Theo Johnson-Freyd, 15 Nov 2011 ①

Notes on Floer Theory / Gromov-Witten TQFT based on conversations with Zack Sylvan.

Physical input:

We will construct a 2-d TQFT with target a symplectic manifold (M, ω).

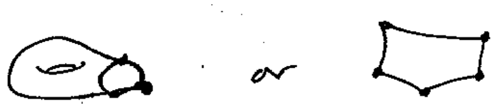
Idea: Boundary conditions = lagrangians in M.

$$\text{Action}(X : \Sigma \to M) = \int_\Sigma X^* \omega .$$

More precisely, for fixed boundary conditions, Fields $= \Pi T(\text{maps } \Sigma \to M \text{ w/ } \partial \text{ conditions})$.
We will use supersymmetry-type arguments to gauge-fix the path integral, and then localize it at certain "classical" solutions.

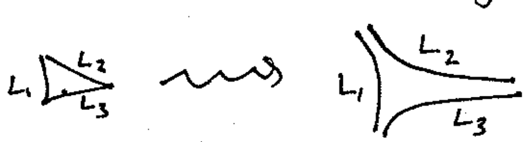
Spacetime:

In the mathematical description, spacetime comes to us as a topological (oriented) surface with boundary and corners:



The corners are labeled either I or O. The edges are labeled by lagrangians $L_1, \ldots \hookrightarrow M$.

It is better to imagine that there are punctures at the corners, so that the necks go off to ∞:



Given such a surface Σ, there is a space of all fields, which is the space of maps $\Sigma \to M$ with $\partial\Sigma \to L \subseteq M$ as prescribed by the diagram.

As physicists, we would like to do the path $\int$ over this space.

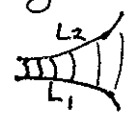
Gauge fixing

Recall that Stokes' Theorem says that we may replace an action by something in the same cohomology class. Usually this breaks manifest symmetry, but improves convergence. If there is enough supersymmetry, we can even localize the integral on the vacua.

For us, the space of *gauge-fixing* *conditions* will be

$$GF = \{\text{metrics on } \Sigma\} \times \{\text{metrics on } M\} \times \{\text{functions on } M\}.$$

Sometimes we ~~will~~ may want the "potential" — the function on M — to also depend on Σ. The metric on M we ask to be compatible with ω, in that $J = g^{-1}\omega$ is an almost complex structure (i.e. $(g^{-1}\omega)^2 = -1$). We will only usually care about the metric on Σ up to conformal isomorphisms — a metric in particular picks out a conformal = complex structure on Σ.

Given $gf \in GF$, the *Euler-Lagrange equations* ~~of~~ for a field $\Sigma \to M$ impose a "modified J-holomorphic curve" equation. We can also calculate the *energy* of a field. Recall that Σ is not compact, and so energy need not be finite. (*Energy* is precisely the gauge-fixed action $\omega + Q\bar{Q}(gf)$, for some "$Q\bar{Q}$".) A finite-energy field is necessarily asymptotic to a Hamiltonian flow near a corner. 

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One expects in the path $\int$ that only finite-energy paths contribute. Also, one expects that in the limit as $\hbar \to 0$ only the vacua = solns to EL eqn = J-holomorphic curves contribute. But SUSY $\Rightarrow$ path $\int$ is indep of $\hbar$. (3)

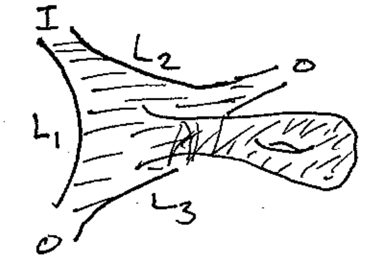
Recall that Σ has labeled ∂. A gauge-fixing condition gf is good if at each corner the set of hamiltonian flows is discrete. Maybe we want some other conditions. One expects goodness to be a dense open condition.

BRST:

One should describe here the details of the FP complex and BRST argument. We will skip them and assert, as above, that in the limit as $\hbar \to 0$ the integral is unchanged but localizes on vacua, and counts (in good situations) vacua with signs. Taking this for granted, the TQFT depends only on the space of vacua.

Mathematical description of the TQFT:

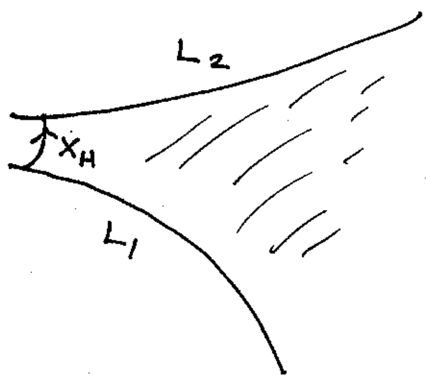
Let Σ be an oriented surface with boundary (labeled by lagrangians) and punctured corners (labeled by signs):

$\Sigma =$ 

For each good gauge-fixing condition

$$gf \in \underbrace{\overset{Z}{\{\text{metrics on } \Sigma\}}}_{\text{"source GF data"}} \times \underbrace{\overset{J}{\{\text{metrics on } M\}}_{\text{compat w/ } \omega} \times \overset{H}{\{\text{potentials on } M\}}}_{\text{"target ~~ga~~ GF data"}}$$

there is a moduli space $\mathcal{M}_\Sigma(gf)$ of finite-energy vacua. By the finite-energy condition, any $v \in \mathcal{M}_\Sigma$ is asymptotically a hamiltonian path for H:

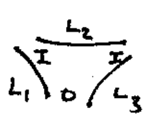


In particular, when $H=0$, asymptotically vacua approach intersections $L_1 \cap L_2$.

Set $CF^\bullet(L_1, L_2)$ = vector space with basis the Hamiltonian (for H) paths $L_1 \rightsquigarrow L_2$.

Then to $\partial\Sigma$ we associate

$$Z(\partial\Sigma) = \mathrm{Hom}\Big(\bigotimes_{I \text{ vertices}} CF^\bullet(L_\ell, L_r), \bigotimes_{O \text{ vertices}} CF^\bullet(L_\ell, L_r)\Big)$$

e.g. $Z\Big($  $\Big) = \mathrm{Hom}\big(CF^\bullet(L_1, L_2) \otimes CF^\bullet(L_2, L_3),\ CF^\bullet(L_1, L_3)\big).$

and we have an <u>evaluation at vertices</u> map

$$\mathcal{M}_\Sigma \to Z(\partial\Sigma) \qquad v \mapsto \bigotimes_{\text{vertices}} v(\text{vertex}).$$

Set $Z(\Sigma) = $ "$\int_{v\in\mathcal{M}_\Sigma} (-1)^{\#} ev(v)$". It depends on the gauge-fixing conditions η, J, H.

Actually, it depends on η only by way of the corresponding conformal structure. Fix J, H. Then we have

$$Z(\Sigma)_{J,H} : \{\text{conformal structures on } \Sigma\} \to Z(\partial\Sigma)_{H}.$$

One expects this to be "continuous" in the sense of extending to

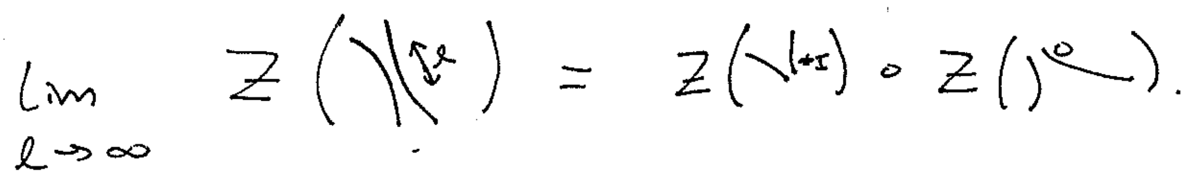
$$Z(\Sigma)_{J,H} : \text{Chains}_{\ast}(\text{conf. str on } \Sigma) \to Z(\partial\Sigma)_H.$$

Some thought must be given to the meaning of "Chains" and to the goodness conditions.

Note that the moduli spaces have cutting/gluing: for fixed J, H, in the limit as η approaches a "long neck" we can cut:

$$\lim_{\ell\to\infty} \mathcal{M}(\ell) = \mathcal{M}(\;) \times_{ev} \mathcal{M}(\;)$$

It follows that we have "composition" law for Z:



E.g.: A_∞ structure

So far we have, for fixed J, H, the space T_Σ of conformal structures on Σ, and over this the space $\mathcal{M}_\Sigma$, so that the fiber of $\begin{matrix}\mathcal{M}_\Sigma \\ \downarrow \\ T_\Sigma\end{matrix}$ at $\eta \in T_\Sigma$ is the moduli space of finite-energy vacua for $gf = (\eta, J, H)$. We have

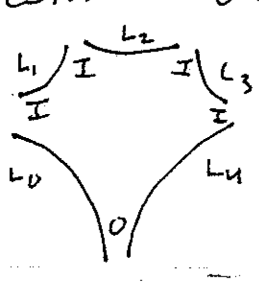
$$\begin{array}{ccc} ev : \mathcal{M}_\Sigma & \longrightarrow & Z(\partial\Sigma)_{H} \\ \downarrow & & \\ T_\Sigma & & \end{array}$$

a distinguished $Z(\partial\Sigma)$-valued cochain on $\mathcal{M}_\Sigma$, and the pushforward

$$Z(\Sigma)_{J,H} : T_\Sigma \to Z(\partial\Sigma)_H$$

using the fact that fibers are unions of manifolds.

We consider now the situation when Σ is a disk with ~~some number of~~ k-many I vertices and one O vertex:



Then T_Σ is a $(K-2)$-dim space.

E.g. $K=0$. $T = \{$ conf. strs on $\}$

$$= B(\mathbb{R}^\times \ltimes \mathbb{R})$$

because $=$ has an action of $PSL(2,\mathbb{R})$, and we've fixed one point.

$$K=1. \quad T = B\mathbb{R}.$$

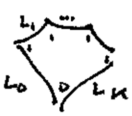
$$K=0. \quad T = \cancel{\emptyset}\ \mathrm{pt}.$$

$$K=2. \quad T = (-\infty, 0).$$

It has a fundamental class which is a $(K-2)$-chain.

We define $a_K(L_0, \dots, L_{K+1}) \in \mathrm{Hom}\left(\bigotimes_{i=1}^{K} CF^\bullet(L_{i-1}, L_i), CF^\bullet(L_0, L_K)\right)$

$$\parallel$$

$Z($  $)$ [fundamental class].

Def: The Fukaya category $\mathcal{F}(M,\omega)_{J,H}$ has objects = Lagrangians and $\hom(L, L') = CF^\bullet(L, L')$. It is an A_∞-category, with A_∞ structure $= \{a_K\}_{K=0}^{\infty}$.

Recall that an A_∞-category is curved if $a_0 \neq 0$ and flat if $a_0 = 0$. If ~~the~~ $\mathcal{F}(M,\omega)_{J,H}$ is flat, then a_1 is a dg structure on $CF^\circ(L, L')$, and we can pass to homology $HF^\circ = H(CF^\circ, a_1)$. This is Floer homology and does not depend on J, H.

What's usually called the TQFT structure

Let's suppose that the Fukaya category is flat. Then we should expect that

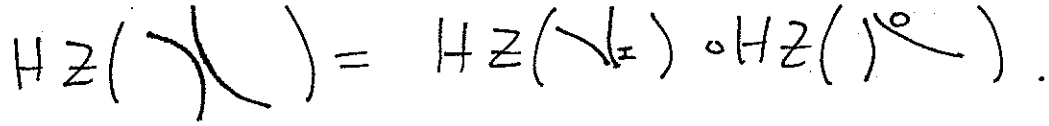
$$Z(\Sigma)_{J,H} : \mathrm{Chains}_*(T_\Sigma) \longrightarrow \mathcal{F}\left(Z(\partial\Sigma)_{J,H}, (a_1)_{J,H}\right)$$

is a map of chain complexes.

Since T_Σ is connected, we thus can define

$$HZ(\Sigma) = \left[Z(\Sigma)[pt]\right] \in H\left(Z(\partial\Sigma)_H, (a_1)\right)$$
$$\|$$
$$\mathrm{Hom}\left(\bigotimes_I HF, \bigotimes_O HF\right).$$

One expects also that this does not depend on J, H. Note that by construction, $HZ(\Sigma)$ is a degree-zero thing. It satisfies a gluing rule at corners:



Defn: The Gromov-Witten invariants are the numbers HZ(closed surfaces). There are also

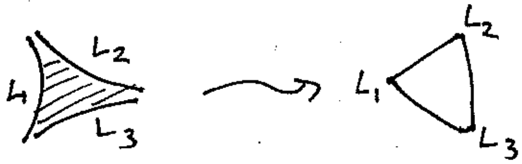
"Gromov-Witten invariants" for surfaces with labeled boundary ~~by~~ but no corners, because

$$\cancel{Z(\partial}\ Z_\partial(\bigcirc^L) = \mathbb{C} \qquad \text{(no corners)}.$$

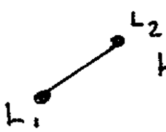
These seem somewhat mysterious.

What's missing?

It would probably make the category theorists happiest if we drew the edge-dual surfaces



because then we can think that $L \mapsto A_\infty\text{-alg } CF^*(L,L)$,



$\mapsto$ bimodule,  $\overset{\text{"HZ"}}{\mapsto}$ ~~[illegible]~~ intertwiner.

This is almost correct, especially if we declare that $\otimes$ of bimodules is given by the $\{a_k\}$s.

But we don't really have a "gluing" across Lagrangians whether we write them as edges or vertices.

The problem is that

$Z(\partial($  $)) = Z(\partial($  $))$,

but $Z($  $)$ and $Z($  $)$ seem

unrelated / able.