Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

The Fundamental Theorem of Calculus, part 2

1. § Use the Fundamental Theorem of Calculus to evaluate the following integrals:

(a) $\int_{-2}^{5} 6\,dx$

(b) $\int_{0}^{1} \left(1 + \frac{1}{2}u^4 - \frac{2}{5}u^9\right) du$

(c) $\int_{-1}^{2} \left(x^3 - 2x\right) dx$

(d) $\int_{1}^{4} \left(5 - 2t + 3t^2\right) dt$

(e) $\int_{0}^{1} x^{4/5}\,dx$

(f) $\int_{1}^{2} \frac{3}{t^4}\,dt$

(g) $\int_{0}^{2} x\left(2 + x^5\right) dx$

(h) $\int_{1}^{9} \frac{x-1}{\sqrt{x}}\,dx$

(i) $\int_{0}^{2} (y-1)(2y+1)\,dy$

(j) $\int_{0}^{\pi/4} \sec^2 t\,dt$

(k) $\int_{-1}^{1} e^{2u}\,du$

(l) $\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt$

2. § What's wrong with the following equations?

$$\int_{-2}^{1} x^{-4}\,dx = \left.\frac{x^{-3}}{3}\right]_{-2}^{1} = -\frac{3}{8}$$

$$\int_{\pi/3}^{\pi} \sec\theta\tan\theta\,d\theta = \sec\theta\Big]_{\pi/3}^{\pi} = -3$$

3. What's wrong with the following equations?

$$\int_{\pi/6}^{\pi/4} \frac{1}{\cos t}\,dt = \left.\frac{1}{\sin t}\right]_{\pi/6}^{\pi/4} = \sqrt{2} - 2$$

$$\int_{0}^{2} \sqrt{3-x}\,dx = \left.\sqrt{3x - \frac{x^2}{2}}\right]_{0}^{2} = 2$$

4. (a) What is the derivative of $g(x) = x\ln x$?
   (b) Use your answer to part (a) to find an antiderivative for $f(x) = \ln x$.
   (c) Use your answer to part (b) to evaluate $\int_1^2 \ln x\,dx$.

5. § Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$ by first recognizing the sum as a Rieman sum for a function defined on $[0, 1]$, and then evaluating the corresponding integral using the Fundamental Theorem of Calculus.

6. (a) § Prove that $\cos x^2 \geq \cos x$ for $0 \leq x \leq 1$. Hint: What's the relationship between x and x^2?

   (b) § Deduce that $\int_0^{\pi/6} \cos x^2 \, dx \geq \frac{1}{2}$.

7. § Prove that $0 \leq \displaystyle\int_5^{10} \frac{x^2}{x^4 + x^2 + 1} \, dx \leq 0.1$ by comparing the integrand to a simpler function.

8. § If f is continuous on $[a, b]$, prove that:

$$2 \int_a^b f(x) f'(x) \, dx = [f(b)]^2 - [f(a)]^2$$

9. § A manufacturing company owns a major piece of equipment that depreciates at the (continuous) rate $f = f(t)$, where t is the time measured in months since its last overhaul. Because a fixed cost A is incurred each time the machine is overhauled, the company wants to determine the optimal time T (in months) between overhauls.

   (a) Explain why $\int_0^t f(s) \, ds$ represents the loss in value of the machine over a period of time t since the last overhaul.

   (b) Let $C = C(t)$ be given by

$$C(t) = \frac{1}{t} \left(A + \int_0^t f(s) \, ds \right)$$

   What does C represent and why would the company want to minimize C?

   (c) Show that C has a minimum value at the numbers $t = T$ where $C(T) = f(T)$.

10. § If f is a differentiable function on $(0, \infty)$ such that $f(x)$ is never 0 and $\int_0^x f(t) \, dt = [f(x)]^2$ for all $x > 0$, find f.

11. § Evaluate $\lim_{x\to 0} \frac{1}{x} \int_0^x (1 - \tan 2t)^{1/t} \, dt$.

12. § Evaluate the following limit by interpreting it as a Riemann sum for a continuous function on the interval $[1, 2]$, and evaluating the integral using by the Fundamental Theorem of Calculus:

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \frac{1}{\sqrt{n}\sqrt{n+3}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+(n-1)}} + \frac{1}{\sqrt{n}\sqrt{n+n}} \right)$$