Math 1B: First Exam
Tuesday, 7 July 2009

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Name: ANSWERS

Problem Number	1a	1b	1c	1d	2a	2b	3	4a	4b	4c	Total
Score											
Maximum	10	10	10	10	10	10	15	5	5	15	100

Please do not begin this test until 2:10 p.m. You may work on the exam until 4 p.m. Please do not leave during the last 15 minutes of the exam time.

You must always justify your answers: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. Please box your final answers.

Calculators are not allowed. Please sign the following honor code:

I, the student whose name and signature appear on this midterm, have completed the exam by myself, without any help during the exam from other people, or from sources other than my allowed one-page hand-written cheat sheet. Moreover, I have not provided any aid to other students in the class during the exam. I understand that cheating prevents me from learning and hurts other students by creating an atmosphere of distrust. I consider myself to be an honorable person, and I have not cheated on this exam in any way. I promise to take an active part in seeing to it that others also do not cheat.

Signature: ________________________________

1. (40 pts – 4 questions, 10 pts each) Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{x\sqrt{x^2+4}}$

We substitute $x = 2\tan\theta$. Then:

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta\, d\theta}{2\tan\theta\sqrt{4\tan^2\theta+4}} \\
&= \int \frac{2\sec^2\theta\, d\theta}{2\tan\theta\, 2\sec\theta} \\
&= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}\, d\theta \\
&= \frac{1}{2}\int \frac{1/\cos\theta}{\sin\theta/\cos\theta}\, d\theta \\
&= \frac{1}{2}\int \csc\theta\, d\theta \\
&= \frac{1}{2}\ln|\csc\theta - \cot\theta| + C \\
&= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{x} - \frac{2}{x}\right| + C \\
&= \boxed{\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C}
\end{aligned}$$

(b) $\displaystyle\int \frac{x+1}{x^2-6x+5}\, dx$

We factor the denominator and expand into partial fractions:

$$\begin{aligned}
\int \frac{x+1}{x^2-6x+5}\, dx &= \int \frac{x+1}{(x-1)(x-5)}\, dx \\
&= \int \left[\frac{-1/2}{x-1} + \frac{3/2}{x-5}\right] dx \\
&= \boxed{-\frac{1}{2}\ln(x-1) + \frac{3}{2}\ln(x-5) + C}
\end{aligned}$$

(c) $\displaystyle\int_0^{\pi/3} x \sec x \tan x \; dx$

We integrate by parts with $u = x$ and $dv = \sec x \tan x \, dx$:

$$\begin{aligned}
\int_0^{\pi/3} x \sec x \tan x \; dx &= x \sec x \Big|_0^{\pi/3} - \int_0^{\pi/3} \tan x \, dx \\
&= \Big[x \sec x\Big]_0^{\pi/3} - \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/3} \\
&= \frac{\pi}{3} \sec \frac{\pi}{3} - 0 - \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| + \ln|1+0| \\
&= \frac{\pi}{3} \cdot 2 - \ln\left(2 + \sqrt{3}\right) \\
&= \boxed{\frac{2}{3}\pi - \ln\left(2+\sqrt{3}\right)}
\end{aligned}$$

(d) $\displaystyle\int (\cos x + \sin x)^2 \, dx$

We expand and use trig identities:

$$\begin{aligned}
\int (\cos x + \sin x)^2 \, dx &= \int \left(\cos^2 x + 2\cos x \sin x + \sin^2 x\right) dx \\
&= \int (1 + 2\cos x \sin x) \, dx \\
&= \int (1 + \sin 2x) \, dx \\
&= \boxed{x - \frac{1}{2}\cos 2x + C}
\end{aligned}$$

2. (20 pts – 2 questions, 10 pts each) Determine whether each of the following improper integrals is convergent or divergent. You do not need to *prove* your answer, and you do not need to *evaluate* the integrals (although you are welcome to do either or both). For example, if I asked you about $\int_1^\infty (x^2+x)^{-1} dx$, you could evaluate the integral (partial fractions), or you could use the comparison test to prove that it converges, but it's enough to write: "$x^2 + x \approx x^2$ as $x \to \infty$, so we expect the integral to converge, since $\int_1^\infty x^{-2} dx$ converges by the p-test."

   (a) $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{x}}\, dx$. Hint: do not try to evaluate.

   The only potential divergence is at $x = 0$. When $x \approx 0$, we have $\cos x \approx 1$, and so

$$\int_0^{\text{small}} \frac{\cos x}{\sqrt{x}}\, dx \approx \int_0^{\text{small}} \frac{1}{\sqrt{x}}\, dx$$

   which converges by the P-test with $p = 1/2$. Thus, we expect the integral to $\boxed{\text{converge}}$.

   (b) $\displaystyle\int_2^\infty \frac{dx}{x \ln x}$. Hint: u-substitution.

   We substitute $u = \ln x$. Then $du = dx/x$ and $\ln \infty = \infty$, so the integral is:

$$\int_2^\infty \frac{dx}{x \ln x} = \int_{\ln 2}^\infty \frac{du}{u}$$

   which diverges by the P-test. Thus the original integral $\boxed{\text{diverges}}$.

3. (15 pts) Let's say you want to know the numerical value of $\ln 3 = \int_1^3 x^{-1} dx$, and you decide to approximate the integral using the trapezoid rule. How many subintervals do you need to take in order to assure that your answer is accurate to within an error of $0.00001 = 10^{-5}$?

We recall the error formula for the Trapezoid Rule:

$$E \leq \frac{K(b-a)^3}{12\,n^2}$$

where K is any number such that $K \geq |f''(x)|$ for all $x \in [a, b]$, where $[a, b] = [1, 3]$ and $f(x) = x^{-1}$. Then $f''(x) = 2x^{-3}$, which is a decreasing function, so it is maximized when $x = 1$, whence $K = 2$ works. Thus, we have:

$$E \leq \frac{2 \cdot 2^3}{12\,n^2} = \frac{4}{3} n^{-2}$$

We want this number to be less than 10^{-5}. Thus, we solve:

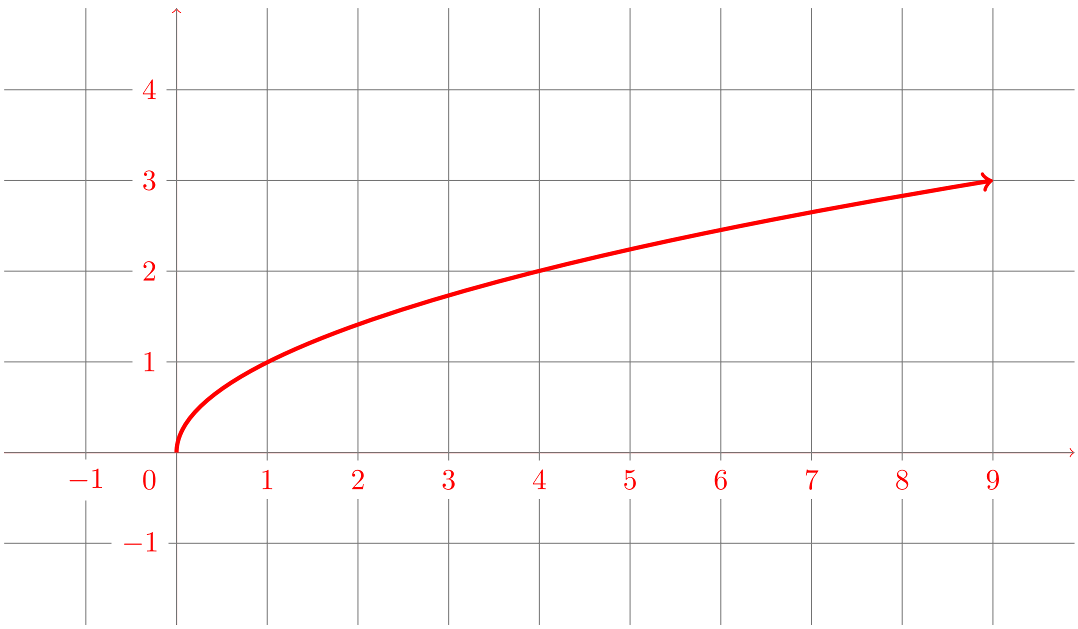
$$\begin{aligned} \frac{4}{3} n^{-2} &\leq 10^{-5} \\ \frac{3}{4} n^2 &\geq 10^5 \\ n^2 &\geq \frac{4}{3} 10^5 \\ n &\geq \sqrt{\frac{4}{3} 10^5} \\ &= \sqrt{40/3} \times 10^2 \end{aligned}$$

We round $40/3 = 13\frac{1}{3}$ up to 16 (a perfect square), and conclude that $\boxed{n = 4 \times 10^2 = 400}$ works.

4. Each of the following questions required the answers to the previous ones. If you answer one incorrectly, I will grade the later problems as if your answer was correct (so that you will not be penalized twice). If you cannot answer one question, you may ask me for the answer, and I will mark your test as such.

   (a) (5 pts) Let a be an arbitrary positive real number. Sketch the curve $\{y = \sqrt{x} : 0 \le x \le a\}$. Write an expression for the arclength as an integral in terms of x.

   We sketch the curve $y = \sqrt{x}$:



   Recalling the arclength formula, the length of the curve for $0 \le x \le a$ is:

$$\boxed{\ell = \int_0^a \sqrt{1 + \frac{1}{4x}}\, dx}$$

   since if $f(x) = \sqrt{x}$ then $f'(x) = 1/2\sqrt{x}$.

   (b) (5 pts) Explain why your expression in part (a) is an improper integral.

   The integral is improper at $x = 0$. The integrand blows up there, since $\lim_{x \searrow 0} \frac{1}{4x} = \infty$.

   We remark that the integral nevertheless converges (which we prove by evaluating it in the next question). Indeed: $1 + \frac{1}{4x} \le \frac{1}{x}$ for $x \le 1/3$, and so $\int_0^{1/3} \sqrt{1 + \frac{1}{4x}}\, dx \le \int_0^{1/3} x^{-1/2}\, dx$ which converges by the P-test.

(c) (15 pts) Evaluate your integral from part (a). Hint: substitute $x = \frac{1}{4}\tan^2\theta$. Never leave any answer in a form like $\sin(\arccos b)$ — use a triangle to find a corresponding algebraic expression.

We are interested in the integral $\int_0^a \sqrt{1+\frac{1}{4x}}\,dx$. We substitute $x = \frac{1}{4}\tan^2\theta$, whence $dx = \frac{1}{2}\tan\theta\sec^2\theta\,d\theta$. Thus:

$$\begin{aligned}
\int_0^a \sqrt{1+\frac{1}{4x}}\,dx &= \int_{x=0}^{x=a} \sqrt{1+\frac{1}{\tan^2\theta}}\,\frac{1}{2}\tan\theta\sec^2\theta\,d\theta \\
&= \int_{x=0}^{x=a} \sqrt{\frac{\tan^2\theta+1}{\tan^2\theta}}\,\frac{1}{2}\tan\theta\sec^2\theta\,d\theta \\
&= \frac{1}{2}\int_{x=0}^{x=a} \sqrt{\frac{\sec^2\theta}{\tan^2\theta}}\tan\theta\sec^2\theta\,d\theta \\
&= \frac{1}{2}\int_{x=0}^{x=a} \frac{\sec\theta}{\tan\theta}\tan\theta\sec^2\theta\,d\theta \\
&= \frac{1}{2}\int_{x=0}^{x=a} \sec^3\theta\,d\theta \\
&= \frac{1}{2}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_{x=0}^{x=a}
\end{aligned}$$

where the last line is from the cheat sheet. When $x = 0$, we have $\theta = 0$, and so $\tan\theta = 0$ and $\sec\theta = 1$. When $x = a$, we have $\tan\theta = 2\sqrt{x} = 2\sqrt{a}$, and $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+4a}$. Thus, we see that:

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_{x=0}^{x=a} \\
&= \frac{1}{4}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]_{x=0}^{x=a} \\
&= \frac{1}{4}\left[\sqrt{1+4a}\,2\sqrt{a} + \ln\left(\sqrt{1+4a}+2\sqrt{a}\right)\right] \\
&= \boxed{\frac{1}{2}\sqrt{a}\sqrt{1+4a} + \frac{1}{4}\ln\left(\sqrt{1+4a}+2\sqrt{a}\right)}
\end{aligned}$$

References: Problems 1 and 2 are modifications of exercises from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart. Problems 3 and 4 are folklore. The honor-code language is adapted from the Stanford Honor Code (`http://www.stanford.edu/dept/vpsa/judicialaffairs/guiding/honorcode.htm`) and from the exams by Zvezda Stankova.

Feel free to use this page for extra scrap work.