Math 1A: Discussion Exercises

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Average Values

1. § Find the average value of the function on the given interval. Verify the Mean Value Theorem by finding a number c in the interval that realizes this value.

   (a) $4x - x^2$, $[0, 4]$
   (b) $\sqrt[3]{x}$, $[1, 8]$
   (c) $\sin x$, $[0, \pi]$
   (d) $(x - 3)^2$, $[2, 5]$
   (e) $2 \sin x - \sin 2x$, $[0, \pi]$
   (f) $2x/(1 + x^2)^2$, $[0, 2]$

2. § If $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on the interval $[1, 3]$.

3. § Prove the Mean Value Theorem For Integrals by applying the MVT For Derivatives to the function $F(x) = \int_a^x f(t)\,dt$.

4. It is a theorem that a continuous function $f(t)$ on a closed interval $[a, b]$ takes a maximum value M and a minimum value m.

   (a) How do the three numbers $(b - a)M$, $(b - a)m$, and $\int_a^b f(t)\,dt$ relate? Put them in ascending order.
   (b) Use the Intermediate Value Theorem to prove the MVT For Integrals.

5. (a) § Let $p(x) = ax^3 + bx^2 + cx + d$ be a cubic function, and assume that $\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$. Prove that $p(x)$ has a root in the interval $[0, 1]$.
   (b) Generalize to polynomials of arbitrary degree.

6. § Let f be a continuous function, and write $f_{\text{ave}}[a, b]$ for the average value of f on the interval $[a, b]$. Let $a < c < b$. Prove that:

$$f_{\text{ave}}[a, b] = \frac{c - a}{b - a} f_{\text{ave}}[a, c] + \frac{b - c}{b - a} f_{\text{ave}}[c, b]$$

   What does the above equation say when $c = \frac{a+b}{2}$?

7. § The velocity v of blood that flows in a blood vessel at a distance r from the central axis is

$$v(r) = \frac{P}{4\nu l}\left(R^2 - r^2\right)$$

   where R is the radius of the vessel, l is the length, P is the pressure difference between the ends of the vessel, and η is the viscosity of blood.

   (a) Find the maximum value of $v(r)$.
   (b) Find the average value of $v(r)$ with respect to r over the interval $0 \le r \le R$.
   (c) The total amount of blood that flows through a blood vessel in unit time is not $\int_0^R v(r)\,dr$, because more blood is at larger radii than at smaller radii. (Why?) Instead, the total amount of blood that flows through the blood vessel in a given amount of time is $2\pi \int_0^R v(r)\, r\,dr$. Find this number.
   (d) Find the average value of $2\pi v(r)\, r$ with respect to r over the interval $0 \le r \le R$. How do your various answers to the different questions relate?