbb{K}[G]$, by extending the construction $\mathbb{K}$-linearly in G; then we replace the set $H \times G$ by the tensor product $H \otimes \mathbb{K}[G]$. On a basis $G \subseteq \mathbb{K}[G]$ we can continue with the formula $(h_1 \otimes g_1)(h_2 \otimes g_2) = (h_1 \cdot g_1.h_1) \otimes (g_1g_2)$, but this is disappointing because it is not linear if extended as written to $g_1 \in \mathbb{K}[G] \smallsetminus G$. Instead, we note that $g_1 \otimes g_1 = \Delta(g_1)$ for $g_1 \in G$. Writing λ_H for the action $\mathbb{K}[G] \otimes H \to H$, we define the multiplication in the semidirect product $H \rtimes \mathbb{K}[G]$ to be given by the map $(m_H \otimes m_G) \circ (\mathrm{id} \otimes \lambda_H \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_G \otimes \mathrm{id} \otimes \mathrm{id})$, where m_H, m_G, and Δ_G are the multiplication on H, the multiplication on G, and the comultiplication on G, respectively.

This suggests the following definition. Let A be a bialgebra acting on an algebra B from the left. We say that A acts on B *by homomorphisms* if the multiplication and unit maps for B are A-module homomorphisms. If $\lambda_B : A \otimes B \to B$ is a left action by homomorphisms, then it is a straightforward calculation to check that the vector space $B \otimes A$ can be made into an algebra with the multiplication:

$$m_{BA} = (m_B \otimes m_A) \circ (\mathrm{id} \otimes \lambda_B \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_A \otimes \mathrm{id} \otimes \mathrm{id})$$

This defines the semidirect product algebra $B \rtimes A$. If B is a bialgebra and the comultiplication and counit maps are also A-module homomorphisms, then $B \rtimes A$ is a bialgebra isomorphic as a coalgebra to $B \otimes A$. If A and B are Hopf, so is $B \rtimes A$. If A acts on B from the right, the construction of $A \ltimes B$ is analogous. When the action is trivial, we recover the tensor product of bialgebras, and when A and B are group algebras, our semidirect product is the group algebra of the semidirect product of groups.

What if bialgebras A and B both act on each other? We recall the *Zappa-Szép product of groups*, also called the *knit product*. Let G, H be two groups such that $\lambda_H : G \times H \to H$ is a left action of sets and $\rho_G : G \times H \to G$ is a right action of sets. Let us demand also that the actions preserve the identity elements and:

$$\lambda_H(g, h_1 h_2) = \lambda_H(g, h_1)\lambda_H(\rho_G(g, h_1), h_2)$$
$$\rho_G(g_1 g_2, h) = \rho_G(g_1, \lambda_H(g_2, h))\rho_G(g_2, h)$$

Then we can define on the set $G \times H$ the a group structure with:

$$(h_1, g_1)(h_2, g_2) = \big(h_1\, \lambda_H(g_1, h_2), \rho_G(g_1, h_2)\, h_2\big)$$
$$(h, g)^{-1} = \big(\lambda_H(h, g)^{-1}, \rho_G(h, g)^{-1}\big)$$

This defines the group $G \bowtie H$. If either λ_H or ρ_G is the trivial action, then the other is an action by automorphisms, and the construction gives the semidirect product.

In light of the knit product of groups, the following definition is natural. Let A and B be two bialgebras so that $\lambda_B : A \otimes B \to B$ is a left A-module and $\rho_A : A \otimes B \to A$ is a right B-module, and such that the unit maps i_A and i_B and the comultiplications d_A and d_B are B-module and A-module homomorphism respectively. Writing m_A, m_B for the multiplication in A and B, we demand moreover that the actions be compatible in the following sense:

$$\lambda_B \circ (\mathrm{id} \otimes m_B) = m_B \circ (\mathrm{id} \otimes \lambda_B) \circ (\lambda_B \otimes \rho_A \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (d_A \otimes d_B \otimes \mathrm{id})$$
$$\rho_A \circ (m_A \otimes \mathrm{id}) = m_A \circ (\rho_A \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \lambda_B \otimes \rho_A) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes d_A \otimes d_B)$$

Then we define the *knit product* or *double cross product* $B \bowtie A$ to be the vector space $B \otimes A$ with multiplication given by:

$$(m_B \otimes m_A) \circ (\mathrm{id} \otimes \lambda_B \otimes \rho_A \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes d_A \otimes d_B \otimes \mathrm{id})$$

That A and B are subalgebras of $B \bowtie A$ is immediate. It is a straightforward but tedious computation, best done in the graphical language, that the multiplication on $B \bowtie A$ is associative. S. Majid provides an algebraic proof in [6]; Majid essentially introduced the knit product of Hopf algebras in his thesis. If on top of the conditions above we demand that:

$$(\rho_A \otimes \lambda_B) \circ (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (d_A \otimes d_B) = \mathrm{flip} \circ (\lambda_B \otimes \rho_A) \circ (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (d_A \otimes d_B)$$

as is definitely the case if e.g. A and B are cocommutative, then $B \bowtie A$ is indeed a bialgebra, isomorphic as a coalgebra to $B \otimes A$. In particular, A, B are subbialgebras of $B \bowtie A$. Moreover, if A and B are Hopf, so is $B \bowtie A$.

We remark that there is also dual construction, going from comodules to a nontrivial coalgebra structure on the algebra $B \otimes A$, called the *double cross coproduct*: A and B are quotient bialgebras rather than subbialgebras. [6] also describes a mixed version, constructing extensions of bialgebras, and calls it the *bicrossed product*; "double cross biproduct" might be a more consistent name.

3 Duals and Doubles

Let A be a Hopf algebra. If A is finite-dimensional over a field $\mathbb{K}$, then its vector space dual A^* is also a Hopf algebra, with multiplication d^*, comultiplication m^*, and antipode s^*. We now specify a convention that is in direct contrast with most of the literature, but that makes much more sense (in particular when moving to the graphical language): we declare that $(A\otimes B)^* = B^*\otimes A^*$, rather than $A^*\otimes B^*$. Of course, these last two vector spaces are isomorphic, but the isomorphism switches, for example, multiplication and comultiplication to their opposites. Put another way, let $\langle,\rangle$ be the pairing $A^*\otimes A\to\mathbb{K}$. Then we define the pairing $A^*\otimes A^*\otimes A\otimes A$ by

$$\langle \mu\otimes\nu, a\otimes b\rangle \stackrel{\text{def}}{=} \langle\mu,b\rangle\langle\nu,a\rangle, \text{ not } \langle\mu,a\rangle\langle\nu,b\rangle$$

Thus what is called A^* in most of the literature we call $(A^*)^{\text{op}}_{\text{op}}$. A justification for the above pairing is that, now writing ev_A for the map $A^*\otimes A\to\mathbb{K}$, we use $\text{ev}_{AB} = \text{ev}_B\circ(\text{id}\otimes\text{ev}_A\otimes\text{id})$, whereas the rest of the world uses $(\text{ev}_A\otimes\text{ev}_B)\circ(\text{id}\otimes\text{flip}\otimes\text{id})$, and from a categorical perspective, the use of the flip map is unnatural.[2] This convention has various implications that do not directly concern us. For example, it is often remarked that if A is a coalgebra and V a right comodule over A, then V is a left module over A^* — in our convention, V is actually a right module over A^*. We will not discuss modules until the last section, and we define them there.

Let A now be an infinite-dimensional (algebraic) Hopf algebra. Then A^* keeps its algebra structure induced by the comultiplication on A, but the comultiplication on A^* induced by the multiplication on A is only topological. There is a construction to find an algebraic Hopf algebra dual to A. Namely, let $m^* : A^*\to A^*\hat\otimes A^*$ be the comultiplication on A^*, and define A° to be the subspace of A^* of elements a such that $m^*(a)\in A^*\otimes A^*$, the algebraic tensor product. Then A° is in fact a subalgebra of A^*, since $m^*(ab) = m^*(a)\,m^*(b)$, now eliding the multiplication map Δ^*. Moreover, the coassociativity of m^* guarantees that $m^*(A^\circ)\subseteq A^\circ\otimes A^\circ$, and A° is trivially preserved by s, so A° is in fact an algebraic Hopf algebra. It is the *Hopf dual* of A. The Hopf dual is the topological dual where A is given a topology by declaring as a base of the topology the kernels of all finite-dimensional algebra quotients of A. Equivalently, A° consists of the linear maps $A\to\mathbb{K}$ that factor through a finite-dimensional quotient of A.[3] In [2], this construction is generalized. Let Σ be any subcategory of the category of A-modules, closed under tensor products, duals, direct sums, and containing the trivial representation; then we define A°_Σ to be consist of all linear maps that factor through $A\to\text{End}(V)$ for $V\in\Sigma$. The algebra A°_Σ is an algebraic Hopf algebra if Σ consists entirely of finite-dimensional modules, and otherwise is at least topological. [2] also provides the following example:

- Let G be a connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then $\mathcal{U}\mathfrak{g}$ and $\mathscr{C}(G)$ are Hopf algebras, as we said above, and pair nondegenerately, so that $\mathcal{U}\mathfrak{g}\hookrightarrow\mathscr{C}(G)^*$ and $\mathscr{C}(G)\hookrightarrow(\mathcal{U}\mathfrak{g})^*$. But in fact $\mathcal{U}\mathfrak{g}\hookrightarrow\mathscr{C}(G)^\circ$ and $\mathscr{C}(G)\hookrightarrow(\mathcal{U}\mathfrak{g})^\circ$. Indeed, $\mathscr{C}(G)^\circ$ is the semidirect product $\mathcal{U}\mathfrak{g}\otimes\mathbb{K}[G]$, whereas if $\mathfrak{g}$ is semisimple then $\mathscr{C}(G) = \mathcal{U}\mathfrak{g}^\circ$.

[2] Equally unnatural is the convention of writing functions to the left of elements, rather than to the right, forcing compositions of functions to be written last-to-first. We grudgingly use this convention in these notes, but remark that one should then write $\text{Hom}(A,B)$ for set of the maps *from B to A* rather than conversely. If needed, we will use $\text{Hom}(A\leftarrow B)$ for this set, and avoid the comma entirely. In the graphical language, which we keep mentioning but will not use here, one generally avoids the problem entirely by writing compositions vertically.

[3] It is not clear to me why A° should necessarily separate points in A, although I cannot think of a counterexample. If A° does separate points, then $(A^\circ)^\circ$ contains A as a subalgebra. The example of $A = \mathcal{U}\mathfrak{g}$ for $\mathfrak{g}$ a semisimple Lie algebra shows that $(A^\circ)^\circ$ may be strictly larger than A.

More generally, we have the following definition. A *non-degenerate pairing of Hopf algebras A* and B is a nondegenerate pairing $\langle,\rangle : A \otimes B \to \mathbb{K}$ such that

$$\begin{aligned} \langle m_A(a_1,a_2), b\rangle &= \langle a_1 \otimes a_2, d_B(b)\rangle \\ \langle d_A(a), b_1 \otimes b_2\rangle &= \langle a, m_A(b_1,b_2)\rangle \\ \langle i_A, b\rangle &= e_B(b) \\ e_A(a) &= \langle a, i_B\rangle \\ \langle s_a(a), b\rangle &= \langle a, s_b(b)\rangle \end{aligned}$$

We remark that, as above, we take the pairing not used in the literature: the pairing $\langle,\rangle : A \otimes A \otimes B \otimes B \to \mathbb{K}$ is given by $\langle a_1 \otimes a_2, b_1 \otimes b_2\rangle = \langle a_1, b_2\rangle\langle a_2, b_1\rangle$. As in the finite-dimensional case, if $(A, B, \langle,\rangle)$ is a nondegenerate pairing of Hopf algebras in our sense, then $(A, B^{\rm op}_{\rm op}, \langle,\rangle)$ is a nondegenerate pairing of Hopf algebras in the sense of, say, [2].

Let $(A, B, \langle,\rangle)$ be a nondegenerate pairing of Hopf algebra. Then we can construct the knit product $A \otimes B^{\rm op}$, where the two actions are:

$$\begin{aligned} B^{\rm op} \otimes A \to A: &\quad (p \otimes \mathrm{id}) \circ (q \otimes \mathrm{flip}) \circ (d_B \otimes d_A^{(3)}) \\ B^{\rm op} \otimes A \to B: &\quad (\mathrm{id} \otimes p) \circ (\mathrm{flip} \otimes q) \circ (d_B^{(3)} \otimes d_A) \end{aligned}$$

Here and below, $p : B \otimes A \to \mathbb{K}$ is the pairing $b \otimes a \mapsto \langle a, b\rangle$ — we have identified $B^{\rm op}$ and B as vector spaces —, and $q = p \circ (\mathrm{id} \otimes s_A) = p \circ (s_B \otimes \mathrm{id})$. As always, we write $d^{(3)} = (d \otimes \mathrm{id}) \circ d = (\mathrm{id} \otimes d) \circ d$ for any comultiplication d. It's straightforward if tedious to check that the above maps satisfy the axioms in part 2. Some care is required: $B^{\rm op}$ has the same multiplication $m_{B^{\rm op}} = m_B$ as has B, but the opposite comultiplication $d_{B^{\rm op}} = d_B^{\rm op}$. The antipode map $s_{B^{\rm op}}$ on $B^{\rm op}$ is the inverse s_B^{-1}.

The knit product $A \otimes B^{\rm op}$ is called the *double* of the pairing $(A, B, \langle,\rangle)$, and denoted by $\mathcal{D}(A, B, \langle,\rangle)$. Since the Hopf algebra B is almost completely determined by A and the requirement that they be in nondegenerate Hopf pairing — the structure maps are completely determined, and only a choice of vector space dual is required — we will shorten the name to $\mathcal{D}(A)$, the *double* of A. The definition of the double in terms of a knit product is from [6]; it is equivalent to the definition given originally by Drinfel'd in [3]. For future convenience, we record the structure of the double directly. We have $\mathcal{D}(A, B, \langle,\rangle) = A \otimes B^{\rm op}$ as a coalgebra, so that ${d_{\mathcal{D}}}_{12}^{1234} = {d_A}_1^{13} {d_B}_2^{42}$, and the multiplication, after simplifying the general knit-product formula, is given by[4]:

$$\begin{aligned} m_{\mathcal{D}} &= (m_A \otimes m_B) \circ (\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes q \otimes \mathrm{id} \otimes \mathrm{id}) \\ &\quad \circ (\mathrm{id} \otimes \mathrm{flip} \otimes p \otimes \mathrm{flip} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes d_B^{(3)} \otimes d_A^{(3)} \otimes \mathrm{id}), \\ \text{i.e. } {m_{\mathcal{D}}}_{1368}^{12} &= ({m_A}_{16}^{1} {m_B}_{38}^{2}) \circ (q_{27} p_{45}) \circ ({d_B}_3^{234} {d_A}_6^{567}), \\ \text{i.e. } {m_{\mathcal{D}}}_{abcd}^{kl} &= {m_A}_{ai}^{k} {m_B}_{fd}^{l} q_{ej} p_{gh} {d_B}_b^{efg} {d_A}_c^{hij}. \end{aligned}$$

When either A or B is cocommutative, the double reduces to a semidirect product. There is a canonical isomorphism $\mathcal{D}(A, B, \langle,\rangle) \cong \mathcal{D}(B, A, \langle,\rangle)^{\rm op}$ — we will see later that the double $\mathcal{D}(A, B, \langle,\rangle)$

[4] If these expressions look long, it's all the more reason to adopt the graphical language.

is the unique Hopf algebra so that A and B^{op} are Hopf subalgebras satisfying an extra natural condition.

Following [6], we remark that the Double construction makes sense at the level of bialgebras. The only requirement on the map $p : B \otimes A \to \mathbb{K}$ is that it be an invertible element of the *convolution algebra* $(B^{\mathrm{op}} \otimes A)^*$, and that $q : B \otimes A \to \mathbb{K}$ be its multiplicative inverse in this algebra. If A and B are Hopf and p is a Hopf pairing, then q must be given by the formula above. In most of the literature, the other convention for Hopf pairings is used. Then their double of a Hopf algebra A is our $\mathcal{D}(A^{\mathrm{op}})^{\mathrm{op}}$.

4 The category of representations of a Hopf algebra

When A is a Hopf algebra, we defined at the start of section 2 the notions of a left A-module, the tensor product of A-modules, and the dual of an A-module. We now specify the dual more precisely.[5]

Let V be an object in a monoidal category. A *left dual* of V is an object ${}^*V$ and morphisms $\epsilon : {}^*V \otimes V \to 1$ and $\eta : 1 \to V \otimes {}^*V$ such that $(\epsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta) = \mathrm{id} : 1 \otimes {}^*V \to {}^*V \otimes 1$ and $(\mathrm{id} \otimes \epsilon) \circ (\eta \otimes \mathrm{id}) = \mathrm{id} : V \otimes 1 \to 1 \otimes V$. A *right dual* is similar. If and object V has a left (right) dual, then it is unique up to isomorphism, an V is the right dual of ${}^*V$. If our category is the category of representations of a bialgebra, then in particular every morphism is $\mathbb{K}$-linear, so the only objects that can have duals are finite-dimensional representations. A category is *rigid* if every object has both a left and a right dual.[6]

Let A be a bialgebra and V a finite-dimensional left A-module. So as to distinguish the vector space dual from a dual in the category of A-modules, we write $\bar{V}$ for the vector-space dual of V. Then $\bar{V}$ is naturally a right A-module, with the action given by $\langle \nu.a, v\rangle = \langle \nu, a.v\rangle$, where $\langle , \rangle$ is the pairing $\bar{V} \otimes V \to \mathbb{K}$. Let $s : A \to A$ be an antihomomorphism of algebras; then $a \otimes \nu \mapsto \nu.s(a)$ defines a left-action on $\bar{V}$. Now suppose that the map $\langle , \rangle : \bar{V} \otimes V \to \mathbb{K}$ is a morphism of left A-modules, where $\mathbb{K}$ is given the trivial action. If V is a faithful representation, then s must satisfy the Hopf identity $m \circ (s \otimes \mathrm{id}) \circ d = i \circ e$. If the identity matrix $I \in \mathrm{End}_{\mathbb{K}}(V)$ defines a left A-module map $\mathbb{K} : V \otimes \bar{V}$, then s must satisfy $m \circ (\mathrm{id} \otimes s) \circ d = i \circ e$. If we have such an s, then it defines a left dual to every finite-dimensional representation. If A has a faithful representation, then the identity ${}^*(V \otimes W) = {}^*W \otimes {}^*V$ implies that s must also be an antihomomorphism of coalgebras, and must be invertible.

Conversely, it's clear that if A is a Hopf algebra, then every finite-dimensional left A-module has a left dual ${}^*V$, which is the vector space dual to V with the action $\langle a.\nu, v\rangle = \langle \nu, s(a).v\rangle$. Every finite-dimensional left A-module also has a right dual V^*; as a vector space this is again the space $\bar{V}$, with the action now given by $\langle a.\nu, v\rangle = \langle \nu, s^{-1}(a).v\rangle$. A Hopf algebra is *involutive* if $s = s^{-1}$ —

[5]We will ignore all issues of strictness of the monoidal structure in these notes: for us there are no associators, since for a Hopf algebra the associators in the category of modules are just the corresponding associators of vector spaces. A *quasiHopf algebra* is like a Hopf algebra, but the coassociativity condition is relaxed; the category of representations of a quasiHopf algebra is monoidal but has nontrivial associators. In the group-theoretic "classical" analogue, the relaxed coassociativity involves a nontrivial cocycle in group cohomology; the quasiHopf axiom similarly is cohomological in flavor.

[6]The notions of "left rigid" and "right rigid" are obvious but not, to my knowledge, useful.

for example, every commutative Hopf algebra is involutive, as is every cocommutative Hopf algerba —, but this does not hold in general. In any monoidal category such that every object has a left dual, any choice of a dual for each object defines a contravariant endofunctor on the category. It's clear that for the category of finite-dimensional representations of a Hopf algebra, this is in fact an antiequivalence, and so the category is *rigid monoidal abelian*.

We have seen, then, that the Hopf axioms by design turn an algebra into one whose category of finite-dimensional modules is rigid monoidal. The philosophy is that the structure functors of the category of modules should correspond to elements and maps in the algebra itself. There is a general theory of *Tannaka-Krein duality*, which says that one can reconstruct an algebra (with structure) from its category (with structure) of modules. We paraphrase from [2, pp. 147-9] a version for general Hopf algebras:

> Let $\mathcal{C}$ a small $\mathbb{K}$-linear rigid monoidal abelian category, and $\Phi : \mathcal{C} \to \text{FinVect}$ a $\mathbb{K}$-linear exact faithful monoidal functor. Then there exists a Hopf algebra A and an equivalence of $\mathbb{K}$-linear categories $\mathcal{C} \to {}_A\text{coMod}$ whose composite with the faithful functor ${}_A\text{coMod} \to \text{FinVect}$ is Φ.

The category ${}_A\text{coMod}$ consists of finite-dimensional comodules of A, with the tensor structure given by the multiplication map. Every comodule over a Hopf algebra A is naturally a module over the dual Hopf algebra A^*, but the converse holds only when A is finite-dimensional. For the above theorem, $\mathbb{K}$ need not be a field, but just a commutative ring. Then the category FinVect consists of all projective finitely-generated $\mathbb{K}$-modules, and ${}_A\text{coMod}$ is the category of A-comodules in FinVect.

Within the category of left A-modules, we have defined a tensor product and left and right duals. If V and W are left A-modules, we can construct modules $V \otimes W$ and $W \otimes V$, but we have not defined an isomorphism between these two modules unless A is cocommutative. Indeed, for a general Hopf algebra A, there is no isomorphism between V and W. A monoidal category for which there are specified coherent natural isomorphisms between any two permutations $V_1 \otimes \cdots \otimes V_n \to V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}$ — coherent in the sense that they depend only on the permutations σ, that the composition of specified isomorphisms is a specified isomorphism, and that the isomorphisms play well with the tensor structure and the other accoutrements of a monoidal category — such a category is called *tensor* or *symmetric monoidal*. Equivalently, a symmetric monoidal category has a natural isomorphism $\sigma : \otimes \to \otimes^{\text{op}}$, where $\otimes^{\text{op}} : (V, W) \mapsto W \otimes V$ is the opposite tensor product, such that $\sigma^2 = \text{id}$ and such that for any U, V, W, we have:

$$\sigma_{U\otimes V,W} = (\sigma_{U,W} \otimes \text{id}_V) \circ (\text{id}_U \otimes \sigma_{V,W})$$
$$\sigma_{U,V\otimes W} = (\text{id}_V \otimes \sigma_{U,W}) \circ (\sigma_{U,V} \otimes \text{id}_W)$$

where $\sigma_{V,W} : V \otimes W \to W \otimes V$ is the natural isomorphism evaluated at the pair (V, W). It's clear that if A is cocommutative, then taking $\sigma = \text{flip}$ makes ${}_A\text{Mod}$ into a symmetric monoidal category.

If A is not cocommutative, what condition still assures that ${}_A\text{Mod}$ is symmetric monoidal? Let λ_V and λ_W be the actions of A on V and W, thought of as maps $\lambda_V : A \to \text{End}(V)$ and $\lambda_W : A \to \text{End}(W)$. Then A acts on $V \otimes W$ by $\big((\lambda_V \otimes \lambda_W) \circ ⫛_A\big) : A \to \big(\text{End}(V) \otimes \text{End}(W)\big) \hookrightarrow \text{End}(V \otimes W)$, whereas it acts on $W \otimes V$ by $\big((\lambda_W \otimes \lambda_V) \circ ⫛_A\big) = \big(\text{flip} \circ (\lambda_V \otimes \lambda_W) \circ ⫛_A^{\text{op}}\big)$. Let us assume that there is a map $\sigma_{V,W} : V \otimes W \to W \otimes V$ that intertwines these two actions, and let's write it in the form $\sigma = \text{flip} \circ \rho$, where $\rho_{V,W} \in \text{End}(V \otimes W)$. In order for σ to be a natural transformation, it suffices for ρ to be of the form $\rho_{V,W} = (\lambda_V \otimes \lambda_W)(r)$, where $r \in A \otimes A$ — it was the same philosophy that suggested that to construct duals of representations of a bialgebra A we look for an antiautomorphism of A. If A is topological and all actions continuous and on complete vector spaces, then we can of course extend $A \otimes A$ to a completed tensor product.

So, to recap, we have an element $r \in A \otimes A$ such that for any $a \in A$:

$$\Big((\lambda_W \otimes \lambda_V) \circ \Delta_A(a)\Big) \circ \text{flip} \circ \Big((\lambda_V \otimes \lambda_W) \circ (r)\Big) = \text{flip} \circ \Big((\lambda_V \otimes \lambda_W) \circ (r)\Big) \circ \Big((\lambda_V \otimes \lambda_W) \circ \Delta_A(a)\Big)$$

We move the flip on the left-hand-side to the front —

$$\text{flip} \circ \Big((\lambda_V \otimes \lambda_W) \circ \Delta_A^{\text{op}}(a)\Big) \circ \Big((\lambda_V \otimes \lambda_W) \circ (r)\Big) = \text{flip} \circ \Big((\lambda_V \otimes \lambda_W) \circ (r)\Big) \circ \Big((\lambda_V \otimes \lambda_W) \circ \Delta_A(a)\Big)$$

—, strip off the flips from both sides, and use the definition of an action:

$$(v \otimes w) \circ (\Delta_A^{\text{op}}(a) \cdot r) = (v \otimes w) \circ (r \cdot \Delta_A(a))$$

Here $\cdot$ is the multiplication in $A \otimes A$. The above equation should hold for any actions $\lambda_V : A \to \text{End}(V)$ and $\lambda_W : A \to \text{End}(W)$. So we suppose that:

$$\Delta_A^{\text{op}}(a) \cdot r = r \cdot \Delta_A(a)$$

The above condition on an element $r \in A \otimes A$ assures that the map $\sigma_{V,W} = \text{flip} \circ \Big((\lambda_V \otimes \lambda_W)(r)\Big) : V \otimes W \to W \otimes V$ is a natural transformation of left A-modules. There are extra conditions required for σ to be a symmetric monoidal structure on ${}_A\text{Mod}$. For $\sigma^2 = \text{id}$, we must have r invertible with $r^{-1} = \text{flip} \circ r$. For the equations decomposing $\sigma_{U \otimes V, W}$ and $\sigma_{U, V \otimes W}$, we must have:

$$(\Delta_A \otimes \text{id}) \circ r = (\text{id} \otimes \text{id} \otimes m_A) \circ (\text{id} \otimes \text{flip} \otimes \text{id}) \circ (r \otimes r) \qquad {\Delta_A}_1^{12}(r^{13}) = r^{13} r^{23}$$

$$(\text{id} \otimes \Delta_A) \circ r = (m_A \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes r \otimes \text{id}) \circ r \qquad {\Delta_A}_3^{23}(r^{13}) = r^{13} r^{12}$$

(On the right-hand-side we have simply repeated the conditions in a more condensed notation; the concatenations $r^{13} r^{23}$ and $r^{13} r^{12}$ refer to multiplication in the third and first components respectively of $A \otimes A \otimes A$.)

We define a bialgebra A along with a choice of invertible element $r \in A \otimes A$ *almost cocommutative* if $\Delta_A^{\text{op}}(a) \cdot r = r \cdot \Delta_A(a)$ for every $a \in A$. An almost cocommutative Hopf algebra is *triangular* if r also satisfies $r^{-1} = \text{flip} \circ r$ and the two quadratic equations in the previous paragraph. We have seen that the category of left modules of a triangular bialgebra is symmetric monoidal, with the symmetry given by $\sigma_{V,W} = \text{flip} \circ \Big((\lambda_V \otimes \lambda_W)(r)\Big)$.

In fact, the condition that a category be symmetric monoidal is very strong, and most interesting Hopf algebras cannot be given a triangular structure. A category is *quasitensor* or *braided monoidal* if it is a monoidal category so that for any collection $V_1, \dots, V_n$ of objects and a permutation σ on n objects, there are natural isomorphisms $V_1 \otimes \cdots \otimes V_n \to V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}$ parameterized by the preimage of σ in the Braid Group on n strands, which has a canonical quotient map to the permutation group. These isomorphisms are required to be coherent in that they should compose correctly and play well with the tensor product. Equivalently, we a braided monoidal category has a natural isomorphism $\sigma : \otimes \to \otimes^{\mathrm{op}}$ such that for any U, V, W, we have

$$\sigma_{U\otimes V,W} = (\sigma_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \sigma_{V,W})$$
$$\sigma_{U,V\otimes W} = (\mathrm{id}_V \otimes \sigma_{U,W}) \circ (\sigma_{U,V} \otimes \mathrm{id}_W),$$

but we do not demand that $\sigma^2 = \mathrm{id}$. Going through the above derivation of the conditions required of the element $r \in A \otimes A$, we see that if we do not require $\sigma^2 = \mathrm{id}$, then we should not require $r^{-1} = \mathrm{flip} \circ r$. We have the following definition: an almost cocommutative bialgebra is *quasitriangular* if r satisfies the quadratic equations $\mathcal{d}_{A1}^{12}(r^{13}) = r^{13}r^{23}$ and $\mathcal{d}_{A3}^{23}(r^{13}) = r^{13}r^{12}$. By design, the category of left modules of a quasitriangular bialgebra is braided. It is really the pair (A, r) that is quasitriangular; an element $r \in A \otimes A$ making (A, r) quasitriangular is an *(universal) R-matrix.*

We list a few facts about quasitriangular bialgebras. The R-matrix r satisfies the *quantum Yang-Baxter equation (QYBE)*:

$$r^{12}r^{13}r^{23} = r^{23}r^{13}r^{12} \quad \text{(multiplication in } A \otimes A \otimes A)$$

We have $(\mathrm{id} \otimes e) \circ r = i = (e \otimes \mathrm{id}) \circ r$. Moreover, if A is Hopf, then $(s \otimes \mathrm{id}) \circ r = r^{-1} = (\mathrm{id} \otimes s^{-1}) \circ r$ and $(s \otimes s) \circ r = r$; in the first equation, r^{-1} is the multiplicative inverse of r in $A \otimes A$ whereas s^{-1} is the inverse map $A \to A$ of the antipode s. These equations are straightforward to check and follow directly from the axioms of a quasitriangular bialgebra. In the category of modules, the QYBE has the interpretation of the Third Reidemeister Move for the Braid group. The condition involving the unit and counit expresses that the braiding gets along with the trivial representation. The Hopf relations express how the braiding interacts with duals.

We describe an extremely important example of a quasitriangular Hopf algebra:

- The double $\mathcal{D}(A)$ of any finite-dimensional Hopf algebra A is quasitriangular. Recall the embeddings $A, A^{*\mathrm{op}} \hookrightarrow \mathcal{D}(A)$, by $\mathrm{id} \otimes i_{A^*}$ and $i_A \otimes \mathrm{id}$. Then the canonical element r in $A^* \otimes A$ corresponding to the pairing $\langle, \rangle : A^* \otimes A \to \mathbb{K}$ is a quasitriangular structure for $\mathcal{D}(A)$, or rather, its image under the embedding $A^{*\mathrm{op}} \otimes A \hookrightarrow \mathcal{D}(A) \otimes \mathcal{D}(A)$. That r satisfies the quadratic equations is immediate — remember that the comultiplication on $\mathcal{D}(A)$ is as $A \otimes A^{*\mathrm{op}}$. That $(\mathcal{D}(A), r)$ is almost commutative is best checked using the graphical language, where it is straightforward.[7]

  When A is infinite-dimensional, the canonical element $r \in (A^* \otimes A)^*$ corresponding to the pairing $\langle, \rangle : A^* \otimes A \to \mathbb{K}$ is not an element of $A^* \otimes A$. The most important case is of quantized universal enveloping algebras, which are complete, topologically free, and finitely generated over $\mathbb{K}[[h]]$, and we also consider only A-modules that are complete and topologically free over $\mathbb{K}[[h]]$. When A is topologically free and finitely generated over $\mathbb{K}[[h]]$, then the canonical element of $(A^* \otimes A)^*$ is in $A^* \hat{\otimes} A$, where we complete the tensor product in the h-adic topology.

We conclude with one more definition and an important result. The definition: an almost cocommutative bialgebra (A, r) is *coboundary* if $r^{-1} = \mathrm{flip} \circ r$ and if $(e \otimes e) \circ r = 1 \in \mathbb{K}$. In particular, a triangular structure is coboundary. The result:

[7] In the other convention for Hopf pairings, the quasitriangular structure r is the canonical element in $A \otimes A^* \hookrightarrow \mathcal{D}(A) \otimes \mathcal{D}(A)$. In that convention, $(A^* \otimes A)^* = A \otimes A^*$, rather than $A^* \otimes A$ for us, so the universal R-matrix r is still the element corresponding to the pairing $\langle, \rangle : A^* \otimes A \to \mathbb{K}$.

- Let A be a topologically free Hopf algebra over $\mathbb{K}[[h]]$ such that $A/hA \cong \mathcal{U}\mathfrak{g}$ for a Lie algebra $\mathfrak{g}$; i.e. let A be a quantized universal enveloping algebra. Recall that A defines a Lie bialgebra structure on $\mathfrak{g}$: $\Delta_A - \Delta_A^{\mathrm{op}}$ is divisible by h, since $\mathcal{U}\mathfrak{g}$ is cocommutative, and the image of $h^{-1}(\Delta_A - \Delta_A^{\mathrm{op}})$ under $h \to 0$ is a coPoisson structure on $\mathcal{U}\mathfrak{g}$, and so its restriction to $\mathfrak{g} \hookrightarrow \mathcal{U}\mathfrak{g}$ is a Lie bialgebra structure on $\mathfrak{g}$. Assume moreover that $r \in A \hat{\otimes} A$ (completed only in the h-adic direction) is a (topological) coboundary/triangular/quasitriangular structure on A, and that the image of r in A/hA is $1 \otimes 1$. Then $h^{-1}(r - 1 \otimes 1)$ defines a coboundary/triangular/quasitriangular structure on $\mathfrak{g}$.

The word "coboundary" is immediate in terms of Lie algebra cohomology. The word "triangular" first arose in physics, where the QYBE with spectral parameters was introduced to understand the factorization of the S-matrix for a three-particle scattering as a product of two-particle S-matrices. There are two such factorizations: for particles a, b, c in $1+1$-dimensional spacetime to change places, either a and b can switch, then a and c, then b and c, or alternately bc, ac, ab. The wordlines of such particles then make either of two triangles, and the QYBE was first called the "triangle equation". From a Hopf algebra point of view, the word "quasitriangular" is natural once "triangular" has been coined.

5 More on the antipode

An almost cocommutative Hopf algebra A is cocommutative up to an inner automorphism of $A \otimes A$. Since a cocommutative Hopf algebra is involutive (the square of the antipode s is the identity), it shouldn't be surprising that an almost cocommutative Hopf algebra is involutive up to an inner automorphism of A. Indeed, let A be a Hopf algebra with almost cocommutative structure $r = r_A$. We define the element $u = u_A \in A$ by $u = m \circ (s \otimes \mathrm{id}) \circ \mathrm{flip} \circ r$, where $s = s_A$ is the antipode map for A. Then u is an invertible element, and for any $a \in A$, $s^2(a) = uau^{-1}$. See [2, pp. 120-1] for a proof; the hint is to start with the almost-commutativity axiom applied to the first two components of a triple comultiplication, apply $\mathrm{id} \otimes s \otimes s^2$, and reverse the order and multiply. The inverse is given by $m \circ (s^{-1} \otimes \mathrm{id}) \circ \mathrm{flip} \circ r^{-1}$. Following [2], we remark that this element $u \in A$ is not unique: $m \circ (s \otimes \mathrm{id}) \circ r^{-1}$, $m \circ (\mathrm{id} \otimes s^{-1}) \circ \mathrm{flip} \circ r$, and $m \circ (\mathrm{id} \otimes s) \circ r^{-1}$ work as well. Indeed, the ratio of any two of these elements is central. When r is quasitriangular, some of these elements agree.

Recall that over any Hopf algebra, a finite-dimensional representation V has a left dual ${}^*V$ and a right dual V^*. As vector spaces, the duals are both the vector space dual of V; as modules, the actions are $\langle a.\nu, v\rangle = \langle \nu, s(a).v\rangle$ for the left dual and $\langle a.\nu, v\rangle = \langle \nu, s^{-1}(a).v\rangle$ for the right dual, where $v \in V$, $\nu : V \to \mathbb{K}$, $a \in A$, and $\langle , \rangle$ is the canonical pairing of a dual vector with a vector. In particular, the left double dual ${}^{**}V$ is the vector space V with the action $a \otimes v \mapsto s^2(a).v$. If $s^2 : A \to A$ is an inner automorphism $s^2(a) = uau^{-1}$, as we showed it is in the almost cocommutative case, then the map $v \mapsto u.v$ is an isomorphism $V \cong {}^{**}V$. Being duals, there are canonical maps $\epsilon : {}^{**}V \otimes {}^*V \to \mathbb{K}$ and $\eta : \mathbb{K} \to V \otimes {}^*V$ — these are just the canonical maps of vector spaces, and the actions of A on ${}^*V$ and ${}^{**}V$ are designed to make ϵ and η into homomorphisms of A-modules. Thus, we can define the *quantum dimension* of V to be the composition $\epsilon \circ (u \otimes \mathrm{id}) \circ \eta$, where by u we mean its image in $\mathrm{End}(V)$; i.e. the quantum dimension of V is just the $\mathbb{K}$-trace of the action of u on V. More generally, we define the *quantum trace* of an element $f \in V \otimes {}^*V = \mathrm{End}(V)$

to be its image under $\epsilon \circ (u \otimes \mathrm{id}) : V \otimes {}^*V \to \mathbb{K}$; i.e. the quantum trace of f is just the trace of $v \mapsto u.f(v)$, or equivalently the trace of $v \mapsto f(u.v)$ by cyclicity of the trace: $\mathrm{qtr}_V f = \mathrm{tr}_V(u \circ f)$. Unfortunately, the notions of "quantum dimension" and "quantum trace" as we have defined them aren't very good: they depend on a choice of element $u \in A$ such that $s^2(a) = uau^{-1}$, and they don't in general respect tensor products. Indeed, if $f \in \mathrm{End}(V)$ and $g \in \mathrm{End}(W)$, we would hope that $\mathrm{qtr}_{V\otimes W}(f \otimes g) = \mathrm{qtr}_V f \ \mathrm{qtr}_W g$. But the left-hand side is $\mathrm{tr}(u \circ f)\,\mathrm{tr}(u \circ g) = \mathrm{tr}\big((u \otimes u) \circ (f \otimes g)\big)$, whereas the right hand side is $\mathrm{tr}\big(d_A u \circ (f \otimes g)\big)$.

When A is quasitriangular, we have defined the element $u \in A$ by $u = m \circ (s \otimes \mathrm{id}) \circ \mathrm{flip} \circ r$, where $r \in A \otimes A$ is the quasitriangular structure. If u were grouplike, it would define a multiplicative quantum trace. In fact, $d_A(u) = \chi \cdot (u \otimes u) = (u \otimes u) \cdot \chi$, where $\chi \in A \otimes A$ is the multiplicative inverse of $(\mathrm{flip} \circ r) \cdot r$ and $\cdot$ is the multiplication in $A \otimes A$. By definition, r is triangular if $\chi = 1 \otimes 1$; thus in a triangular Hopf algebra the quantum trace is multiplicative.

If A is quasitriangular but not triangular, we try to find an invertible element $v \in A$ such that $d(v) = \chi \cdot (v \otimes v)$ — then uv^{-1} is grouplike. If in addition v is central, then $s^2(a) = (uv^{-1})a(uv^{-1})^{-1}$. Then we can redefine the quantum trace to $\mathrm{qtr}_V(f) = \mathrm{tr}_V(uv^{-1}f)$, and this redefined quantum trace is multiplicative on tensor products and still a homomorphism of A-modules $\mathrm{End}(V) \to \mathbb{K}$. We can rescale v, so we may as well demand that $e_A(v) = 1$. For the quantum trace to play well with duals, we ask also that $s(v) = v$ and that $v^2 = u \cdot s(u)$ — in any quasitriangular Hopf algebra, the element $u \cdot s(u)$ is central. A quasitriangular Hopf algebra (A, r) with a choice of v satisfying the conditions listed in this paragraph is a *ribbon Hopf algebra* (c.f. [8]). We list three examples:

- We do not prove here that the standard quantization of a semisimple Lie algebra is (topologically) quasitriangular and in fact ribbon. See [2, chapter 8] for a proof.

- Let A be any quasitriangular Hopf algebra, and define u as above. We formally adjoin a square root of $u \cdot s(u)$. Indeed, consider the vector space $A \oplus Av$, where v is a formal symbol. We give it an algebra structure by demanding that v be central and $v^2 = u \cdot s(u)$. We give it a coalgebra structure by demanding that $d(v) = \chi \cdot (v \otimes v)$. Then this bialgebra is Hopf with $s(v) = v$, and indeed ribbon.

- Any cocommutative Hopf algebra can be made into a ribbon Hopf algebra by taking $r = 1 \otimes 1$ and $v = 1$. Then the quantum trace is just the ordinary trace. Indeed, when r is a triangular structure, then u is grouplike, as we said above, and moreover $u^{-1} = s(u)$, so we can again take $v = 1$.

Given a quasitriangular Hopf algebra (A, r), we have defined a natural isomorphism of finite-dimensional A-modules $V \to {}^{**}V$ given by the action of $u \in A$. When (A, r, v) is ribbon, we have argued that a better choice of isomorphism $V \to {}^{**}V$ is given by the action of uv^{-1}. We have also defined a natural isomorphism $\sigma_{V,W} : V \otimes W \to W \otimes V$ for any modules V, W, given by the action of $r \in A \otimes A$ on $V \otimes W$, followed by the $\mathbb{K}$-linear flip map. Consider the natural homomorphism $\epsilon_V \circ \sigma_{V,{}^*V} : V \otimes {}^*V \to \mathbb{K}$; it corresponds to a natural homomorphism $\zeta_V : V \to {}^{**}V$ — in fact, this map ζ makes sense in any rigid braided monoidal category. A rigid monoidal category is *ballanced* if it is equipped with a natural automorphism $\beta_V : V \to V$ such that $\beta_V \otimes \beta_W = \sigma^2 \circ \beta_{V\otimes W}$, $\beta_{{}^*V} = {}^*(\beta_V)$, and $\beta_1 = \mathrm{id}$, where 1 is the monoidal unit and ${}^*(\beta_V)$ is the homomorphism ${}^*V \to {}^*V$

dual to $\beta_V : V \to V$. A monoidal category that is both ballanced and braided is *ribbon*; in a ribbon category, the natural homomorphism $\kappa_V = \zeta_V \circ \beta_V^{-1} : V \to {}^{**}V$ satisfies $\kappa_{V\otimes W} = \kappa_V \otimes \kappa_W$. In the case of a ribbon Hopf algebra (A, r, v), one can check directly that the action of v satisfies the axioms of the braiding β, whence κ is precisely the action of uv^{-1}.

There is a ribbon category TANGLE, whose objects are n-tuples of points in the plane $\mathbb{R}^2$, and whose morphisms are isotopy classes of smooth embeddings of line segments and circles into $\mathbb{R}^2 \times [0,1]$, such that every segment ends on $\mathbb{R}^2 \times \{0,1\}$. Composition is by gluing. There is a "directed" version of this, where the "strings" are oriented and the points are coherently signed; this is the category dTANGLE. There is a "framed" version, where to each point we assign a unit vector and each string carries sections of the "unit normal" circle bundle; then we can expand the strings a little bit and consider them as "ribbons" with one black side and one white side, defining the category RIBBON. Of course, we can have directed and framed tangles, comprising the category dRIBBON. Moreover, we can consider "generalized (directed, framed) tangles", in which the strings can also land on "coupons", flat oriented bigons in space; in [2] the category of ribbons with coupons is called $\widetilde{\text{dRIBBON}}$.[8] If X is any set, one can form the category dRIBBON(X) (or TANGLE(X), etc.), whose points and strings are colored by elements of X; given two sets X_0 and X_1, one can color the strings of $\widetilde{\text{dRIBBON}}$ by elements of X_0 and the coupons by elements of X_1. If $\mathcal{X} = (X_0, X_1)$ is itself a ribbon category, then one should demand that the coupons are colored by morphisms between the appropriate tensor products. The category dRIBBON is the universal ribbon category in the following sense: if $\mathcal{X}$ is any ribbon category, then there is a unique ribbon functor $\widetilde{\text{dRIBBON}}(X) \to X$ that sends colored points and colored coupons to their colors. Therefore, a choice of category $\mathcal{X}$ and coloring of strings provides an invariant of ribbon tangles, a result due to Reshetikhin and Turaev [8]. In particular, given a ribbon category $\mathcal{C}$ with a chosen object V, one can decide to color every string by that object. The induced functor dRIBBON $\to \mathcal{C}$ sends a ribbon tangle to a morphism $V^{\otimes m} \to V^{\otimes n}$. In particular, it sends a knot to an element in $\mathrm{Hom}(1,1) = 1$.

- The category of representations of $\mathcal{U}_q\mathfrak{sl}_2$ is ribbon. Choosing the two-dimensional "defining representation" with which to construct the functor dRIBBON $\to \mathcal{C}$, one recovers the *Jones polynomial*. See [8].

One can also consider tangles inside three-manifolds with boundary, such that the strings end on the boundaries. See [1] and references therein.

References

[1] B. Bakalov and A. Kirillov. *Lectures on Tensor Categories and Modular Functor*. 2000.

[2] V. Chari and A. Pressley. *A Guide to Quantum Groups*. Cambridge University Press, Cambridge, UK, 1994.

[8]The category $\widetilde{\text{dRIBBON}}$ is really more like a "planar algebra": one can compose not just by stacking, but also by stick morphisms into coupons.

[3] V. Drinfel'd. Quantum Groups. *Journal of Mathematical Sciences*. 1988 vol. 41 (2) pp. 898-915.

[4] P. Etingof and O. Schiffmann. *Lectures on quantum groups*. Second edition. Lectures in Mathematical Physics. International Press, Somerville, MA, 2002.

[5] T. Johnson-Freyd. Poisson Lie linear algebra in the graphical language. Draft available at `http://math.berkeley.edu/~theojf/GraphicalLanguage.pdf`.

[6] S. Majid. Some Remarks on the Quantum Double. 1994. Available at `http://arxiv.org/abs/hep-th/9409056v1`.

[7] N. Reshetikhin. *Quantum Groups*, UC Berkeley, Spring 2009. Unedited lecture notes by T. Johnson-Freyd available at `http://math.berkeley.edu/~theojf/QuantumGroups.pdf`.

[8] N. Reshetikhin and V. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Communications in Mathematical Physics*. 1990 vol. 127 (1) pp. 1-26.