

You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (10 pts) Determine whether the following series converges or diverges. If it converges, find the limit.

$$\sum_{n=1}^{\infty} \left( \cos \frac{\pi}{n+1} - \cos \frac{\pi}{n} \right)$$

We compute the  $N$ th partial sum  $s_N$ :

$$\begin{aligned} s_N &= \sum_{n=1}^N \left( \cos \frac{\pi}{n+1} - \cos \frac{\pi}{n} \right) = \left( \cos \frac{\pi}{2} - \cos \frac{\pi}{1} \right) + \left( \cos \frac{\pi}{3} - \cos \frac{\pi}{2} \right) + \cdots + \\ &\quad + \left( \cos \frac{\pi}{N-1+1} - \cos \frac{\pi}{N-1} \right) + \left( \cos \frac{\pi}{N+1} - \cos \frac{\pi}{N} \right) \\ &= -\cos \frac{\pi}{1} + \cos \frac{\pi}{N+1} \end{aligned}$$

Thus, the infinite series converges if and only if  $\lim_{N \rightarrow \infty} s_N$  exists, and this limit gives the value. We have:

$$\begin{aligned} \lim_{N \rightarrow \infty} s_N &= \lim_{N \rightarrow \infty} \left( -\cos \pi + \cos \frac{\pi}{N+1} \right) \\ &= -\cos \pi + \lim_{N \rightarrow \infty} \cos \frac{\pi}{N+1} \\ &= -(-1) + \cos 0 = \boxed{2} \end{aligned}$$

In particular, the series converges.

2. (bonus) Explain why it's wrong to write:

$$\sum_{n=1}^{\infty} \left( \cos \frac{\pi}{n+1} - \cos \frac{\pi}{n} \right) = \left( \sum_{n=1}^{\infty} \cos \frac{\pi}{n+1} \right) - \left( \sum_{n=1}^{\infty} \cos \frac{\pi}{n} \right)$$

We cannot write the above equation because the sums on the right-hand-side each diverge. Indeed,

$$\lim_{n \rightarrow \infty} \cos \frac{\pi}{n+1} = \cos 0 = 1$$

and so the sums each fail the divergence test.